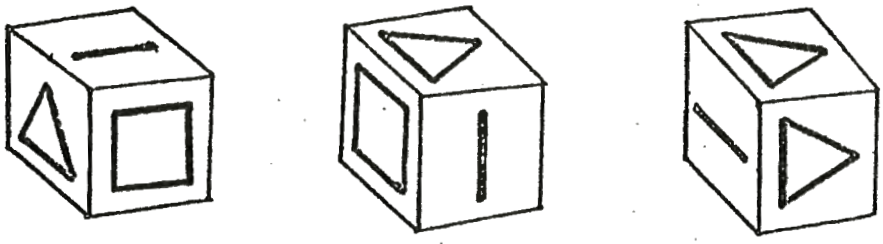
SCHOOL MATHEMATICS COMPETITION 1981

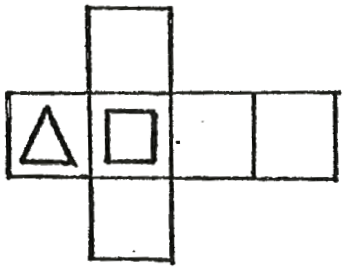
JUNIOR DIVISION

1. *(The cube problem)*

Each face of a cube is marked with a square, a triangle or a line segment. Three views of the cube are given below:

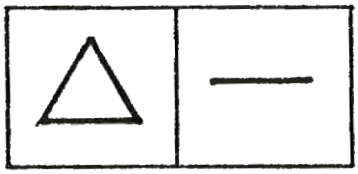


Complete the "exploded diagram" of the cube given below (in your answer book).

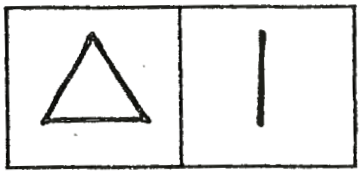


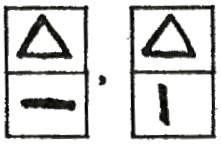
Solution.

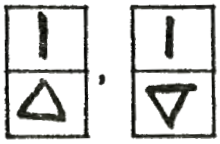
It is necessary to consider the orientation of the symbols. For example,



is different from



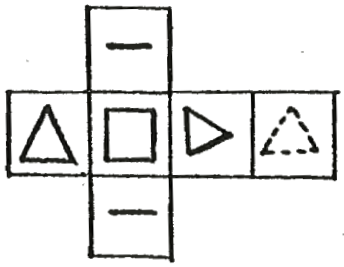
There must be at least 3 Δ's since  and  all occur.

Similarly there must be 2 1's since  and  all occur.

(continued over)

The 6th face must be marked with a □.

The four faces around the □ are ▷– and △– . This determines



and hence the final face is as dotted in.

2. *(The 101 problem)*

101 is prime. Are there any other numbers 10101 ... 101 (all written in base 10) which are prime?

Solution.

Let $A = 10101 \ldots 101$

$$= 10^{2n} + 10^{2n-2} \ldots + 10^2 + 1$$

$$100A = 10^{2n+2} + 10^{2n} \ldots + 10^4 + 10^2$$

$$99A = 10^{2n+2} - 1 = (10^{n+1} + 1)(10^{n+1} - 1)$$

If $n > 1$ then $10^{n+1} - 1 > 99$ hence A has no proper factorization and is not prime.

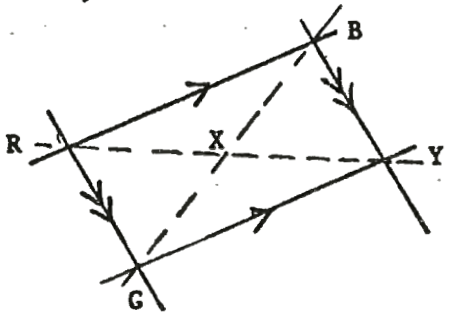
3. *(The 4 colour problem)*

Colour each point of the plane by one of the 4 colours red, blue, green and yellow with each colour used for at least one point. Prove that there will always be a line containing 3 points which are differently coloured. Find a colouring involving all 4 colours but with no line containing all 4 colours.

Solution.

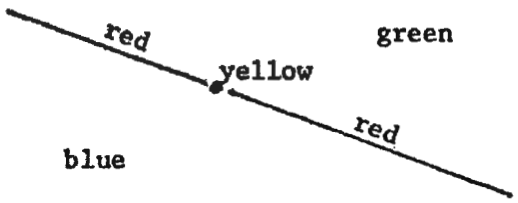
There are at least 4 points R, B, G, Y coloured red, blue, green and yellow respectively.

Assume there is no line containing 3 colours. Then RB, RG, RY, BG, BY and GY are 6 distinct lines. Either RB meets GY, RG meets BY or RY meets BG. For if RB∥GY and RG∥BY then RY and BG are diagonals of a parallelogram which must thus meet. Without loss of generality assume RY meets BG at X. By assumption



(continued over)

X is red or yellow but X is also blue or green, a contradiction. Hence any colouring includes a line containing 3 points which are differently coloured. Here is one example of a colouring involving all 4 colours but with no line containing all 4 colours.



4. *(The 39 problem)*

Prove that among 39 consecutive natural numbers there always exists one for which the sum of its digits (when written in base 10) is divisible by 11. Find the smallest set of 38 consecutive natural numbers for none of which the sum of the digits is divisible by 11.

Solution.

In this solution we replace n by the sum of its digits taken modulo 11. e.g. 86 = 3, 999999 = 10, 1000000 = 1.

S_1 S_2 S_3 S_4

7 8|0 1 2 3 4 5 6 7 8 9|1 2 3 4 5 6 7 8 9 10|2 3 4 5 6 7 8 9 10 0|3 4 5 6 7 8 9 10 0 1|4 5 . . . . .

The above diagram shows a section of the positive integers where | denotes a boundary between ab ... n9 and ab ... (n + 1)0. If ab ... n9 = x (mod 11) then ab ... (n + 1)0 = x + 3 (mod 11). Similarly if ab ... n99 = x (mod 11) then ab ... (n + 1)00 = x + 5 (mod 11); if ab ... n999 = x (mod 11) then ab ... (n + 1)000 = x + 7 (mod 11) etc.

Now the only decad (between two |'s) which does not contain a number divisible by 11 is S_2. Hence a maximal possible sequence of decades is S_1, S_2, S_2, S_3 with the boundary between the 2 S_2's being x, x + 2 i.e. 10, 1. This occurs first between 999999 and 1000000. Hence there are at most 38 consecutive numbers not divisible by eleven and the smallest such set is 999981 → 1000018.

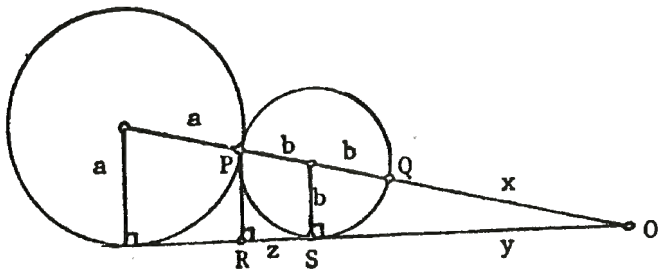
5. *(The 2 spheres problem)*

Two spheres of radii a,b (a > b) are glued together at a point P. The solid so formed is rolled (without slipping) on a horizontal surface. Show that P describes a circle of radius r where $r^2 = \frac{16a^3b^3}{(a^2 - b^2)^2}$

Solution.

The diagram on the right will facilitate solution.

Denote: QO = x, RS = z, SO = y.



By ratio and proportion

$$\frac{a}{b} = \frac{a + 2b + x}{b + x} = \frac{a + b}{b + x} + 1$$

Hence $b + x = (a + b)/(\frac{a}{b} - 1) = b(a + b)/(a - b)$

Similarly $\frac{z + y}{y} = \frac{2b + x}{b + x} = \frac{b}{b + x} + 1$ and

$$y = (z + y)(b + x)/((b + x) + b)$$
$$= (z + y)b(a + b)/(b(a + b) + b(a - b))$$
$$= (z + y)(a + b)/2a$$

By Pythagoras $y^2 + b^2 = (b + x)^2$

$$\frac{(z + y)^2(a + b)^2}{4a^2} = \frac{b^2(a + b)^2}{(a - b)^2} - b^2 = \frac{b^2 \cdot 4ab}{(a - b)^2}$$

and

$$(z + y)^2 = \frac{16a^3b^3}{(a^2 - b^2)^2}$$

Now it is clear that P describes a circle radius $z + y$, hence, the problem is solved.

SENIOR DIVISION

1. See Junior Division question 5.

2. *(1st 1981 problem)*

Denote by a_n the integer nearest to $\sqrt{n}$. Find the value of

$$\sum_{i=1}^{1981} \frac{1}{a_i} = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{1981}}$$

Solution.

$a_1 = a_2 = 1$ $a_3 = a_4 = a_5 = a_6 = 2$ and $a_7 = 3$ since $(2\frac{1}{2})^2 = 6\frac{1}{4}$.

In general, $(m + \frac{1}{2})^2 = m^2 + m + \frac{1}{4}$ hence $a_{m^2+m} = m$ but $a_{m^2+m+1} = m + 1$.

Now $\sqrt{1981} \approx 44.51$ and $44^2 + 44 = 1980$. The number of integers for which $a_n = m$

(continued over)

is $m^2 + m - ((m-1)^2 + (m-1) + 1) + 1 = 2m$.

Hence $\sum_{i=1}^{1981} \frac{1}{a_i} = \sum_{m=1}^{44} \frac{2m}{m} + \frac{1}{45} = 88\frac{1}{45}$.

3. *(2nd 1981 problem)*

The function $f(m,n)$ is defined for positive integers m,n and satisfies the following conditions:

a) $f(x,x) = x$

b) $f(x,y) = f(y,x)$

c) $(x + y).f(x,y) = y.f(x, x + y)$

What is the value of $f(1981, 2830)$?

Solution.

From the given conditions we may write:

$$f(x, x + y) = \frac{x + y}{y} f(x,y) .$$

Hence $f(1981,2830) = \frac{2830}{849} f(1981,849) = \frac{2830}{849} f(849,1981) =$

$= \frac{2830}{849}\frac{1981}{1132} f(849,1132) = \frac{2830}{849}\frac{1981}{1132}\frac{1132}{283} f(849,283) = \frac{19810}{849} f(283,849) =$

$= \frac{19810}{849}\frac{849}{566} f(283,566) = \frac{19810}{566}\frac{566}{283} f(283,283) = \frac{19810.283}{283} = 19810 .$

Indeed $f(x,y)$ = l.c.m. of x and $y = xy/(x,y)$ where (x,y) is the h.c.f. of x and y .

4. *(The Cricket Match problem)*

It is estimated that in a cricket match University of New South Wales will win over Sydney Unversity with probability 2/3 and University of New South Wales will win over Macquarie University with probability 9/10. University of New South Wales must play the others alternatively in a 3 match series and try to win 2 matches in a row. If University of New South Wales has choice of first opponent what should they do? Discuss the corresponding problem of winning 3 in a row during a similar 4 match series.

Solution.

The interest in this problem is that the result for a 3 match series is perhaps (?) counterintuitive.

If University of New South Wales plays Sydney-Macquarie-Sydney the chance of a success is $2/3 \times 9/10 + 1/3 \times 9/10 \times 2/3 = 4/5 = .8$.

(continued over)

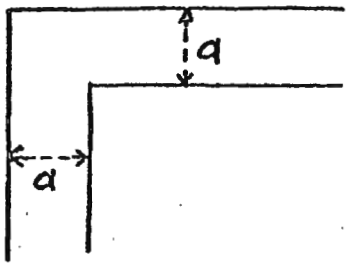
If University of New South Wales plays Macquarie-Sydney-Macquarie the chance of a success is $9/10 \times 2/3 + 1/10 \times 2/3 \times 9/10 = 66/100 = .66$.

That is, the better strategy is to play the stronger team twice!

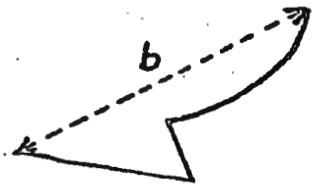
To win 3 matches in a row in a 4 match series it does not matter which opponent University of New South Wales plays first as the second and third matches must be won and at least one of the first and last matches must be won which are against both of the teams.

For the record, the chance of success is
$= 2/3 \times 9/10(1 - 1/3.1/10) = \frac{87}{150} = \frac{29}{50} = .58.$

5. *(The Corridor problem)*



This is a 2 dimensional problem, i.e. it is a problem in the plane. The effective length b of a pipe is the distance between the end points. Find the rigid pipe of longest effective length that can be taken around the corner in the above corridor.

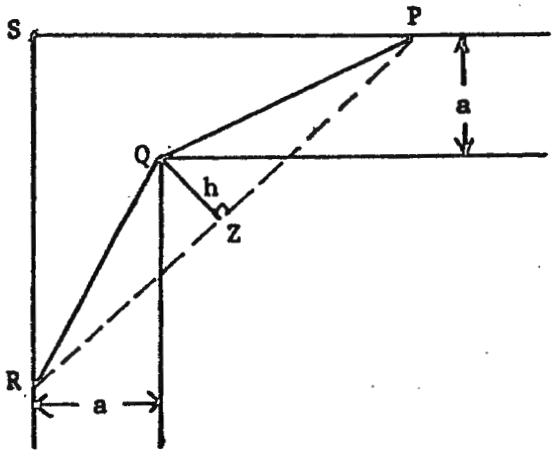


Is there a unique shape such a piece of pipe must have?

Solution.

Consider a piece of pipe PQR (with effective length PR). Now if the pipe fits into the arm of the corridor then h must be $\leq$ a.

The critical situation occurs (by symmetry) when PS = PR and h = a. QS = $\sqrt{2}$a and SZ = a(1 + $\sqrt{2}$). Hence PR = 2(1 + $\sqrt{2}$)a.



This is the longest possible effective length but the above argument does not prove it can be achieved. For example, a pipe consisting of the two line segments PQ, QR will not move at all.

(continued over)

However consider the circle PQR· then $h(2r - h) = (RZ)^2$. Therefore $2r = \frac{a^2(1 + \sqrt{2})^2}{a} + a$ and $r = (2 + \sqrt{2})a = a + (1 + \sqrt{2})a$ i.e. if O is the centre of the circle PQR then OZ = SZ and PQR is a quarter-circle!! Hence PQR will roll around smoothly and the maximum effective length $2(1 + \sqrt{2})a$ can be achieved.

However, the shape is not unique - the pipe can be altered considerably away from the midpoint and the 2 end points.

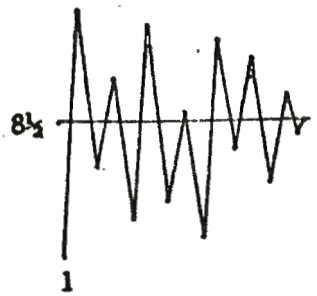
6. *(The Elevator problem)*

The mathematics building at the University of New South Wales has 15 floors numbered level 1 to level 15 which are h metres apart. Imagine starting at level 1 and taking the lift to each of the other 14 levels in turn. What is the longest trip that one can take? Prove that any trip that achieves the above maximum ends up on the same level as any other. How many different maximal trips are there?

Solution.

The trip 1, 15, 2, 14, 3, 13, 4, 12, 5, 11, 6, 10, 7, 9, 8 has length $(14 + 13 + \ldots + 3 + 2 + 1)h = 105h$ metres, is maximal, and finishes on level 8. To see why this is maximal note that any trip can only travel between levels 14 and 15 at most twice, between levels 13 and 14 at most 4 times, etc. Also the lift goes between levels 1 and 2 at most once, between 2 and 3 at most 3 times, etc. (see table and diagram below), i.e. any trip is of length $\leq 105h$ metres and also every move must include a crossing from level 8 to level 9 (or 9 to 8). Thus each maximal trip starts from 1 and alternatively visits an upper level (from 9 to 15) and then a lower level (from 2 to 8).

Level No.	Level No.	No. of Trips
14	15	2
1	2	1
13	14	4
2	3	3
12	13	6
3	4	5
11	12	8
4	5	7
10	11	10
5	6	9
9	10	12
6	7	11
8	9	14
7	8	13
		105



Such a trip which visits each upper level and each lower level in any order and then returns to level 8½ has length:

$$[7\tfrac{1}{2} + 2(\tfrac{1}{2} + 1\tfrac{1}{2} + \ldots + 6\tfrac{1}{2}) + 2(\tfrac{1}{2} + 1\tfrac{1}{2} + \ldots + 6\tfrac{1}{2})]h = 105\tfrac{1}{2}h \text{ metres.}$$

Hence any maximal trip must finish at level 8, a surprising result! Also the number of such maximal trips = $7! \times 6! = 3{,}628{,}800$.

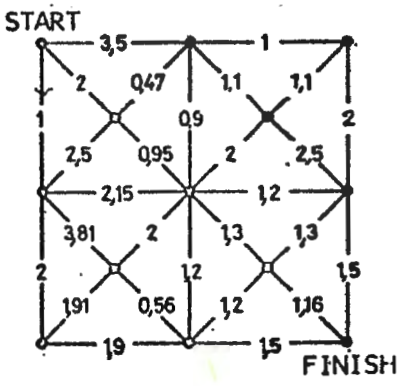
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COMPUTER-COLUMN

"CRITICAL PATH".

This game can be played by any number of players, each with his own calculator.

We are given a diagram which consists of points connected with straight line segment bearing a number. We may regard these segments as partial paths of the network.



We take the first step from the point marked START and we have to reach the point FINISH by taking any of the partial paths; none of them can be traversed more than once. As we progress we multiply the numbers assigned to the partial paths we have chosen. The player with the smallest final number is the winner.

"AVERAGE".

This game can be played by 3 or more players.

Every player writes down a natural number consisting at most of 6 digits. The numbers are read out and those who wrote down the same number are eliminated from the game. The rest of the players calculate the average of the numbers read out. The winner is the person whose number is closest to the average.

If there is only one player left after the elimination process then clearly he is the winner. If everybody is eliminated, then there is no winner. If more than one player has a number equally close to the average, these all win.