SCHOOL MATHEMATICS COMPETITION 1979

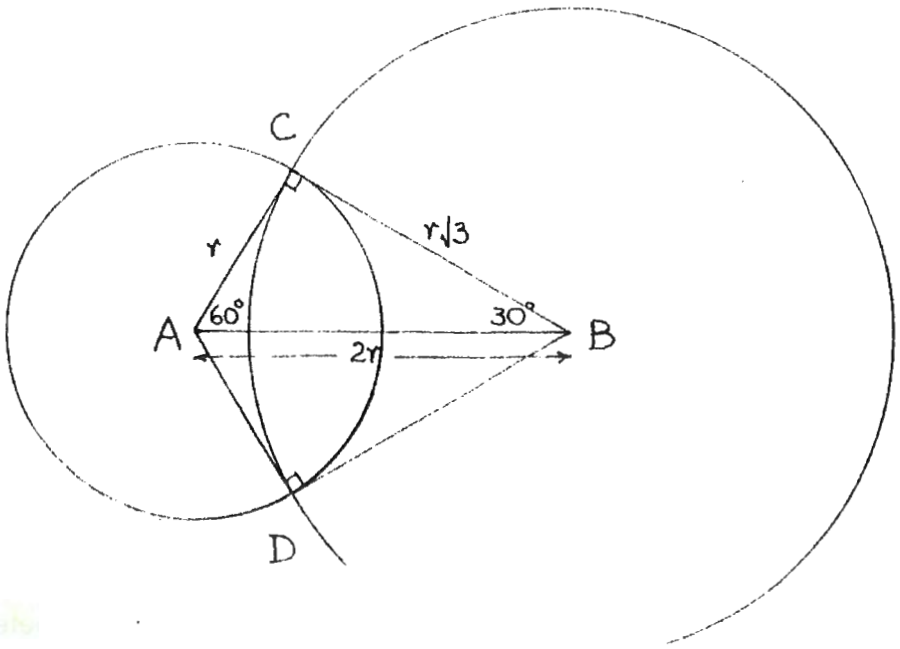
Junior Division

1. We start with 24 sheets of paper. Some are selected and cut up into 10 pieces each. Some of these smaller sheets are again cut up into 10 pieces each, and so on. After this operation has been carried out several times, someone claims that there are exactly 1979 pieces of paper altogether, while someone else claims that there are exactly 1980 pieces. Can either number be correct? If so, which?

Solution.

Each time a piece of paper is cut into 10, the number of pieces grows by 9, so the number of pieces is necessarily of the form $24 + 9n$, with n a non-negative integer. Both people must have miscounted. ☆

2. Two circular discs, of radii r and $r\sqrt{3}$, are placed on a plane so that their edges cross at right angles. Show that the area of their overlap is $\{(5/6)\pi - \sqrt{3}\}r^2$. Hence find the area covered by their union.



Solution.

Let A, B be the centres of the two discs and C, D be the points where their edges cross. From the figure, angle CAD is 120° and angle CBD is 60°, so the area of overlap is

$$\begin{aligned}&\text{area of sector ACD} + \text{area of sector BCD} - \text{area of quadrilateral ACBD}\\ &= (1/3)\pi r^2 + (1/6)\pi(r\sqrt{3})^2 - 2(1/2)r(r\sqrt{3})\\ &= \{(5/6)\pi - \sqrt{3}\}r^2.\end{aligned}$$

The area covered by the union is

$$\pi r^2 + \pi(r\sqrt{3})^2 - \{(5/6)\pi - \sqrt{3}\}r^2 = \{(19/6)\pi + \sqrt{3}\}r^2. \quad \therefore$$

3. Let a(1), a(2), a(3), . . . be the positive integers not divisible by 2, 3, or 5 listed in increasing order, and let b(1), b(2), b(3), . . . be the positive integers divisible by 6, 10, or 15 listed in increasing order. Show that

$$a(1) < b(1) < a(2) < b(2) < a(3) < b(3) < \ldots < a(n) < b(n) < a(n+1) < \ldots .$$

Solution.

The a-sequence is 1, 7, 11, 13, 17, 19, 23, 29|31, 37, 41,. . . while the b-sequence is 6, 10, 12, 15, 18, 20, 24, 30|36, 40,. . . . Note that adding 30 to an a-term gives another, while adding 30 to a b-term also gives another. In technical jargon, both sequences are periodic modulo 30, and since they "interlock" up to 30, that is

$$1 = a(1) < b(1) < a(2) < b(2) < \ldots < a(8) < b(8) = 30,$$

they continue to do so. $\therefore$

4. $\lfloor x \rfloor$ denotes the "integer part" of x, that is, the largest integer not greater than x, and $\{x\} = x - \lfloor x \rfloor$. Prove that for all x

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{3} \rfloor + \lfloor x + \tfrac{2}{3} \rfloor,$$

and

$$\{3x\} = \{x\} + \{x + \tfrac{1}{3}\} + \{x + \tfrac{2}{3}\} - 1.$$

Solution I.

We consider three cases: $0 \leqslant \{x\} < \tfrac{1}{3}$, $\tfrac{1}{3} \leqslant \{x\} < \tfrac{2}{3}$, $\tfrac{2}{3} \leqslant \{x\} < 1$.

In the first case, $\lfloor x \rfloor \leqslant x < \lfloor x \rfloor + \tfrac{1}{3}$ and $3\lfloor x \rfloor \leqslant 3x < 3\lfloor x \rfloor + 1$, which gives $\lfloor 3x \rfloor = 3\lfloor x \rfloor$. Also

$$\lfloor x \rfloor + \tfrac{1}{3} \leqslant x + \tfrac{1}{3} < \lfloor x \rfloor + \tfrac{2}{3} \text{ and } \lfloor x \rfloor + \tfrac{2}{3} \leqslant x + \tfrac{2}{3} < \lfloor x \rfloor + 1,$$

so $\lfloor x + \tfrac{1}{3} \rfloor = \lfloor x \rfloor$ and $\lfloor x + \tfrac{2}{3} \rfloor = \lfloor x \rfloor$. Hence

$$\lfloor x \rfloor + \lfloor x + \tfrac{1}{3} \rfloor + \lfloor x + \tfrac{2}{3} \rfloor = 3\lfloor x \rfloor = \lfloor 3x \rfloor.$$

In the second case,

$$\lfloor 3x \rfloor = 3\lfloor x \rfloor + 1 = \lfloor x \rfloor + \lfloor x \rfloor + (\lfloor x \rfloor + 1) = \lfloor x \rfloor + \lfloor x + \tfrac{1}{3} \rfloor + \lfloor x + \tfrac{2}{3} \rfloor,$$

while in the third case,

$$\lfloor 3x \rfloor = 3\lfloor x \rfloor + 2 = \lfloor x \rfloor + (\lfloor x \rfloor + 1) + (\lfloor x \rfloor + 1) = \lfloor x \rfloor + \lfloor x + \tfrac{1}{3} \rfloor + \lfloor x + \tfrac{2}{3} \rfloor.$$

To prove the second equation, note that

$$3x = x + (x + \tfrac{1}{3}) + (x + \tfrac{2}{3}) - 1 \text{ and } \lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \tfrac{1}{3} \rfloor + \lfloor x + \tfrac{2}{3} \rfloor,$$

so, subtracting, we obtain

$$\{3x\} = \{x\} + \{x + \tfrac{1}{3}\} + \{x + \tfrac{2}{3}\} - 1.$$

Solution II.

There is a more attractive solution. Can you find it? Try drawing graphs of the functions occurring in the question. ☆

5. A rectangular pine plantation consists of trees planted in equidistant rows in the north-south and east-west directions. A bird can fly from any tree to a neighbouring tree in the N, SE, or SW directions. Prove that the bird can get from any tree to any other tree. Can the bird leave its nest and return in exactly 50 flights?

Solution.

Suppose the trees are at points in the plane with integer coordinates and that the bird is at (x,y). To reach the 8 neighbouring trees, it can proceed as follows:

to reach $(x, y+1)$, it need only travel N, i.e. one step north;
to reach $(x+1, y+1)$, move N, SE, N;
to reach $(x+1, y)$, move N, SE, or SE, N;
to reach $(x+1, y-1)$, move SE;
to reach $(x, y-1)$, move SE, N, SW, or SW, N, SE;
to reach $(x-1, y-1)$, move SW;
to reach $(x-1, y)$, move N, SW, or SW, N;
to reach $(x-1, y+1)$, move N, SW, N.

Thus the bird can travel from any tree to any neighbouring tree (provided the plantation is not a single line of trees), and thence to any other tree.

In flying away from and back to its nest, the bird must make equal numbers, say k, of SW and SE flights. These carry the bird 2k units southwards, so the bird must make 2k N-flights, making 4k flights altogether. The equation $4k = 50$ does not have a solution in integers, so the bird cannot leave the nest and return in exactly 50 flights. ☆

Senior Division

1. $\lfloor x \rfloor$ denotes the "integer part" of x, that is, the largest integer not greater than x, and $\{x\} = x - \lfloor x \rfloor$. Solve the system of equations

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 1.1 \\ \{x\} + y + \lfloor z \rfloor &= 2.2 \\ \lfloor x \rfloor + \{y\} + z &= 3.3 \end{aligned}$$

Solution.

The only solution is $x = 1$, $y = 0.2$ and $z = 2.1$. For, adding the three equations gives $2x + 2y + 2z = 6.6$, that is

$$x + y + z = 3.3.$$

Comparing this with the equation $[x] + \{y\} + z = 3.3$ and noting that $[x] \leqslant x$ and $\{y\} \leqslant y$, we obtain

$$[x] = x, \quad \{y\} = y, \quad \text{and so } \{x\} = 0, \quad [y] = 0.$$

Now the first two equations become

$$[x] + \{z\} = 1.1, \quad \{y\} + [z] = 2.2.$$

It follows that $[x] = 1$, $\{z\} = 0.1$, $[z] = 2$ and $\{y\} = 0.2$, hence

$$x = [x] = 1, \quad y = \{y\} = 0.2 \text{ and } z = [z] + \{z\} = 2.1 \quad \star$$

2. The cubic polynomials P, Q, and R satisfy $P(x) \leqslant Q(x) \leqslant R(x)$ for all x. Suppose further that $P(u) = R(u)$ for some u. Prove that, for some c with $0 \leqslant c \leqslant 1$, we have $Q(x) = cP(x) + (1 - c)R(x)$ for all x.

Solution.

P and Q are cubics and $P(x) \leqslant Q(x)$ for all x. Now $Q - P$ is cubic at most and is non-negative, so must be quadratic at most. Since $Q - P$ is zero at u, we must have $Q(x) - P(x) = A(x-u)^2$ for some $A \geqslant 0$. Similarly, $R(x) - Q(x) = B(x-u)^2$ for some $B \geqslant 0$ and so $R(x) - P(x) = (A+B)(x-u)^2$. If $A + B = 0$, then $P = Q = R$ and any c will do. Otherwise we have $R(x) - Q(x) = c\{R(x) - P(x)\}$ with $c = B/(A+B)$. Then $0 \leqslant c \leqslant 1$ and $Q(x) = cP(x) + (1-c)R(x)$. $\star$

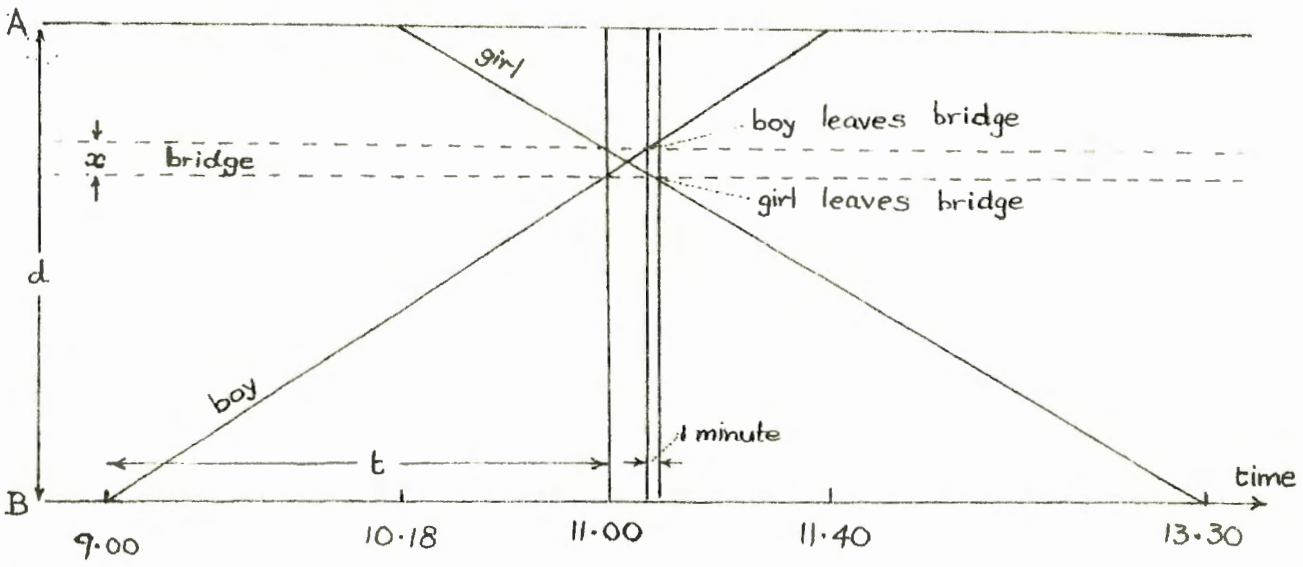
3. A girl leaves a small town A at 10.18 a.m. and, walking at uniform speed, arrives at town B at 1.30 p.m. On the same day, a boy leaves town B at 9.00 a.m. and, walking along the same route at uniform speed, arrives at A at 11.40 a.m. The road crosses a wide river and they both arrive at the bridge on their respective sides at exactly the same moment. The girl leaves the bridge one minute later than the boy. When did they arrive at the bridge?

Solution.

If we denote the distance between the towns by d, then the girl's speed is d/192 and the boy's is d/160 units per minute. If we denote the length of the bridge by x, then $192x/d - 160x/d = 1$, so $32x/d = 1$, that is $x = d/32$. Suppose they arrive at the bridge t minutes after 9 a.m. The boy has travelled $td/160$ and the girl $(t-78)d/192$, and so

$$td/160 + (t-78)d/192 + d/32 = d.$$

Multiplying this equation by 960/d gives $6t + 5(t-78) + 30 = 960$ and so $t = 120$. So they arrive at the bridge at 11 a.m. The travel graph may make the solution clearer. $\star$



4. Given four points in the plane, not on the same circle and no three on a line, how many circles K are there with the property that K is equidistant from all four points? (The distance between a circle and a point P is defined as follows: Draw the ray from the centre of the circle through the point P. This ray intersects the circle in a point Q. The distance PQ is the distance between the circle and the point P.)

Solution.

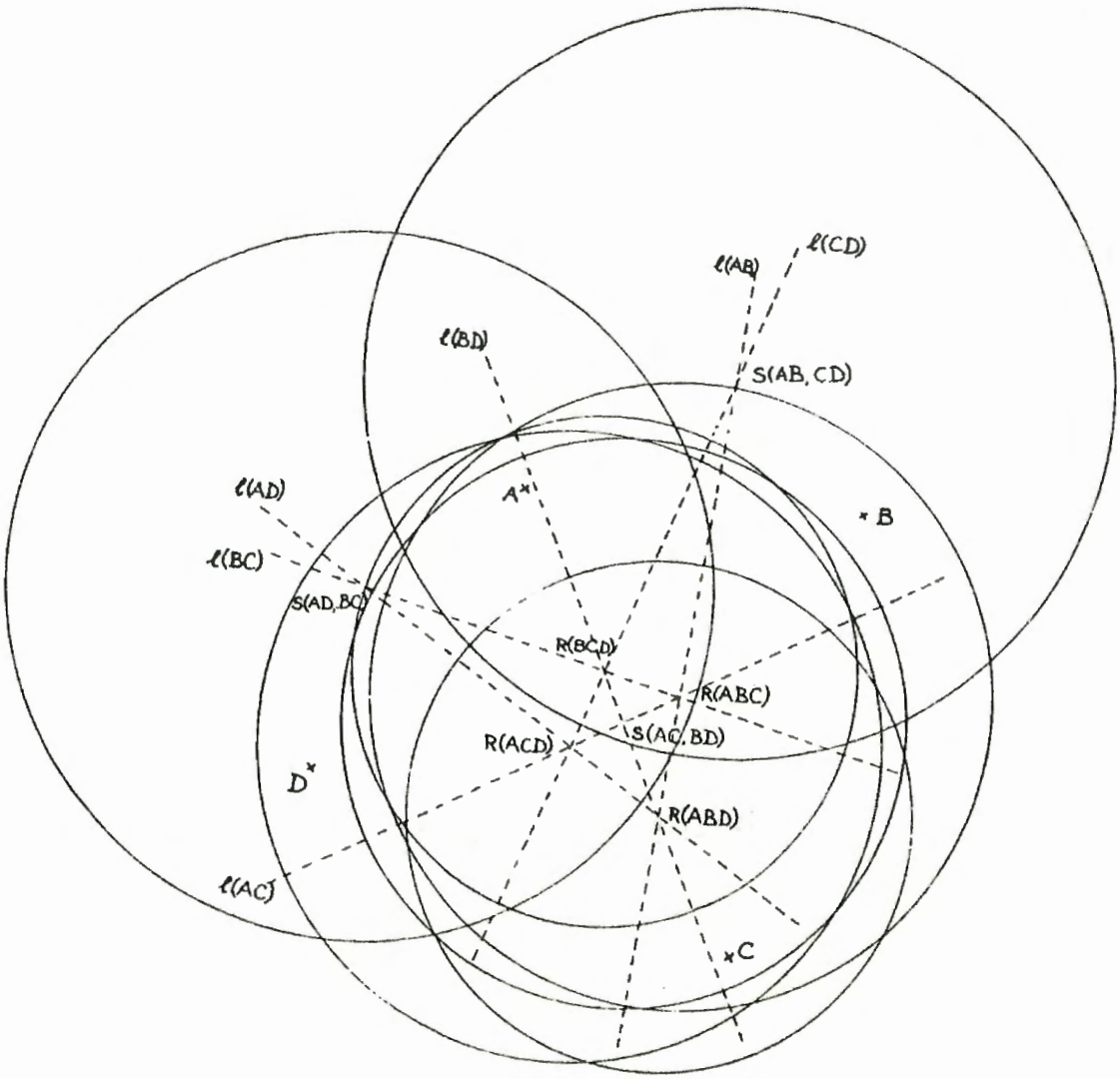
In general, there are seven such circles.

For, let the 4 points be A, B, C, D. Let ℓ(AB) be the perpendicular bisector of the join of A to B. In the same way, define ℓ(AC), ℓ(AD), ℓ(BC), ℓ(BD) and ℓ(CD). Since A, B, C, D do not all lie on the one circle, it follows that no two of these six lines coincide (i.e. along their entire length). The lines ℓ(AB), ℓ(AC) and ℓ(BC) are concurrent at a point (the "circumcentre" of the triangle ABC) which we will denote R(ABC). In this way we can define R(ABC), R(ABD), R(ACD) and R(BCD). Since A, B, C and D do not all lie on the one circle, no two of these four points coincide. Unless AB is parallel to CD, the lines ℓ(AB) and ℓ(CD) intersect at a point which we will denote S(AB,CD). In this way we can define S(AB,CD), S(AC,BD), S(AD, BC). In general all three points exist, and they are distinct from one another and the four points R. The four points R and the three points S are the centres of the seven circles equidistant from the points A, B, C, D. For R(ABC) is equidistant from A, B and C. Call this distance r, and let r′ be the distance of R(ABC) from D. The circle centre R(ABC) and radius $\frac{1}{2}(r + r')$ is equidistant from A, B, C and D. Next S(AB,CD) is equidistant from A, B (call this distance r) and from C, D (call this distance r′). The circle centre S(AB,CD) radius $\frac{1}{2}(r + r')$, is equidistant from A, B, C, D.

That there can be no other circle equidistant from A, B, C, D is shown as follows: If K is such a circle, then we cannot have all four points inside (or outside) K. For then we could shrink (or expand) K till it passes through all four points, a situation which was disallowed. It follows that either three points are on one side of K, one on the other, or two points are on one side, two on the other. In the first case, shrinking (or expanding) the circle causes it to pass through three of the

points, and so its centre is the circumcentre of the triangle they form (one of the points R). In the second case, shrinking the circle causes it to pass through two points, expanding it causes it to pass through the other two, and so its centre is the point of intersection of two bisectors (one of the points S).

If the points A, B, C, D form a trapezium, one of the circles becomes a straight line, while if they form a parallelogram, two of the circles become straight lines. ☆



5. We have 64 cubes, each with one black face, placed on a chessboard with squares the same size as the faces of the cubes. Prove that, by using only rotations of a full vertical or horizontal row of 8 cubes, we can achieve the situation in which all top faces are black.

Solution.

It is sufficient to show how one cube can be rotated in such a way that any face can be brought to the top, while all the other cubes are left as they are.

Consider the cube in the i-th row and j-th column, C(ij). Let H(i) be the rotation of the i-th row, as

shown in Figure 1, V(j) the rotation of the j-th column as shown. The result of applying the operations V(j) followed by H(i) followed by V(j) $^{-1}$ followed by H(i) $^{-1}$ (that is P = H(i) $^{-1}$V(j) $^{-1}$H(i)V(j)) is to move the bottom face of C(ij) to the position A shown in Figure 1, and leave all other cubes as they were. The result of Q = V(j) $^{-1}$H(i) $^{-1}$V(j)H(i) is to move the bottom face of C(ij) to the position B shown in Figure 1 and leave all other cubes as they were.

Now consider the cube C(ij), labelled as in Figure 2, with face 1 on top. The operation P puts face 3 on top, P^2 puts face 6 on top, $P^3 = P^{-1}$ puts face 4 on top, Q puts face 2 on top, $Q^2 = P^2$ puts face 6 on top, and $Q^3 = Q^{-1}$ puts face 5 on top. In each case, the operation leaves all the other cubes as they were.

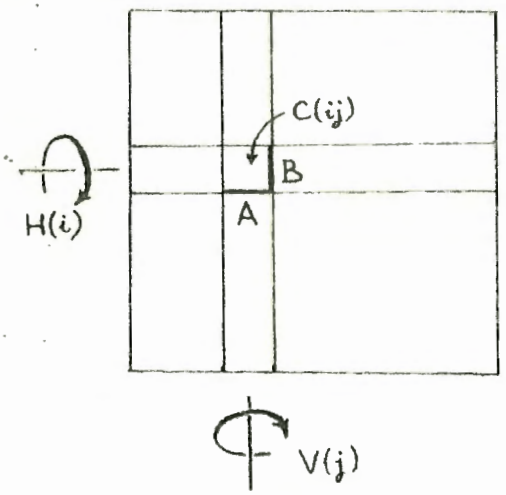


Figure 1

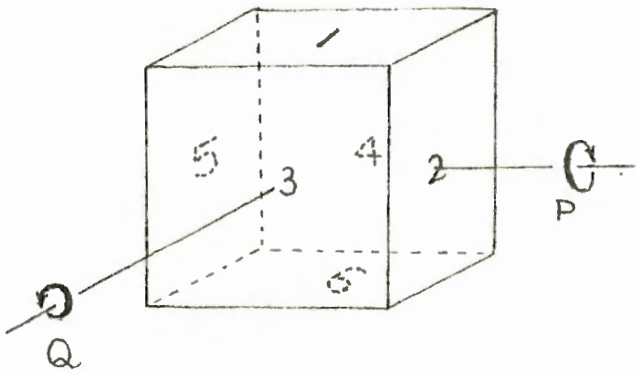


Figure 2

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THEOREMA PLUMBUM

Mathematics is full of theorems, not to mention their attendant lemmas and corollaries. One style, which is very popular in some quarters, goes as follows:

"$b^2 > 4ac$ is a necessary and sufficient condition for the quadratic $ax^2 + bx + c$ to have two real roots," or

"a television set is a necessary and sufficient condition for watching old movies".

Unfortunately, such theorems often suffer from the drawback that they are hard to prove. It gives us great pleasure to reveal a new style of theorem free from every spot or stain of a vexatious proof. For example:

"$a^2 > 4bc$ is an unnecessary and insufficient condition for the quadratic $ax^2 + bx + c$ to have two real roots," or

"bread is unnecessary and insufficient for French peasants" (Marie Antoinette), or

"politicians are unnecessary and insufficient" (General de Gaulle).