1. UNSW 52nd Annual School Mathematics Competition
Junior Division

Problem 1.1: Suppose that x, y, z are non-zero integers with no common factor except 1 such that

$$x^2 + y^2 = z^2.$$

Prove that exactly one of these integers is a multiple of 5.

Solution. Let $x, y, z \in \mathbb{Z}$, $x, y, z \neq 0$

$$\gcd(x, y, z) = 1.$$

Now

$$\begin{array}{rcccccc} x & \equiv & 0 & 1 & 2 & 3 & 4 \mod 5 \\ x^2 & \equiv & 0 & 1 & 4 & 4 & 1 \mod 5 \\ 2x^2 & \equiv & 0 & 2 & 3 & 3 & 2 \mod 5 \end{array}$$

So, if $x \equiv y \mod 5$, then

$$z^2 = x^2 + y^2 \equiv 2x^2 \mod 5$$

is possible if and only if

$$x \equiv y \equiv z \equiv 0 \mod 5.$$

This contradicts to

$$\gcd(x, y, z) = 1.$$

So $x \not\equiv y \mod 5$. Thus, we have

$$\begin{array}{rll} x^2 + y^2 \equiv & 0 + 1 \text{(in some order)} & \equiv 1 \quad (1) \\ \text{or} \equiv & 0 + 4 \ldots & \equiv 4 \quad (2) \\ \text{or} \equiv & 1 + 4 \ldots & \equiv 0 \quad (3) \\ \text{or} \equiv & 4 + 4 \ldots & \equiv 3 \quad (4) \end{array}$$

Since $z^2 \equiv 3$ is impossible, (4) cannot occur. So either (1), (2) or (3) occurs.

In (1) or (2) exactly one of $x, y \equiv 0 \mod 5$; and in (3) neither of $x, y \equiv \mod 5$ and $z \equiv 0 \mod 5$. □

Problem 1.2: How many integers between 100 and 100000 have exactly three identical digits in their decimal representation. For example, 11123 and 10020 are such numbers but 12121 is not.

Solution. The only 6-digit number in the range is 100000 fails the condition so the required numbers have 3, 4 or 5 digits.

Let α represents the number of integers with repeating 0 and β the number of integers with repeating non-zero digit.

For the integers computed by α, the leading digit is non-zero, so

$$\alpha = \alpha_4 + \alpha_5,$$

where α_j is the number of j-digit integers counted by α. So

$$\alpha_j = \underbrace{9}_{A} \times \underbrace{\binom{j-1}{3}}_{B} \times \underbrace{P(8, j-4)}_{C},$$

where A stands for the choice for the leading digit; B stands for the choice of the places for the repeating zero; and C represents filling the remaining $j-4$ positions with other digits with no repetitions. So, we compute

$$\begin{aligned} \alpha &= \alpha_4 + \alpha_5 \\ &= 9 \times 1 \times 1 + 9 \times \binom{4}{3} \times 8 \\ &= 9 + 9 \times 4 \times 8 \\ &= 9 \times 33 = 297. \end{aligned}$$

We split the integers counted by β into three subgroups:

$$\beta = \beta_3 + \beta_4 + \beta_5,$$

where β_j is the number of j-digit integers within β.

Within integers counted by β, repeating digit may or may not include leading digit, so we compute

$$\begin{aligned} \beta_j = \underbrace{9}_{A} \times \Bigg[& \underbrace{1}_{B} \times \underbrace{\binom{j-1}{2}}_{C} \times \underbrace{P(9, j-3)}_{D} \\ & + \underbrace{8}_{B'} \times \underbrace{\binom{j-1}{3}}_{C'} \times \underbrace{P(8, j-4)}_{D'} \Bigg], \end{aligned}$$

where A chooses the non-zero repeating digit; B puts that digit into the leading position; C chooses two other positions for the remaining repeating digits; D fills the rest with no repetition; B' chooses another non-zero digit for the leading position; C' chooses positions for the repeating digit; D' fills the rest with no repetition.

Skipping some elementary computations, we conclude that

$$\beta = 6516 \quad \text{and} \quad \alpha + \beta = 6813.$$

□

Problem 1.3: 1) Find the number which is divisible by 2 and 9 and which has 14 divisors (including 1 and the number itself).

2) Show that there are multiple solutions in the case when the number has 15 divisors.

Solution. Let

$$n = 2^{\alpha_1} 3^{\alpha_2} p_3^{\alpha_3} \cdot \ldots \cdot p_r^{\alpha_r}$$

be factorisation of n into distinct primes

$$2 < 3 < p_3 < \ldots < p_r$$

such that all $\alpha_j > 0$ and $r \geq 2$. So

$$\begin{aligned} 2|n &\rightarrow \alpha_1 \geq 1 \\ 3|n &\rightarrow \alpha_2 \geq 2 \end{aligned}$$

Observe that the number of divisors of n is

$$d(n) = (\alpha_1 + 1)(\alpha_2 + 1) \cdot \ldots \cdot (\alpha_r + 1)$$

and

$$\alpha_1 + 1 \geq 2 \text{ and } \alpha_2 + 1 \geq 3.$$

1) If $d(n) = 14 = 2 \times 7$, then $r = 2$ and $\alpha_1 + 1 = 2$, $\alpha_2 + 2 = 7$. So

$$\alpha_1 = 1 \text{ and } \alpha_2 = 6$$

and

$$n = 2^1 \times 3^6 = 1458.$$

2) If $d(n) = 15 = 3 \times 5$, then $r = 2$ and

$$\alpha_1 + 1 = 3 \text{ and } \alpha_2 + 2 = 5$$

or

$$\alpha_1 + 1 = 5 \text{ and } \alpha_2 + 2 = 3.$$

So

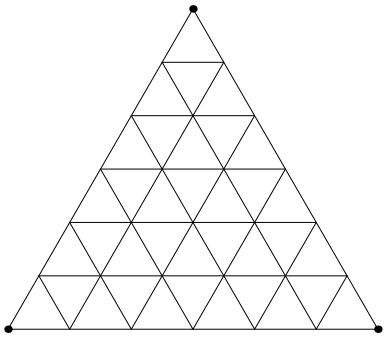
$$\alpha_1 = 2, \quad \alpha_2 = 4 \text{ and } n = 2^2 \cdot 3^4 = 324$$

or

$$\alpha_1 = 4, \quad \alpha_2 = 2 \text{ and } n = 2^4 \times 3^2 = 144.$$

□

Problem 1.4: 1) The plan of an art museum is an equilateral triangle consisting of 36 triangular exhibition halls (see the diagram). Each hall has passages into all adjacent halls. Prove that you can visit at most 31 halls if you plan to enter each hall at most once.



2) Subject to the condition that each hall is visited at most once, find the largest number of halls that can be visited in the case that the museum is an equilateral triangle and that it has k^2 exhibition halls, where k is an integer.

Solution. This is the common problem with Senior Division. The solution is given in Senior Division. □

Problem 1.5: We wish to label the vertices of a regular polygon with 45 sides using the digits $0, 1, \ldots, 9$. Can this be done such that for every pair of digits there is an edge labelled by these digits?

Solution. Such labelling cannot be done. A digit 'a' forms 9 pairs with other digits. To accommodate these pairs, the digit 'a' must be next to at least 5 vertices. So, at least 50 vertices are required. □

Problem 1.6: You are given n different weights and a balance. On every move, you add one more weight to one of the sides of the balance.

1) Prove that there is a strategy such that the left hand side of the balance is lower after the first move; the right hand side of the balance is lower after the second move; the left hand side is lower after the third move; and so on.
2) We assign the "word" LRLRLR... to the strategy above. The letter 'L' means that the left hand side is lower; the letter 'R' means that the right hand side is lower.

   For any "word" of length n of letters 'L' and 'R', find a strategy which will correspond to that word according to the rules above.

Solution. The strategy is based on the following lemma **Lemma**: *Let* $m_1 < m_2 < \ldots < m_k$. *If*

$$S_{\text{odd}} = m_1 + m_3 + \ldots \quad \textit{and} \quad S_{\text{even}} = m_2 + m_4 + \ldots,$$

then

$$S_{\text{even}} > S_{\text{odd}} \quad \textit{if } k \textit{ even}$$

and

$$S_{\text{odd}} > S_{\text{even}} \quad \textit{if } k \textit{ odd.}$$

That is, the sum which has the heaviest weight is the largest.

Proof of Lemma. The proof has two parts. If k is even, then we add the following inequalities:

$$\begin{array}{rcl} m_1 & < & m_2 \\ m_3 & < & m_4 \\ & \vdots & \\ m_{k-1} & < & m_k \end{array}$$

On the other hand, if k is odd, then we add the following inequalities

$$\begin{array}{rcl} 0 & < & m_1 \\ m_2 & < & m_3 \\ & \vdots & \\ m_{k-1} & < & m_k \end{array}$$

□

Now the required strategy is described as follows. It is convenient to describe the strategy in reverse. That is, we give the final distribution of weights first and then we explain how to choose the weight to remove on each step until the balance is empty.

1) Let
$$m_1 < m_2 < \ldots < m_n$$
be the weights;
2) put all odd indexed weights on one side of the balance and all even indexed weights on the other side such that the heaviest weight is on the left hand side, if the last letter of the word is 'L'; or the right hand side if the letter is 'R';
3) on every move, you remove the last letter from the word and a weight from the balance as follows:
   a) you remove the heaviest if the letter at the back changes from 'L' to 'R' or from 'R' to 'L';
   b) otherwise, you remove the lightest weight if the letter at the back of the word does not change.

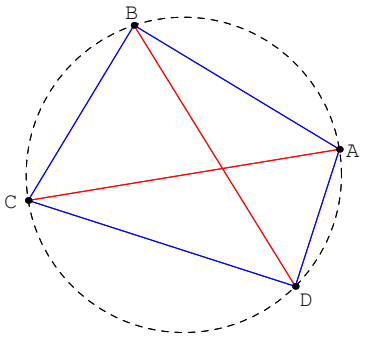
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2. UNSW 52nd Annual School Mathematics Competition
Senior Division

Problem 2.1: In this problem you may assume the following result known as Ptolemy's Theorem:

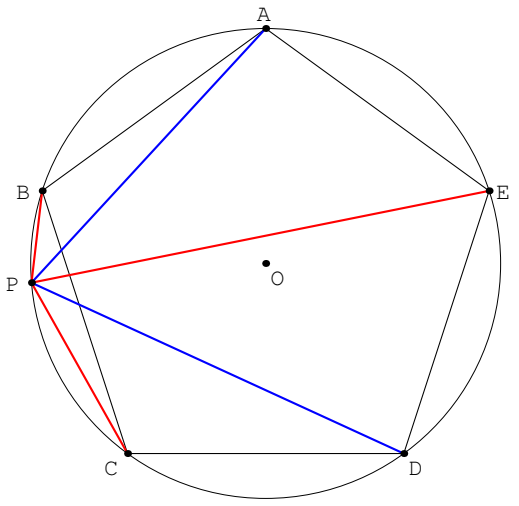
Theorem: *If $ABCD$ is a cyclic quadrilateral, that is a quadrilateral inscribed in a circle, then*

$$BD \cdot AC = AD \cdot BC + AB \cdot DC.$$



Suppose $ABCDE$ is a regular pentagon inscribed in a circle. Let P be any point on the arc BC. Prove that

$$PA + PD = PB + PC + PE.$$



Solution. Since we have to prove

$$PA + PD = PB + PC + PE$$

we may assume (by applying a dilation) that

$$AB = BC = CD = DE = EA = 1.$$

The five diagonals of the regular pentagon have the same length so assume that

$$AC = BD = CE = DA = EB = d.$$

With six points on the circle there are

$$\binom{6}{4} = 15$$

quadruples of points to which Ptolemy's Theorem can be applied. Fortunately the 5 sets corresponding to the points of that pentagon give the same equation. Consider the quadrilateral $ABCD$:

$$AC \cdot BD = AB \cdot CD + AD \cdot BC \quad d^2 = 1 + d \qquad (5)$$

The exact value of d,

$$d = \frac{1}{2}(1 + \sqrt{5}),$$

is not needed.

Apply Ptolemy's Theorem to $PBAE$:

$$PA \cdot BE = PB \cdot AE + PE \cdot AB, \quad PA \cdot d = PB + PE \quad (6)$$

and to $PCDE$:

$$PD \cdot CE = PC \cdot DE + PE \cdot CD, \quad PD \cdot d = PC + PE. \quad (7)$$

Add (6) and (7)

$$PB + PC + 2PE = PA \cdot d + PD \cdot d \qquad (8)$$

Apply Ptolemy's to $PAED$:

$$PE \cdot AD = PA \cdot ED + PD \cdot AE, \quad PE \cdot d = PA + PD,$$

so

$$PE = \frac{1}{d}PA + \frac{1}{d}PD.$$

Substitute in (8)

$$PB + PC + PE = PA(d - \frac{1}{d}) + PD(d - \frac{1}{d}).$$

But from (5)

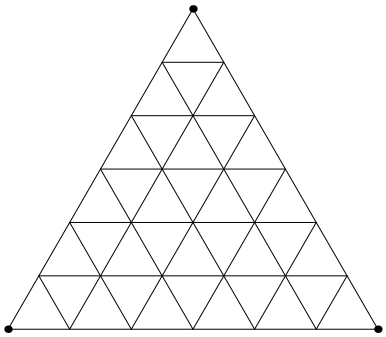
$$d - \frac{1}{d} = 1$$

and

$$PB + PC + PE + PA + PD$$

as required. □

Problem 2.2: 1) The plan of an art museum is an equilateral triangle consisting of 36 triangular exhibition halls (see the diagram). Each hall has passages into all adjacent halls. Prove that you can visit at most 31 halls if you plan to enter each hall at most once.

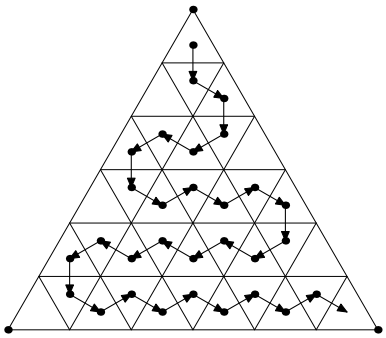


2) Subject to the condition that each hall is visited at most once, find the largest number of halls that can be visited in the case that the museum is an equilateral triangle and that it has k^2 exhibition halls, where k is an integer.

Solution. Starting from the top and working systematically along each row it appears that one can only visit

$$36 - 5 = 31$$

rooms.



Let a_k be the number of rooms which can be visited, then apparently

k	=	1	2	3	4	5	6
a_k	=	1	3	7	13	21	31

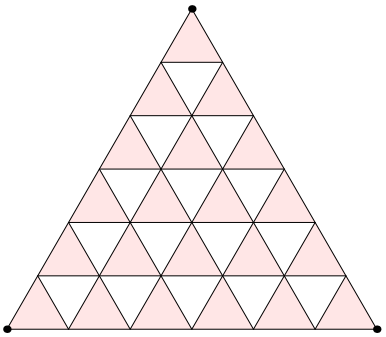
and for k, in general,

$$a_k = k^2 - (k-1) = k^2 - k + 1.$$

Indeed, the method described above shows that

$$a_6 \geq 31 \quad \text{and} \quad a_k \geq k^2 - k + 1,$$

in general. To see that 31 is the most possible, colour the up triangles red and the down triangles white.



Each path must go ...RWRW... There are 21 red triangles and 15 white triangles so a maximal path can contain no more than 16 red and 15 white triangles, so $a_6 \leq 31$. Thus,

$$a_6 = 31.$$

If there are k^2 exhibition halls, then the number of red halls is

$$1 + 2 + \ldots + k = \frac{1}{2}k(k+1)$$

and the number of white halls is

$$1 + 2 + \ldots + (k-1) = \frac{1}{2}k(k-1).$$

So the maximal possible path visits

$$\frac{1}{2}k(k-1) + \frac{1}{2}k(k+1) = k^2 - k + 1$$

halls.

□

Problem 2.3: Given an integer n, find the largest integer k such that 3^k is a factor of

$$2^{3^n} + 1.$$

Give an argument to justify your answer.

Solution. For a few initial values of n, we have

$$n = 0: \quad 2^{3^0} + 1 = 3,$$

$$n = 1: \quad 2^3 + 1 = 8 + 1 = 9,$$

$$n = 2: \quad 2^{3^2} + 1 = 512 + 1 = 27 \times 19.$$

Thus, we hypothesise

$$k = n + 1$$

and a proof by induction is available. Assume

$$2^{3^n} + 1 = 3^{n+1} \times A_n$$

with $n \geq 2$ and $(3, A_n) = 1$. Then,

$$\begin{aligned} 2^{3^{n+1}} + 1 &= (2^{3^n})^3 + 1 \\ &= \left(2^{3^n} + 1\right)\left((2^{3^n})^2 - 2^{3^n} + 1\right) \\ &= 3^{n+1}A_n\left((3^{n+1}A_n - 1)^2 - (3^{n+1}A_n - 1) + 1\right) \\ &= 3^{n+1}A_n\left(3^{n+1}\left(3^{n+1}A_n - 2A_n - A_n\right) + 3\right) \\ &= 3^{n+2}A_n\left(3^n\left(3^{n+1}A_n - 3A_n\right) + 1\right) \end{aligned}$$

which is divisible by exactly 3^{n+2} but not 3^{n+3} as required. Since $k = n + 1$ for $n = 0, 1, 2, \ldots$ the proof by induction is complete. □

Problem 2.4: Suppose that x_1, x_2, x_3 and y_1, y_2, y_3 are positive real numbers such that

$$0 < x_1y_1 < x_2y_2 < x_3y_3$$

and

$$x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3.$$

Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}.$$

Solution. Unfortunately this question is wrong as can be seen by taking x_1 small relative to y_1, y_2 and y_3. For example, let

$$(x_1, x_2, x_3) = (1, 4, 4) \text{ with } \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{3}{2}$$

and

$$(y_1, y_2, y_3) = (2, 3, 4) \text{ with } \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} = \frac{13}{12},$$

then

$$\begin{aligned} &0 < x_1y_1 < x_2y_2 < x_3y_3, \\ &\qquad x_1 + x_2 + x_3 \geq y_1 + y_2 + y_3 \\ &\qquad \text{and } \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} > \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3}. \end{aligned}$$

□

Problem 2.5: Each of the numbers $x_1, \ldots, x_n$ is either 1 or -1 and furthermore

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1 = 0.$$

Prove that n is a multiple of 4.

Solution. Let

$$y_1 = x_1x_2,\ y_2 = x_2x_3, \ldots, y_n = x_nx_1.$$

Clearly,

$$y_j = \pm 1, \quad j = 1, 2, \ldots, n.$$

Now the sum of an odd number of odd numbers is odd, so if

$$\sum_{j=1}^{n} y_j = 0,$$

then n is even.

Let us show that n is a multiple of 4. If

$$\sum_{j=1}^{n} y_j = 0$$

with $n = 2k$, then k of the y_j's are 1 and k of the y_j's are -1. Consider,

$$y_1y_2 \cdot \ldots \cdot y_n = x_1^2x_2^2 \cdot \ldots \cdot x_n^2 = 1,$$

since each $x_j^2 = 1$. Thus, the number of negative y_j's, which is k, is even. So $n = 2k$ is a multiple of 4. □

Alternative Solution. We may assume $x_1 = 1$. Consider the signs of x_1, x_2, …, x_n, x_1 as

$$+ + \ldots + - - \ldots - + \ldots + - \ldots - \ldots + \ldots +$$

The number of negative y_j's is the number of times $+-$ or $-+$ occurs in the sequence. Clearly $+-$ and $-+$ occur the same number of times so k is even, and $n = 2k$ is a multiple of 4. □

Problem 2.6: Seven people have access to a bank vault. The vault is secured with a number of locks, and for each lock, some but not all of the seven people have a key. Find the minimal number of locks needed so that any 4 of them can open the vault whereas any 3 of them cannot. For the minimal number of locks, specify the required distribution of the keys among the people.

Solution. The required number of locks is

$$n = \binom{7}{3} = 35.$$

Assume first that we have some amount of locks installed and some distribution of keys between the people such that the conditions of the problem are met. That is, every group of three people cannot open the vault and any group of four can open the vault.

In this case, for every group of three people, there is a unique lock such that this group open this lock, i.e., none in the group has a key to the lock. Indeed, if this was not so, that is, if there was two different groups of three people, say A and B, and a lock, say L, such that both A and B cannot open the lock L, then, by joining a person from B to the group A, we end up with a group of four people which cannot open the lock L. This is in contradiction with the assumption above. Hence, every group of three people has a unique lock associated with it and so the number of locks is at least

$$\binom{7}{3}.$$

Let us show that the the latter amount is sufficient to ensure the requirement of the problem. Indeed, given that that we install

$$\binom{7}{3}$$

locks, we associate uniquely a lock with a group of three people and we give keys to this lock to everyone but the associated group of three. □