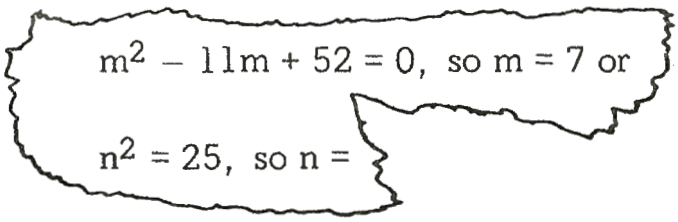
SCHOOL MATHEMATICS COMPETITION 1977

Junior Division

Question 1: Amidst the ruins of Atlantis was found the following fragment:



It is believed that the Atlanteans used a different base from us. Can you complete the fragment?

Answer: The complete fragment was

$$m^2 - 11m + 52 = 0, \text{ so } m = 7 \text{ or } 16$$
$$n^2 = 25, \text{ so } n = 7 \text{ or } -7.$$

Here 16, or whatever it was, was their digit for our decimal number 16. For, suppose the Atlanteans used base B, (while we use base 10). Then the first equation really says

$$m^2 - (B + 1)m + (5B + 2) = 0.$$

We know 7 is a solution, so

$$7^2 - 7(B + 1) + (5B + 2) = 0$$
$$49 - 7B - 7 + 5B + 2 = 0$$
$$2B = 44$$
$$B = 22.$$

So in our notation, the equations become

$$m^2 - 23m + 112 = 0, \text{ so } m = 7 \text{ or } 16$$
$$n^2 = 49, \text{ so } n = 7 \text{ or } -7.$$

Question 2:

i) 1977 is written as the sum of two natural numbers. Find the greatest possible value of the product of these numbers.

ii) 1977 is written as the sum of more than 2 natural numbers. Find the greatest possible value of the product of these numbers.

Answer (i): The greatest possible value of the product is 988 x 989 = 977132. For, let the smaller number be ½ x 1977 − x, and the larger be ½ x 1977 + x. Since they are natural numbers, x is at least ½. Their product is

$$(½ \times 1977 - x)(½ \times 1977 + x) = (½ \times 1977)^2 - x^2$$

which is at most

$$(½ \times 1977)^2 - (½)^2 = (½ \times 1977 - ½)(½ \times 1977 + ½)$$
$$= 988 \times 989 = 977132.$$

Answer (ii): The greatest possible value of the product is 3^{659} . For, suppose we write $1977 = x_1 + x_2 + \ldots + x_n$, where the x's are all natural numbers.

(a) If any of the x's is 1, we can increase their product by changing the x's. Indeed suppose $x_1 = 1$. Then the product of the x's is $1 \times x_2 \times x_3 \times \ldots \times x_n$. If we add x_1 and x_2 we have

$$1977 = (x_1 + x_2) + x_3 + \ldots + x_n,$$

and the product becomes

$$(x_1 + x_2) \times x_3 \times \ldots \times x_n = (1 + x_2) \times x_3 \times \ldots \times x_n$$

which is greater than it was before.

(b) If any of the x's, say x_1, is greater than 3, and if we split x_1 into 2 and $x_1 - 2$, then we have

$$1977 = 2 + (x_1 - 2) + x_2 + \ldots + x_n,$$

and the product becomes

$$2 \times (x_1 - 2) \times x_2 \times \ldots \times x_n,$$

which is at least as great as it was before, since $2 \times (x_1 - 2)$ is at least as great as x_1.

(c) If three or more of the x's are 2's, and we combine three 2's into two 3's, thus

$$2 + 2 + 2 = 3 + 3$$

then the product will be greater than before since

$$2 \times 2 \times 2 < 3 \times 3.$$

In the light of (a), (b), and (c), and since we can write

$$1977 = 3 + 3 + \ldots + 3 \quad (659\ 3\text{'s}),$$

the product of the x's can be no more than 3^{659}. ☆

Question 3: Tiles in the shapes of regular pentagons (5 sides) and regular decagons (10 sides), all the sides of length 1, are available. Is it possible to tile a plane with these tiles without gaps or overlaps?

Answer: No, it is not possible.

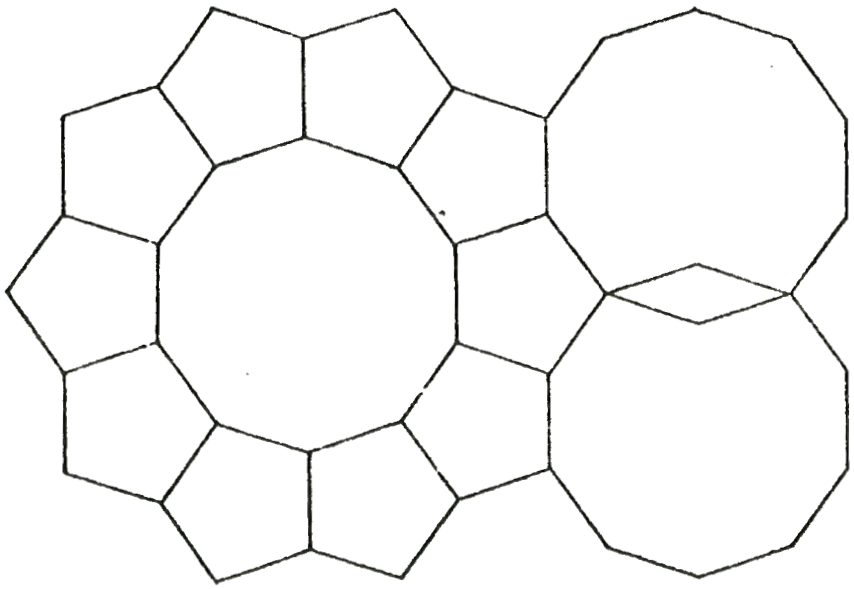
You should know that the sum of the angles in a polygon with n sides is $(n-2) \times 180°$. Thus, the sum of the angles in a pentagon is $(5-2) \times 180° = 3 \times 180° = 540°$, and in a decagon is $(10-2) \times 180° = 8 \times 180° = 1440°$.

In a regular polygon, all the angles are equal, so in a regular pentagon are $540°/5 = 108°$, in a regular decagon are $1440°/10 = 144°$. There is only one way in which angles of 108° and 144° can be combined to make an angle of 360°, namely

$$2 \times 108° + 144° = 360°.$$

This means that if we are to tile a plane with these tiles, we must place two pentagons at each corner of a decagon, and wherever two pentagons meet we must place a decagon.

We can begin to tile the plane quite happily; however, we soon find (see the figure) that an overlap is forced.



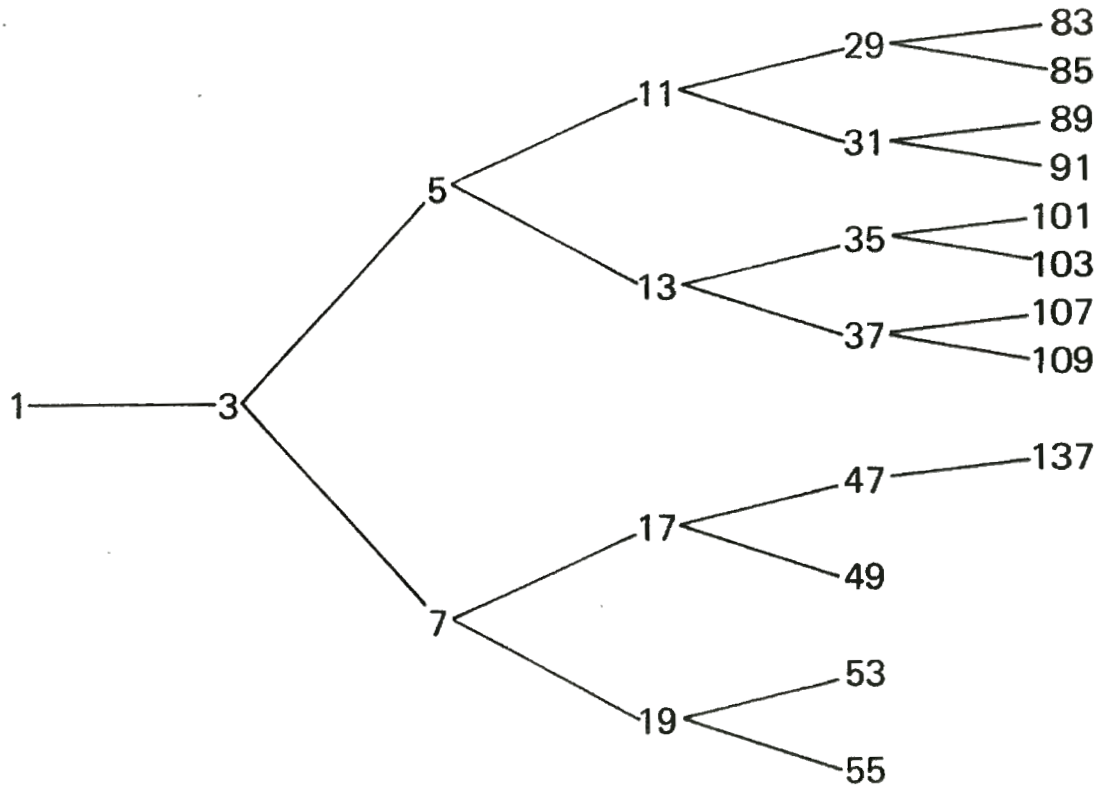
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Question 4: Fewer than 135 players entered a tournament in which each match had three contestants and one winner. Before each round the names of all the survivors were drawn to provide as many matches as possible, with byes being given to any players left out. It turned out that the tournament was won outright by a player who contested only one match. What was the greatest possible number who took part?

Answer: The greatest number who took part is 109.

Since there was an outright winner, there were precisely three players left with one round to go, one of whom was the eventual winner, this being his only match. So in the earlier rounds he, at least, was given a bye, so at no earlier stage was the number of players divisible by 3. Indeed, if, after any round but the last, there were n players left, then before that round there must have been either $2 + 3(n-2) = 3n-4$ or $1 + 3(n-1) = 3n-2$ players.

The "tree" below shows the possible numbers of players who entered the tournament so that the above might occur.



Since there were fewer than 135 players, the greatest possible number who took part was 109. ☆

Question 5:

i) The product of all the divisors of a natural number is $2^{120}3^{60}5^{90}$. What is the number?

ii) The product of all the divisors of a natural number is $2^m 3^{49}5^n$, where m and n are greater than 0. What is the number?

Answer (i): The number is 18000.

For, let the number be k. Clearly, the factorisation of k into primes is

$$k = 2^a 3^b 5^c.$$

The number of divisors of k is

$$s = (a + 1)(b + 1)(c + 1),$$

since every divisor of k is of the form $2^x 3^y 5^z$, with $x \in \{0,1,2,\ldots,a\}$, $y \in \{0,1,2,\ldots b\}$, $z \in \{0,1,2,\ldots,c\}$, independent of one another.

Also, the product of the divisors of k, P say, is given by

$$P^2 = k^s,$$

because if the divisors of k are, in increasing order,

$$d_1, d_2, \ldots, d_s \text{ (where } d_1 = 1, d_s = k),$$

then

$$\begin{aligned} P^2 &= (d_1 \times d_2 \times \ldots \times d_s) \times (d_s \times d_{s-1} \times \ldots \times d_1) \\ &= (d_1 \times d_s) \times (d_2 \times d_{s-1}) \times \ldots \times (d_s \times d_1) \\ &= k \times k \times \ldots \times k \\ &= k^s. \end{aligned}$$

So, we have

$$(2^{120}3^{60}5^{90})^2 = (2^a 3^b 5^c)^{(a+1)(b+1)(c+1)}.$$

So

$$\begin{aligned} a(a + 1)(b + 1)(c + 1) &= 240, \\ b(a + 1)(b + 1)(c + 1) &= 120, \\ c(a + 1)(b + 1)(c + 1) &= 180. \end{aligned}$$

So $a : b : c = 240 : 120 : 180 = 4 : 2 : 3.$

Let $a = 4r, b = 2r, c = 3r$, where r is certainly a positive rational number.

Then $r(4r + 1)(2r + 1)(3r + 1) = 60.$

The only solution of this equation with r a positive number is r = 1, which gives a = 4, b = 2, c = 3, and

$$k = 2^4 \times 3^2 \times 5^3 = 16 \times 9 \times 125 = 18000.$$

Answer (ii): The number is 3000000.

For, let the number be $k = 2^a 3^b 5^c$. As before we find

$$a(a + 1)(b + 1)(c + 1) = 2m,$$
$$b(a + 1)(b + 1)(c + 1) = 98,$$
$$c(a + 1)(b + 1)(c + 1) = 2n.$$

The second equation says that b is a positive integer such that

$$98 \text{ is divisible by } b(b + 1).$$

The only possible value for b is 1.

So $(a + 1)(c + 1) = 49.$

Since m and n are positive, a and c cannot be 0, so must both be 6.

So $k = 2^6\ 3^1\ 5^6 = 3 \times 10^6 = 3000000.$ ☆

Senior Division

Question 1:

i) Prove that the equation $x^7 + y^9 = z^8$ has infinitely many solutions in positive integers x, y and z, all powers of 2.

ii) Prove that the equation $w^{1977} + x^7 + y^9 = z^8$ has infinitely many solutions in positive integers w, x, y and z.

Answer (i): $x = 2^{9+72t}$, $y = 2^{7+56t}$, $z = 2^{8+63t}$ is a solution for every non-negative integer t, since

$$\begin{aligned} x^7 + y^9 &= 2^{63+504t} + 2^{63+504t} \\ &= 2 \times 2^{63+504t} \\ &= 2^{64+504t} = z^8. \end{aligned}$$

For, let $x = 2^a$, $y = 2^b$, $z = 2^c$.

Then $x^7 = 2^{7a}$, $y^9 = 2^{9b}$, $z^8 = 2^{8c}$.

It is now a fact that

$$7a = 9b$$

since, if, for example, $7a < 9b$, we would have

$$2^{7a}(1 + 2^{9b-7a}) = 2^{8c}$$

which is not possible, since $1 + 2^{9b-7a}$ is odd.

So $a = 9d$, $b = 7d$

so $x^7 + y^9 = 2^{63d} + 2^{63d}$

$= 2 \times 2^{63d}$

$= 2^{63d+1} = z^8 = 2^{8c}$,

so $63d + 1 = 8c$

or, $8c - 63d = 1$.

The solutions of this diophantine equation are

$c = 8 + 63t$, $d = 1 + 8t$,

so finally, $a = 9d = 9 + 72t$,

$b = 7d = 7 + 56t$,

$c = 8 + 63t$.

Answer (ii): $w = 3^{63+168t}$, $x = 3^{17793+47448t}$, $y = 3^{13839+36904t}$,

$z = 3^{15569+41517t}$ is a solution for every non-negative integer t, since

$w^{1977} + x^7 + y^9 = 3^{124551+332136t} + 3^{124551+332136t} + 3^{124551+332136t}$

$= 3 \times 3^{124551+332136t}$

$= 3^{124552+332136t}$

$= z^8$.

For, let $w = 3^a$, $x = 3^b$, $y = 3^c$, $z = 3^d$.

Then $w^{1977} = 3^{1977a}$, $x^7 = 3^{7b}$, $y^9 = 3^{9c}$, $z^8 = 3^{8d}$.

Then $1977a = 7b = 9c$

so $a = 21e$, $b = 5931e$, $c = 4613e$,

$$w^{1977} + x^7 + y^9 = 3 \times 3^{41517e} = 3^{1+41517e} = z^8 = 3^{8d}.$$

So $8d - 41517e = 1,$

from which it follows that

$$d = 15569 + 41517t, \quad e = 3 + 8t.$$

So

$$a = 63 + 168t,$$
$$b = 17793 + 47448t,$$
$$c = 13839 + 36904t. \quad \star$$

Question 2: Find all the solutions in positive integers of the equation

$$\frac{x + y}{x^2 - xy + y^2} = \frac{8}{73}.$$

Answer: (x,y) = (7,17) or (17,7) are the only solutions in positive integers.

For, let $x + y = s$.

Then the equation becomes

$$\frac{s}{s^2 - 3xy} = \frac{8}{73} \text{ or, } xy = (8s^2 - 73s)/24$$

Now, $0 < xy = x(s-x) = \frac{1}{4}s^2 - (x - \frac{1}{2}s)^2 \leqslant \frac{1}{4}s^2$

so $0 < (8s^2 - 73s)/24 \leqslant s^2/4,$

from which it follows that $10 \leqslant s \leqslant 36$.

Also $8s^2 - 73s$ is divisible by 24,

so $8s^2 - 73s$ is divisible by 8

so s is divisible by 8

so s = 16, 24 or 32.

But $8 \times 16^2 - 73 \times 16 = 880$ is not divisible by 24, so s = 24 or 32.

If s = 32, $xy = (8 \times 32^2 - 73 \times 32)/24 = 244,$

$x(32-x) = 244$

$(x-16)^2 = 12$. But this has no integer solutions.

If s = 24, $xy = (8 \times 24^2 - 73 \times 24)/24 = 119,$

$x(24-x) = 119,$

$(x-12)^2 = 25, \quad x = 7 \text{ or } 17.$

If x = 7, y = 17, while if x = 17, y = 7. ★

Question 3: If the midpoint of each of the sides of a parallelogram is joined to the two opposite vertices, the eight lines so formed determine an octagon. Prove that the area of the octagon is 1/6 of the area of the parallelogram.

Answer:

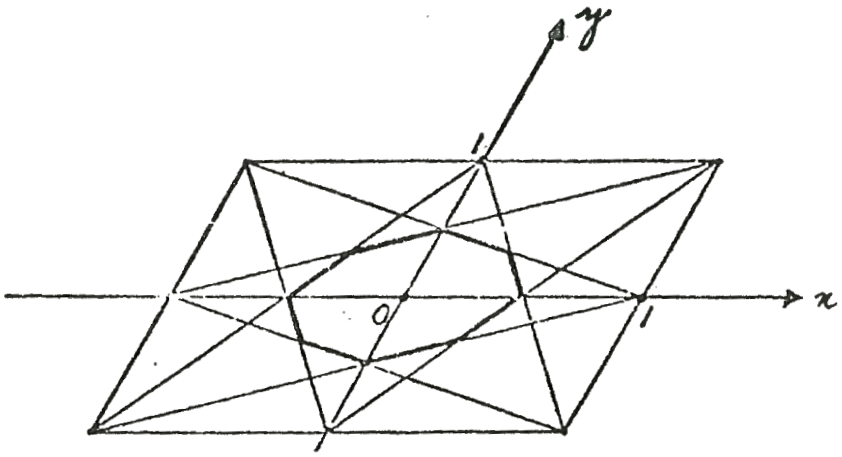


Figure 1

Construct coordinate axes as shown in Fig. 1 parallel to the sides of the parallelogram, with origin at the centre of the parallelogram, and consider that portion of the diagram in the (+,+) quadrant.

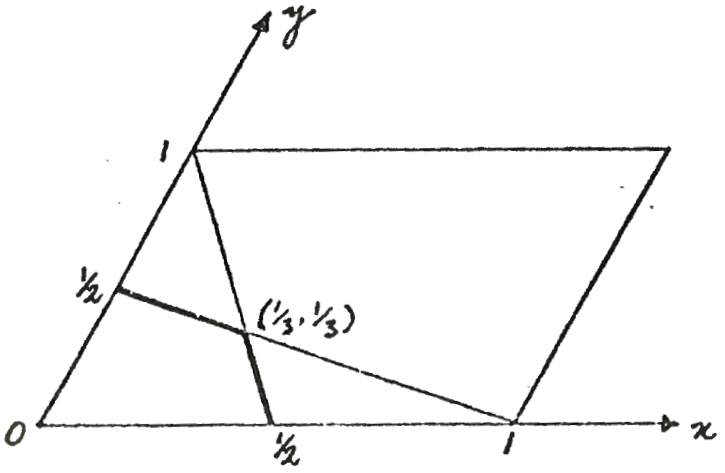


Figure 2

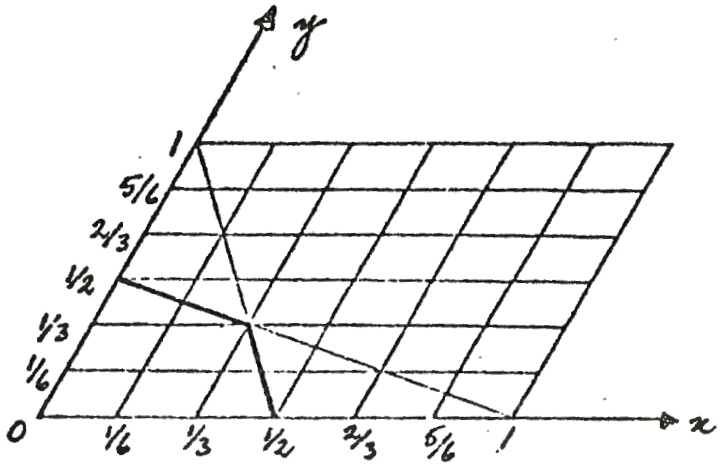


Figure 3

The two lines shown in Fig. 2 have equations $x + 2y = 1$, $2x + y = 1$, and so meet at (1/3,1/3). Now draw the lines $x = 1/6, 1/3, 1/2, 2/3, 5/6$, and $y = 1/6, 1/3, 1/2, 2/3, 5/6$ (see Fig. 3).

The result now becomes clear. ☆

Question 4: Given three infinite sequences of natural numbers,

$a_1, a_2, \ldots, a_n, \ldots,$ $\quad b_1, b_2, \ldots, b_n, \ldots,$ $\quad c_1, c_2, \ldots, c_n, \ldots,$

prove that there exist p and q such that $a_p \geqslant a_q$, $b_p \geqslant b_q$, $c_p \geqslant c_q$.

Answer: To simplify the notation, we will write a(n) instead of a_n, b(n) instead of b_n and c(n) instead of c_n. Now consider the first sequence of numbers a(1), a(2), a(3), ...

There are two possibilities: (i) there is a number A such that $a(n) < A$ for every n. (ii) there is no such number A. In the first case there must be a number a, and an infinite sequence $n_1 < n_2 < \ldots$ such that $a(n_1) = a(n_2) = \ldots = a$; in the second case there must be an infinite sequence $n_1 < n_2 < \ldots$ such that $a(n_1) < a(n_2) < a(n_3) < \ldots$.

In either case we confine our attention to the triples $(a(n_1), b(n_1), c(n_1))$, $(a(n_2), b(n_2), c(n_2))$, ... Either there is a B such that $b(n_i) < B$ for all n_i, or there is not. As above, there is an infinite sequence $n_1' < n_2' < \ldots$, such that $b(n_1') = b(n_2') = \ldots$, or there is an infinite sequence $n_1' < n_2' < \ldots$ such that $b(n_1') < b(n_2') < \ldots$, and we also have $a(n_1') = a(n_2') = \ldots$ or $a(n_1') < a(n_2') < \ldots$. In any case, we confine our attention to the triples $(a(n_1'), b(n_1'), c(n_1'))$, $(a(n_2'), b(n_2'), c(n_2'))$, ...

As above, we find that there is an infinite sequence $n_1'' < n_2'' < \ldots$ such that $c(n_1'') = c(n_2'') = \ldots$ or there is an infinite sequence $n_1'' < n_2'' < \ldots$ such that $c(n_1'') < c(n_2'') < \ldots$ and also $a(n_1'') = a(n_2'') = \ldots$ or $a(n_1'') < a(n_2'') < \ldots$, and $b(n_1'') = b(n_2'') = \ldots$ or $b(n_1'') < b(n_2'') < \ldots$.

So, if p,q are any two of n_1'', n_2'', . . . , with $q < p$, then $a(p) \geqslant a(q)$, $b(p) \geqslant b(q)$, $c(p) \geqslant c(q)$. ✡

Question 5: Let $a_1, a_2, \ldots, a_n$ be the numbers 1, 2, . . . , n in some order. Show that

$$(1-a_1)^2 + (2-a_2)^2 + \ldots + (n-a_n)^2 \leqslant (n-1)n(n+1)/3$$

and that

$$1a_1 + 2a_2 + \ldots + na_n \geqslant n(n+1)(n+2)/6.$$

Answer: We need the following facts:

$1 + 2 + \ldots + n = \frac{1}{2}n(n+1)$, $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

Let $S_1 = (1-a_1)^2 + (2-a_2)^2 + \ldots + (n-a_n)^2$,

$$S_2 = 1a_1 + 2a_2 + \ldots + na_n.$$

Then $S_1 + 2S_2 = 1^2 + 2^2 + \ldots + n^2 + a_1^2 + a_2^2 + \ldots + a_n^2$

$= 2(1^2 + 2^2 + \ldots + n^2)$

$= 2 \times n(n+1)(2n+1)/6$

$= n(n+1)(2n+1)/3.$

Now let $b_i = n + 1 - a_i$, $i = 1,2, \ldots, n$. Then $b_1, b_2, \ldots, b_n$ are also $1,2, \ldots, n$ in some order.

Let $T_1 = (1-b_1)^2 + (2-b_2)^2 + \ldots + (n-b_n)^2$,

$T_2 = 1b_1 + 2b_2 + \ldots + nb_n.$

Then, as before,

$$T_1 + 2T_2 = n(n + 1)(2n + 1)/3.$$

Also, $S_2 + T_2 = 1a_1 + 2a_2 + \ldots na_n + 1b_1 + 2b_2 + \ldots + nb_n$

$= 1(a_1 + b_1) + 2(a_2 + b_2) + \ldots + n(a_n + b_n)$

$= 1(n + 1) + 2(n + 1) + \ldots + n(n + 1)$

$= (n + 1)(1 + 2 + \ldots + n)$

$= (n + 1) \times \frac{1}{2}n(n + 1)$

$= \frac{1}{2}n(n + 1)^2.$

So $S_1 + T_1 = (S_1 + 2S_2) + (T_1 + 2T_2) - 2(S_2 + T_2)$

$= 2 \times n(n + 1)(2n + 1)/3 - 2 \times n(n + 1)^2/2$

$= n(n + 1)[2(2n + 1) - 3(n + 1)]/3$

$= (n-1)n(n + 1)/3.$

Now, clearly $T_1 \geqslant 0$, so $S_1 \leqslant (n-1)n(n + 1)/3$,

so $2S_2 \geqslant [n(n + 1)(2n +1) - (n- 1)n(n + 1)]/3$

$= n(n + 1)[(2n + 1) - (n-1)]/3$

$= n(n + 1)(n + 2)/3,$

so $S_2 \geqslant n(n + 1)(n + 2)/6.$

Both equalities hold if and only if $T_1 = 0$,

that is, $b_i = i,\ i = 1, 2, \ldots, n,$

or $a_i = n + 1 - i,\ i = 1, 2, \ldots, n.$ ☆