Parabola Volume 37, Issue 2 (2001)

UNSW SCHOOL MATHEMATICS COMPETITION 2001

SOLUTIONS

JUNIOR DIVISION

1. Travelling from Goulburn to Canberra at a uniform speed I noticed a distance marker showing a two-digit distance from Goulburn. Thirty minutes later another marker showed a distance with the previous two digits reversed, and another thirty minutes later another marker with the original two digits separated by a 0. What was my speed?

   Solution:
   Suppose that the tens digit and the units digit on the first marker are x and y respectively. The distances of the markers from Goulburn are $(10x+y)$ km, $(10y+x)$ km and $(100x+y)$ km. Since the speed is uniform, we have

   $$\begin{aligned}(10y+x)-(10x+y) &= (100x+y)-(10y+x)\\ 9y-9x &= 99x-9y\\ y &= 6x.\end{aligned}$$

   As x and y are digits, we conclude that $x=1$ and $y=6$. Hence the speed is $106-16=90$ km/hr.

2. See Senior (1).

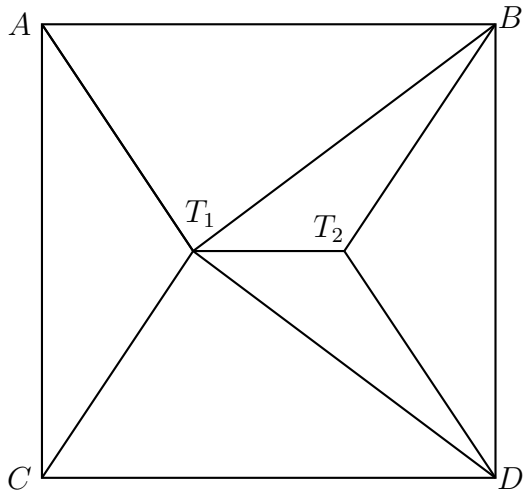
3. Let $ABCD$ be a square with side length 20. Let $T_1, T_2, \cdots T_{2000}$ be points in its interior, so that no three points from the set

   $$S=\{A,\ B,\ C,\ D,\ T_1,\ T_2, \cdots, T_{2000}\}$$

   lie on the same line. Prove that at least one triangle with vertices in S has area less than $\frac{1}{10}$.

   Solution:
   We first divide $ABCD$ into non-overlapping triangles with vertices in S as follows.

   By joining T_1 to A, B, C and D, the square $ABCD$ is divided into four triangles. Since no three points in S are collinear, the next point T_2 must be inside one of these triangles. After joining T_2 to the vertices of the triangle in which it lies, the number of triangular regions is increased by 2.



At each subsequent stage, join T_i to the vertices of the triangle in which it lies, $i = 3, 4, \ldots, 2000$.

The square will then be divided into 4004 triangles. If none of these triangles has area less than $\frac{1}{10}$, then

$$\text{Area of } ABCD \geq 4004 \times \frac{1}{10} = 400\frac{2}{5} \text{ square units},$$

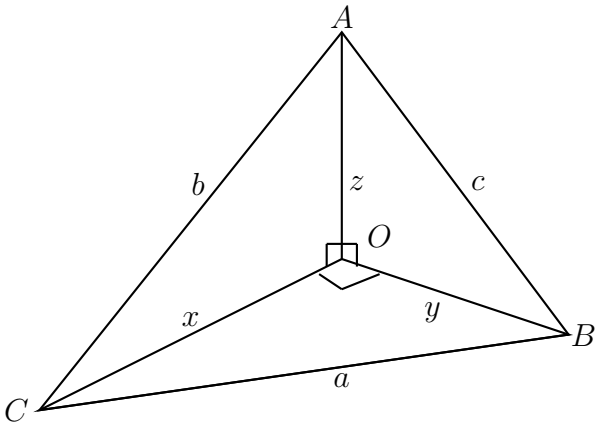
which contradicts the fact that the area of $ABCD$ is 400 square units.

4. Suppose that a triangle XYZ has side lengths x, y, z with $s = (x + y + z)/2$. If the area of the triangle is denoted by $|XYZ|$, then Heron's Formula states that

$$|XYZ|^2 = s(s - x)(s - y)(s - z).$$

Let $OABC$ be a right-angled tetrahedron, with all angles at the vertex O being right angles as shown. Using Heron's Formula or otherwise, prove the three dimensional Pythagoras' Theorem:

$$|OAB|^2 + |OBC|^2 + |OCA|^2 = |ABC|^2.$$



Solution:

Let the lengths of OA, OB and OC by z, y and x respectively, and let the lengths of AB, BC and CA be c, a and b respectively.

Since $\quad |OAB| = \frac{1}{2}yz, \quad |OBC| = \frac{1}{2}xy \quad$ and $\quad |OCA| = \frac{1}{2}xz,$

$$|OAB|^2 + |OBC|^2 + |OCA|^2 = \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2).$$

By Heron's Formula,

$$\begin{aligned}
|ABC|^2 &= (\frac{a+b+c}{2})(\frac{a+b+c}{2} - a)(\frac{a+b+c}{2} - b)(\frac{a+b+c}{2} - c) \\
&= \frac{1}{16}(a+b+c)(-a+b+c)(a-b+c)(a+b-c) \\
&= \frac{1}{16}[(b+c)^2 - a^2][a^2 - (b-c)^2] \\
&= \frac{1}{16}(2bc + b^2 + c^2 - a^2)(a^2 - b^2 - c^2 + 2bc) \\
&= \frac{1}{16}[4b^2c^2 - (a^2 - b^2 - c^2)^2] \\
&= \frac{1}{16}[4(x^2+z^2)(y^2+z^2) - (x^2 + y^2 - x^2 - z^2 - y^2 - z^2)^2] \\
&= \frac{1}{16}[4(x^2y^2 + x^2z^2 + y^2z^2 + z^4) - 4z^4] \\
&= \frac{1}{4}(x^2y^2 + x^2z^2 + y^2z^2) \\
&= |OAB|^2 + |OBC|^2 + |OCA|^2.
\end{aligned}$$

5. Let a and b be real numbers such that

$$a^3 - 3ab^2 = 44, \qquad b^3 - 3a^2b = 8.$$

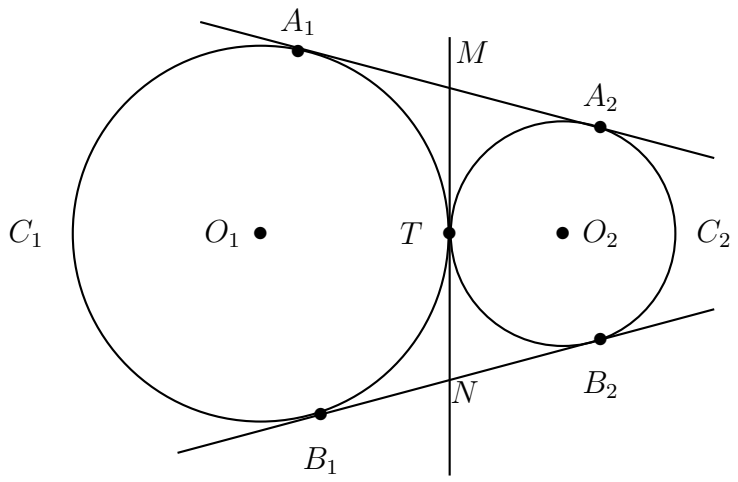
Determine the value of $a^2 + b^2$.

Solution:

$$\begin{aligned}
(a^3 - 3ab^2)^2 + (b^3 - 3a^2b)^2 &= 44^2 + 8^2 \\
a^6 - 6a^4b^2 + 9a^2b^4 + b^6 - 6a^2b^4 + 9a^4b^2 &= 2000 \\
a^6 + 3a^4b^2 + 3a^2b^4 + b^6 &= 2000 \\
(a^2 + b^2)^3 &= 2000 \\
a^2 + b^2 &= \sqrt[3]{2000}
\end{aligned}$$

6. Two circles, C_1 and C_2, with centers O_1, O_2, are externally tangent to each other at T. Their common tangents meet them at A_1, A_2 and B_1, B_2 respectively. Prove that the circles with diameters A_1A_2 and B_1B_2 are tangent to each other at T.

Solution:



Construct the common tangent at T to the circles C_1 and C_2. Suppose that this tangent cuts A_1A_2 and B_1B_2 at M and N respectively. By tangents from an external point theorem, we have

$$A_1M = MT = A_2M \qquad \text{and} \qquad B_1N = NT = B_2N.$$

Hence the circle C_3 with diameter A_1A_2 and the circle C_4 with diameter B_1B_2 pass through the point T. Furthermore M, N are centres of C_3 and C_4. Since M, T, N are collinear, and $MN =$ sum of the radii of C_3 and C_4, we conclude that C_3 and C_4 are tangential to each other at the point of intersection T.

SENIOR DIVISION

1. Find all positive integer solutions of the equation $xy + yz + zx - xyz = 2$.

   Solution: (Based on Maiyuran Arumugam's solution)
   Without loss of generality, assume $x \geq y \geq z$. Since x and z are positive integers, we have $xy \geq xz \geq yz$. Hence

   $$\begin{aligned} 3xy &\geq xy + xz + yz \\ 3xy - xyz &\geq xy + xz + yz - xyz = 2 > 0 \\ xy(3 - z) &> 0. \end{aligned}$$

   Therefore z is less than 3.
   If $z = 1$, the equation becomes

   $$\begin{aligned} xy + y + x - xy &= 2 \\ x + y &= 2. \end{aligned}$$

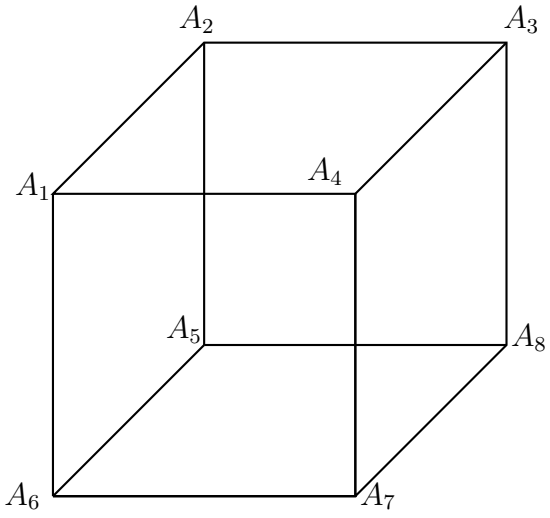
   Thus $x = y = 1$.
   If $z = 2$, the equation becomes

   $$\begin{aligned} xy + 2y + 2x - 2xy &= 2 \\ xy - 2x - 2y + 4 &= 2 \\ (x - 2)(y - 2) &= 2 \end{aligned}$$

   As $x \geq y \geq z = 2$, the only possible solution is $x = 4$ and $y = 3$. Thus the only positive integer solutions are $x = y = z = 1$, or (x, y, z) is a permutation of $(2, 3, 4)$.

2. A bug crawls from vertex to vertex on a cube along edges, choosing each subsequent edge at random from the three emanating from the vertex it is currently at. Show that there is no chance that after 5 such random steps, the bug is back where it started. What is the probability that after 6 such random steps, the bug is back where it started?

   Solution: (Based on Maiyuran Arumugam's solution)

In each move, the bug goes from an odd vertex (i.e. A_1, A_3, A_5 or A_7) to an even vertex (i.e. A_2, A_4, A_6, A_8) or from an even vertex to an odd vertex. If the bug starts from A_1, it can only be at an even vertex after 5 moves. Hence there is no chance that it is back to A_1 after 5 moves.

Suppose the bug starts at A_1 and let the probability that it is at A_1 after $2n$ moves be P_n. Thus, the probability that it is not at A_1 after $2n$ moves is $1 - P_n$.

If it is at A_1 after $2n$ moves, the probability that it will be at A_1 after 2 more moves is $\frac{1}{3}$.

If it is not at A_1 after $2n$ moves, it must be at A_3, A_5 or A_7. At any one of these vertices, the chance of moving to A_2, A_4 or A_6 is $\frac{2}{3}$. Thus the probability of returning to A_1 after 2 more moves is $\frac{2}{3} \times \frac{1}{3}$. Hence

$$P_{n+1} = \frac{1}{3} P_n + \frac{2}{3} \times \frac{1}{3}(1 - P_n) = \frac{2}{9} + \frac{P_n}{9}$$

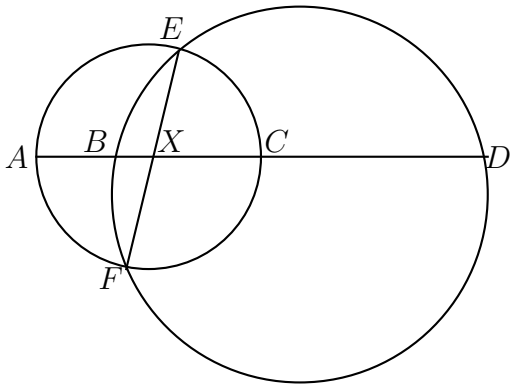
i.e.

$$P_{n+1} = \frac{2}{9} + \frac{1}{9} P_n.$$

Since $P_0 = 1$, we get $P_1 = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$, $P_2 = \frac{2}{9} + \frac{1}{9} \times \frac{1}{3} = \frac{7}{27}$ and $P_3 = \frac{2}{9} + \frac{1}{9} \times \frac{7}{27} = \frac{61}{243}$.

3. A, B, C, D are points on a straight line (in the given order) such that $BC = 2AB, CD = AC$. Draw a circle through A and C and another circle through B and D. Prove that the common chord of the two circles intersects the given line in the midpoint of AC.

Solution:



Suppose that the two circles intersect at E and F and the line EF cuts BC at X.

By product of intercepts theorem for intersecting chords, we have

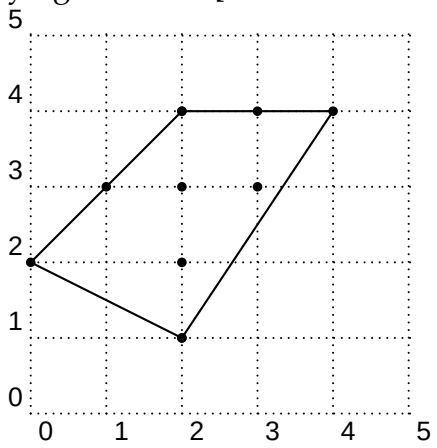
$$\begin{aligned} AX \cdot XC = FX \cdot XE &= BX \cdot XD \\ (AB + BX)(BC - BX) &= BX(BD - BX) \\ (AB + BX)(2AB - BX) &= BX(5AB - BX) \\ 2AB^2 + AB \cdot BX - BX^2 &= 5AB \cdot BX - BX^2 \\ 2AB^2 &= 4AB \cdot BX \\ \therefore \qquad BX &= \frac{1}{2} AB. \end{aligned}$$

Hence $AX = AB + BX = \frac{3}{2}AB = \frac{1}{2}AC$. Thus X is the mid-point of AC.

4. A polytope P in the plane is called a lattice polytope if all its vertices lie on integral lattice points: those of the form (n, m) with n, m integers. It is known that there exist rational numbers a, b and c such that for any lattice polytope P its area is

$$\text{area}(P) = aE + bI + c$$

where E is the number of lattice points lying on the edges of P and I is the number of lattice points lying inside P. [Thus in the example $E = 6$, $I = 4$].

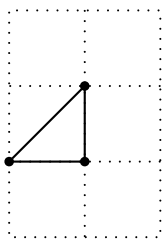


(i) Find a, b and c.

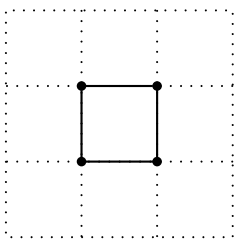
(ii) Prove the formula is valid for any lattice triangle.

Solution:

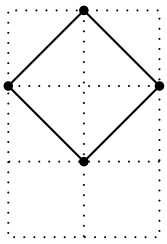
(i) By considering a right angled triangle of sides 1, a square of side length 1 and a diamond of side length $\sqrt{2}$, we get the following equations for a, b and c.



$3a + c = \frac{1}{2}$



$4a + c = 1$



$4a + b + c = 2$

Hence $a = \frac{1}{2}$, $b = 1$ and $c = -1$.

(ii) The following is a proof taken from Coxeter's "Introduction to Geometry" (Wiley, 1969) that

$$\text{area}(P) = \frac{1}{2}E + I - 1.$$

We first observe that the expression $\frac{1}{2}E+I-1$ is additive when two polygons are juxtaposed. In fact, if two polygons, involving $E_1 + I_1$ and $E_2 + I_2$ lattice points respectively, have a common side containing $n(\geq 0)$ lattice points in addition to the two vertices at its ends, then the values of E and I for the combined polygon are

$$E = E_1 + E_2 - 2n - 2, \quad I = I_1 + I_2 + n,$$

so that

$$\frac{1}{2}E + I - 1 = \left(\frac{1}{2}E_1 + I_1 - 1\right) + \left(\frac{1}{2}E_2 + I_2 - 1\right).$$

Next, the formula holds for a parallelogram joining 4 lattice points and having no other lattice points on its edges (so that $E = 4$ and the expression reduces to $I + 1$). For, when N such parallelograms are fitted together, four at each vertex, to fill a large region, the number of lattice points involved (apart from a negligible peripheral error) is $N(I + 1)$, and this must be the same as the number of unit cells needed to fill the same region.

Splitting the parallelogram into two congruent triangles by means of a diagonal, we see that the formula holds also for a triangle having no lattice points on its sides. A triangle that does have lattice points on a side can be dealt with by joining such points to the opposite vertex so as to split the triangle into smaller triangles.

5. Seventeen primes $p_1 < p_2 < \cdots < p_{17}$ have the property that the sum of their squares is also a square. Prove that $p_{17}^2 - p_{16}^2$ is divisible by p_1.

Solution:

If $p_1 > 3$, then for each $1 \leq k \leq 17$, p_k is not a multiple of 3. Thus $p_k \equiv 1 \pmod 3$ or $p_k \equiv 2 \pmod 3$. In both cases, $p_k^2 \equiv 1 \pmod 3$. Hence

$$p_1^2 + p_2^2 + \cdots + p_{17}^2 \equiv 17 \pmod 3 \equiv 2 \pmod 3.$$

Noting that any square number is equivalent to 0 or 1 modulo 3, we then conclude that $p_1^2 + p_2^2 + \cdots + p_{17}^2$ is not a square number. As a result, if the sum of squares of these 17 primes is a square, then $p_1 = 2$ or 3.

On the other hand, p_{16} and p_{17} are odd and thus $p_{17}^2 - p_{16}^2$ is even. Furthermore,

$$p_{17}^2 - p_{16}^2 \equiv 1 - 1 \equiv 0 \pmod 3,$$

Then $p_{17}^2 - p_{16}^2$ is divisible by 3. The result then follows.

6. Twelve knights $k_1,\ k_2, \cdots, k_{12}$ are seated in anti-clockwise order around a circular table. What is the minimal number of swaps required to change their order to a clockwise one, if any swap can be made only between adjacent knights? What is the answer if there are thirteen knights?

Solution:

We shall solve this problem for the general case that there are n knights $k_1, k_2, \ldots, k_n$ where $n \geq 3$. Let us label the seats such that, initially, the knights $k_1, k_2, \ldots, k_n$ take the seats $s_1, s_2, \ldots, s_n$ respectively. Let the minimum number of adjacent swaps required to change the knights from anti-clockwise to clockwise order be M_n. We first divide the knights into two sets $\{k_1, \ldots, k_a\}$ and $\{k_{a+1}, \ldots, k_n\}$, where $a = \left\lfloor \frac{n}{2} \right\rfloor$, the greatest integer that is less than or equal to $\frac{n}{2}$. We then reverse the order of each set as follows. First move k_1 to s_a by successively swapping k_1 with k_2, k_3 and so on using $a-1$ swaps. Then move k_2 to s_{a-1} by $a-2$ swaps, and so on. Hence, we can reverse the order of the first set by $1+2+\cdots+(a-1)$ swaps. Similarly, we can reverse the order of the second set by $1 + 2 + \cdots + (n-a-1)$ swaps. Thus, we have an algorithm to change the order to a clockwise one using

$$[1 + 2 + \cdots + (a-1)] + [1 + 2 + \cdots + (n-a-1)] = \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor$$

swaps. When $n = 12$, this gives $(1 + 2 + \cdots + 5) + (1 + 2 + \cdots + 5) = 30$ swaps, and when $n = 13$, this gives $(1 + 2 + \cdots + 5) + (1 + 2 + \cdots + 6) = 36$ swaps.

We now show these numbers are minimal, i.e. $M_n \leq \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor$.

Suppose that we have an algorithm to change the order to a clockwise one by M_n swaps and that, after these swappings, the seat s_i is taken by $k_{f(i)}$, for each $1 \leq i \leq n$. Let $d_i \equiv i - f(i) \pmod{n}$, where $0 \leq d_i \leq n-1$. If s_j is the seat next to s_i in anti-clockwise direction, then

$$\begin{aligned} d_j &\equiv (i+1) - (f(i) - 1) \pmod{n} \\ &\equiv d_i + 2 \pmod{n} \end{aligned}$$

Let $D = \{d_1, \ldots d_n\}$. When n is even, the set D is either $\{0, 2, 4, \ldots, n-2\}$ or $\{1, 3, 5, \ldots, n-1\}$ so there exists i such that $d_i = \frac{n}{2}$ or $\frac{n-2}{2}$. When n is odd, there exists i such that $d_i = \frac{n-1}{2}$. In each of these cases, we call the $f(i)$'th knight, the phantom knight. By relabelling the knights and seats, we may assume that k_n is the phantom knight. When n is odd, the final position of the phantom knight will be directly opposite his original position. When n is even, the final position will be directly opposite to the original position or one next to it. In all cases, the phantom knight needs at least $\left\lfloor \frac{n-1}{2} \right\rfloor$ swaps to get from the initial to the final position.

From this algorithm of M_n swaps, we can induce an algorithm which changes $(n-1)$ knights to clockwise order by removing all the swaps involving the phantom. For instance, from the four swaps algorithm for 5 knights

$$k_1 \leftrightarrow k_2, \quad k_3 \leftrightarrow k_4, \quad k_3 \leftrightarrow k_5, \quad k_4 \leftrightarrow k_5,$$

we can get an algorithm for 4 knights

$$k_1 \leftrightarrow k_2, \quad k_3 \leftrightarrow k_4.$$

Let the number of swaps in the induced algorithm for $(n-1)$ knights be P. Thus

$$M_{n-1} \leq P \leq M_n - \left\lfloor \frac{n-1}{2} \right\rfloor$$

$$\text{i.e.} \qquad M_n \geq M_{n-1} + \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Inductively,

$$M_n \geq \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor + \cdots + \left\lfloor \frac{3}{2} \right\rfloor + M_3.$$

It is obvious that $M_3 = 1$, and hence $\quad M_n \geq \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor$.

Thus

$$M_n = \sum_{r=2}^{n-1} \left\lfloor \frac{r}{2} \right\rfloor.$$