The Forty-Sixth Annual William Lowell Putnam Competition
Saturday, December 7, 1985

A–1 Determine, with proof, the number of ordered triples (A_1, A_2, A_3) of sets which have the property that

(i) $A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5,6,7,8,9,10\}$, and

(ii) $A_1 \cap A_2 \cap A_3 = \emptyset$.

Express your answer in the form $2^a 3^b 5^c 7^d$, where a,b,c,d are nonnegative integers.

A–2 Let T be an acute triangle. Inscribe a rectangle R in T with one side along a side of T. Then inscribe a rectangle S in the triangle formed by the side of R opposite the side on the boundary of T, and the other two sides of T, with one side along the side of R. For any polygon X, let $A(X)$ denote the area of X. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where T ranges over all triangles and R,S over all rectangles as above.

A–3 Let d be a real number. For each integer $m \geq 0$, define a sequence $\{a_m(j)\}$, $j = 0,1,2,\ldots$ by the condition

$$a_m(0) = d/2^m,$$
$$a_m(j+1) = (a_m(j))^2 + 2a_m(j), \qquad j \geq 0.$$

Evaluate $\lim_{n\to\infty} a_n(n)$.

A–4 Define a sequence $\{a_i\}$ by $a_1 = 3$ and $a_{i+1} = 3^{a_i}$ for $i \geq 1$. Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many a_i?

A–5 Let $I_m = \int_0^{2\pi} \cos(x)\cos(2x)\cdots\cos(mx)\,dx$. For which integers m, $1 \leq m \leq 10$ is $I_m \neq 0$?

A–6 If $p(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a polynomial with real coefficients a_i, then set

$$\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.$$

Let $F(x) = 3x^2 + 7x + 2$. Find, with proof, a polynomial $g(x)$ with real coefficients such that

(i) $g(0) = 1$, and

(ii) $\Gamma(f(x)^n) = \Gamma(g(x)^n)$

for every integer $n \geq 1$.

B–1 Let k be the smallest positive integer for which there exist distinct integers m_1, m_2, m_3, m_4, m_5 such that the polynomial

$$p(x) = (x-m_1)(x-m_2)(x-m_3)(x-m_4)(x-m_5)$$

has exactly k nonzero coefficients. Find, with proof, a set of integers m_1, m_2, m_3, m_4, m_5 for which this minimum k is achieved.

B–2 Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1) f_n(x+1)$$

for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

B–3 Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m,n).

B–4 Let C be the unit circle $x^2 + y^2 = 1$. A point p is chosen randomly on the circumference C and another point q is chosen randomly from the interior of C (these points are chosen independently and uniformly over their domains). Let R be the rectangle with sides parallel to the x and y-axes with diagonal pq. What is the probability that no point of R lies outside of C?

B–5 Evaluate $\int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})}\,dt$. You may assume that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

B–6 Let G be a finite set of real $n \times n$ matrices $\{M_i\}$, $1 \leq i \leq r$, which form a group under matrix multiplication. Suppose that $\sum_{i=1}^r \mathrm{tr}(M_i) = 0$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{i=1}^r M_i$ is the $n \times n$ zero matrix.

The Forty-Seventh Annual William Lowell Putnam Competition
Saturday, December 6, 1986

A–1 Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying $x^4 + 36 \leq 13x^2$.

A–2 What is the units (i.e., rightmost) digit of

$$\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor ?$$

A–3 Evaluate $\sum_{n=0}^{\infty} \operatorname{Arccot}(n^2+n+1)$, where $\operatorname{Arccot} t$ for $t \geq 0$ denotes the number θ in the interval $0 < \theta \leq \pi/2$ with $\cot\theta = t$.

A–4 A *transversal* of an $n \times n$ matrix A consists of n entries of A, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices A satisfying the following two conditions:

(a) Each entry $a_{i,j}$ of A is in the set $\{-1, 0, 1\}$.

(b) The sum of the n entries of a transversal is the same for all transversals of A.

An example of such a matrix A is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1 b_1^n + a_2 b_2^n + a_3 b_3^n + a_4,$$

where the a_i's and b_i's are rational numbers.

A–5 Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of n real variables $x = (x_1, \ldots, x_n)$ with continuous second-order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants c_{ij} such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij}$$

for all i and j, $1 \leq i \leq n$, $1 \leq j \leq n$. Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \partial g/\partial x_i$ is linear for all i, $1 \leq i \leq n$. (A linear function is one of the form

$$a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n.)$$

A–6 Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and n (but independent of $a_1, a_2, \ldots, a_n$).

B–1 Inscribe a rectangle of base b and height h in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of h do the rectangle and triangle have the same area?

B–2 Prove that there are only a finite number of possibilities for the ordered triple $T = (x-y, y-z, z-x)$, where x, y, z are complex numbers satisfying the simultaneous equations

$$x(x-1) + 2yz = y(y-1) + 2zx = z(z-1) + 2xy,$$

and list all such triples T.

B–3 Let Γ consist of all polynomials in x with integer coefficients. For f and g in Γ and m a positive integer, let $f \equiv g \pmod{m}$ mean that every coefficient of $f-g$ is an integral multiple of m. Let n and p be positive integers with p prime. Given that f, g, h, r and s are in Γ with $rf + sg \equiv 1 \pmod{p}$ and $fg \equiv h \pmod{p}$, prove that there exist F and G in Γ with $F \equiv f \pmod{p}$, $G \equiv g \pmod{p}$, and $FG \equiv h \pmod{p^n}$.

B–4 For a positive real number r, let $G(r)$ be the minimum value of $|r - \sqrt{m^2 + 2n^2}|$ for all integers m and n. Prove or disprove the assertion that $\lim_{r\to\infty} G(r)$ exists and equals 0.

B–5 Let $f(x,y,z) = x^2 + y^2 + z^2 + xyz$. Let $p(x,y,z), q(x,y,z), r(x,y,z)$ be polynomials with real coefficients satisfying

$$f(p(x,y,z), q(x,y,z), r(x,y,z)) = f(x,y,z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$, where the number of minus signs is 0 or 2.

B–6 Suppose A, B, C, D are $n \times n$ matrices with entries in a field F, satisfying the conditions that AB^T and CD^T are symmetric and $AD^T - BC^T = I$. Here I is the $n \times n$ identity matrix, and if M is an $n \times n$ matrix, M^T is its transpose. Prove that $A^T D - C^T B = I$.

The Forty-Eighth Annual William Lowell Putnam Competition
Saturday, December 5, 1987

A–1 Curves A, B, C and D are defined in the plane as follows:

$$A = \left\{ (x,y) : x^2 - y^2 = \frac{x}{x^2+y^2} \right\},$$
$$B = \left\{ (x,y) : 2xy + \frac{y}{x^2+y^2} = 3 \right\},$$
$$C = \left\{ (x,y) : x^3 - 3xy^2 + 3y = 1 \right\},$$
$$D = \left\{ (x,y) : 3x^2y - 3x - y^3 = 0 \right\}.$$

Prove that $A \cap B = C \cap D$.

A–2 The sequence of digits

$$123456789101112131415161718192021\ldots$$

is obtained by writing the positive integers in order. If the 10^n-th digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed, define $f(n)$ to be m. For example, $f(2) = 2$ because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

A–3 For all real x, the real-valued function $y = f(x)$ satisfies

$$y'' - 2y' + y = 2e^x.$$

(a) If $f(x) > 0$ for all real x, must $f'(x) > 0$ for all real x? Explain.

(b) If $f'(x) > 0$ for all real x, must $f(x) > 0$ for all real x? Explain.

A–4 Let P be a polynomial, with real coefficients, in three variables and F be a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y-x, z-x) \quad \text{for all real } x, y, z, u,$$

and such that $P(1,0,0) = 4$, $P(0,1,0) = 5$, and $P(0,0,1) = 6$. Also let A, B, C be complex numbers with $P(A,B,C) = 0$ and $|B-A| = 10$. Find $|C-A|$.

A–5 Let

$$\vec{G}(x,y) = \left(\frac{-y}{x^2+4y^2}, \frac{x}{x^2+4y^2}, 0 \right).$$

Prove or disprove that there is a vector-valued function

$$\vec{F}(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$$

with the following properties:

(i) M, N, P have continuous partial derivatives for all $(x,y,z) \neq (0,0,0)$;

(ii) $\operatorname{Curl} \vec{F} = \vec{0}$ for all $(x,y,z) \neq (0,0,0)$;

(iii) $\vec{F}(x,y,0) = \vec{G}(x,y)$.

A–6 For each positive integer n, let $a(n)$ be the number of zeroes in the base 3 representation of n. For which positive real numbers x does the series

$$\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$$

converge?

B–1 Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

B–2 Let r, s and t be integers with $0 \leq r$, $0 \leq s$ and $r+s \leq t$. Prove that

$$\frac{\binom{s}{0}}{\binom{t}{r}} + \frac{\binom{s}{1}}{\binom{t}{r+1}} + \cdots + \frac{\binom{s}{s}}{\binom{t}{r+s}} = \frac{t+1}{(t+1-s)\binom{t-s}{r}}.$$

B–3 Let F be a field in which $1+1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with x and y in F is given by $(x,y) = (1,0)$ and

$$(x,y) = \left(\frac{r^2-1}{r^2+1}, \frac{2r}{r^2+1} \right)$$

where r runs through the elements of F such that $r^2 \neq -1$.

B–4 Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist.

B–5 Let O_n be the n-dimensional vector $(0, 0, \cdots, 0)$. Let M be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex z_i, not all zero, then at least one of the z_i is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\operatorname{re}\left[M \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \right] = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.$$

(Note: if C is a matrix of complex numbers, $\operatorname{re}(C)$ is the matrix whose entries are the real parts of the entries of C.)

B–6 Let F be the field of p^2 elements, where p is an odd prime. Suppose S is a set of $(p^2-1)/2$ distinct nonzero elements of F with the property that for each $a \neq 0$ in F, exactly one of a and $-a$ is in S. Let N be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that N is even.

The Forty-Ninth Annual William Lowell Putnam Competition
Saturday, December 3, 1988

A–1 Let R be the region consisting of the points (x,y) of the cartesian plane satisfying both $|x|-|y|\le 1$ and $|y|\le 1$. Sketch the region R and find its area.

A–2 A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a,b) and a nonzero function g defined on (a,b) such that this wrong product rule is true for x in (a,b).

A–3 Determine, with proof, the set of real numbers x for which

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\csc\frac{1}{n}-1\right)^x$$

converges.

A–4 (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?

(b) What if "three" is replaced by "nine"?

A–5 Prove that there exists a *unique* function f from the set R^+ of positive real numbers to R^+ such that

$$f(f(x)) = 6x - f(x)$$

and

$$f(x) > 0$$

for all $x > 0$.

A–6 If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

B–1 A *composite* (positive integer) is a product ab with a and b not necessarily distinct integers in $\{2,3,4,\dots\}$. Show that every composite is expressible as $xy+xz+yz+1$, with x,y,z positive integers.

B–2 Prove or disprove: If x and y are real numbers with $y \ge 0$ and $y(y+1) \le (x+1)^2$, then $y(y-1) \le x^2$.

B–3 For every n in the set $\mathrm{N} = \{1,2,\dots\}$ of positive integers, let r_n be the minimum value of $|c - d\sqrt{3}|$ for all nonnegative integers c and d with $c+d=n$. Find, with proof, the smallest positive real number g with $r_n \le g$ for all $n \in \mathrm{N}$.

B–4 Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers, then so is $\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}$.

B–5 For positive integers n, let M_n be the $2n+1$ by $2n+1$ skew-symmetric matrix for which each entry in the first n subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is -1. Find, with proof, the rank of M_n. (According to one definition, the rank of a matrix is the largest k such that there is a $k \times k$ submatrix with nonzero determinant.)

One may note that

$$M_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}.$$

B–6 Prove that there exist an infinite number of ordered pairs (a,b) of integers such that for every positive integer t, the number $at+b$ is a triangular number if and only if t is a triangular number. (The triangular numbers are the $t_n = n(n+1)/2$ with n in $\{0,1,2,\dots\}$.)

The Fiftieth Annual William Lowell Putnam Competition
Saturday, December 2, 1989

A–1 How many primes among the positive integers, written as usual in base 10, are alternating 1's and 0's, beginning and ending with 1?

A–2 Evaluate $\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\, dy\, dx$ where a and b are positive.

A–3 Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here z is a complex number and $i^2 = -1$.)

A–4 If α is an irrational number, $0 < \alpha < 1$, is there a finite game with an honest coin such that the probability of one player winning the game is α? (An honest coin is one for which the probability of heads and the probability of tails are both $\frac{1}{2}$. A game is finite if with probability 1 it must end in a finite number of moves.)

A–5 Let m be a positive integer and let $\mathscr{G}$ be a regular $(2m+1)$-gon inscribed in the unit circle. Show that there is a positive constant A, independent of m, with the following property. For any points p inside $\mathscr{G}$ there are two distinct vertices v_1 and v_2 of $\mathscr{G}$ such that

$$\big||p - v_1| - |p - v_2|\big| < \frac{1}{m} - \frac{A}{m^3}.$$

Here $|s - t|$ denotes the distance between the points s and t.

A–6 Let $\alpha = 1 + a_1x + a_2x^2 + \cdots$ be a formal power series with coefficients in the field of two elements. Let

$$a_n = \begin{cases} 1 & \text{if every block of zeros in the binary expansion of } n \text{ has an even number of zeros in the block} \\ 0 & \text{otherwise.} \end{cases}$$

(For example, $a_{36} = 1$ because $36 = 100100_2$ and $a_{20} = 0$ because $20 = 10100_2$.) Prove that $\alpha^3 + x\alpha + 1 = 0$.

B–1 A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form $\dfrac{a\sqrt{b}+c}{d}$, where a, b, c, d are integers.

B–2 Let S be a non-empty set with an associative operation that is left and right cancellative ($xy = xz$ implies $y = z$, and $yx = zx$ implies $y = z$). Assume that for every a in S the set $\{a^n : n = 1, 2, 3, \ldots\}$ is finite. Must S be a group?

B–3 Let f be a function on $[0, \infty)$, differentiable and satisfying

$$f'(x) = -3f(x) + 6f(2x)$$

for $x > 0$. Assume that $|f(x)| \le e^{-\sqrt{x}}$ for $x \ge 0$ (so that $f(x)$ tends rapidly to 0 as x increases). For n a non-negative integer, define

$$\mu_n = \int_0^\infty x^n f(x)\, dx$$

(sometimes called the nth moment of f).

a) Express μ_n in terms of μ_0.

b) Prove that the sequence $\{\mu_n \frac{3^n}{n!}\}$ always converges, and that the limit is 0 only if $\mu_0 = 0$.

B–4 Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

B–5 Label the vertices of a trapezoid T (quadrilateral with two parallel sides) inscribed in the unit circle as A, B, C, D so that AB is parallel to CD and A, B, C, D are in counterclockwise order. Let s_1, s_2, and d denote the lengths of the line segments AB, CD, and OE, where E is the point of intersection of the diagonals of T, and O is the center of the circle. Determine the least upper bound of $\frac{s_1 - s_2}{d}$ over all such T for which $d \neq 0$, and describe all cases, if any, in which it is attained.

B–6 Let $(x_1, x_2, \ldots x_n)$ be a point chosen at random from the n-dimensional region defined by $0 < x_1 < x_2 < \cdots < x_n < 1$. Let f be a continuous function on $[0, 1]$ with $f(1) = 0$. Set $x_0 = 0$ and $x_{n+1} = 1$. Show that the expected value of the Riemann sum

$$\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})$$

is $\int_0^1 f(t)P(t)\, dt$, where P is a polynomial of degree n, independent of f, with $0 \le P(t) \le 1$ for $0 \le t \le 1$.

The 51st William Lowell Putnam Mathematical Competition
Saturday, December 8, 1990

A–1 Let

$$T_0 = 2, T_1 = 3, T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where $\{A_n\}$ and $\{B_n\}$ are well-known sequences.

A–2 Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$ $(n, m = 0, 1, 2, \ldots)$?

A–3 Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area greater than or equal to 5/2.

A–4 Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

A–5 If **A** and **B** are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

A–6 If X is a finite set, let X denote the number of elements in X. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer.

B–1 Find all real-valued continuously differentiable functions f on the real line such that for all x,

$$(f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2]\, dt + 1990.$$

B–2 Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) P_j = 0,$$

where P_j is

$$\frac{(1-z)(1-zx)(1-zx^2)\cdots(1-zx^{j-1})}{(z-x)(z-x^2)(z-x^3)\cdots(z-x^j)}.$$

B–3 Let S be a set of 2×2 integer matrices whose entries a_{ij} (1) are all squares of integers and, (2) satisfy $a_{ij} \leq 200$. Show that if S has more than 50387 $(= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

B–4 Let G be a finite group of order n generated by a and b. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots, g_{2n}$$

such that

(1) every element of G occurs exactly twice, and

(2) g_{i+1} equals $g_i a$ or $g_i b$ for $i = 1, 2, \ldots, 2n$. (Interpret g_{2n+1} as g_1.)

B–5 Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

B–6 Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K_1, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K,t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K,t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.)

The 52nd William Lowell Putnam Mathematical Competition
Saturday, December 7, 1991

A–1 A 2×3 rectangle has vertices as $(0,0),(2,0),(0,3)$, and $(2,3)$. It rotates 90° clockwise about the point $(2,0)$. It then rotates 90° clockwise about the point $(5,0)$, then 90° clockwise about the point $(7,0)$, and finally, 90° clockwise about the point $(10,0)$. (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1).

A–2 Let $\mathbf{A}$ and $\mathbf{B}$ be different $n\times n$ matrices with real entries. If $\mathbf{A}^3=\mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B}=\mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2+\mathbf{B}^2$ be invertible?

A–3 Find all real polynomials $p(x)$ of degree $n\geq 2$ for which there exist real numbers $r_1<r_2<\cdots<r_n$ such that

1. $p(r_i)=0, \qquad i=1,2,\ldots,n$, and
2. $p'\left(\frac{r_i+r_{i+1}}{2}\right)=0 \qquad i=1,2,\ldots,n-1,$

where $p'(x)$ denotes the derivative of $p(x)$.

A–4 Does there exist an infinite sequence of closed discs $D_1,D_2,D_3,\ldots$ in the plane, with centers $c_1,c_2,c_3,\ldots$, respectively, such that

1. the c_i have no limit point in the finite plane,
2. the sum of the areas of the D_i is finite, and
3. every line in the plane intersects at least one of the D_i?

A–5 Find the maximum value of

$$\int_0^y \sqrt{x^4+(y-y^2)^2}\,dx$$

for $0\leq y\leq 1$.

A–6 Let $A(n)$ denote the number of sums of positive integers

$$a_1+a_2+\cdots+a_r$$

which add up to n with

$$a_1>a_2+a_3, a_2>a_3+a_4,\ldots,$$
$$a_{r-2}>a_{r-1}+a_r, a_{r-1}>a_r.$$

Let $B(n)$ denote the number of $b_1+b_2+\cdots+b_s$ which add up to n, with

1. $b_1\geq b_2\geq\cdots\geq b_s$,
2. each b_i is in the sequence $1,2,4,\ldots,g_j,\ldots$ defined by $g_1=1$, $g_2=2$, and $g_j=g_{j-1}+g_{j-2}+1$, and
3. if $b_1=g_k$ then every element in $\{1,2,4,\ldots,g_k\}$ appears at least once as a b_i.

Prove that $A(n)=B(n)$ for each $n\geq 1$.

(For example, $A(7)=5$ because the relevant sums are $7, 6+1, 5+2, 4+3, 4+2+1$, and $B(7)=5$ because the relevant sums are $4+2+1, 2+2+2+1, 2+2+1+1+1, 2+1+1+1+1+1, 1+1+1+1+1+1+1$.)

B–1 For each integer $n\geq 0$, let $S(n)=n-m^2$, where m is the greatest integer with $m^2\leq n$. Define a sequence $(a_k)_{k=0}^{\infty}$ by $a_0=A$ and $a_{k+1}=a_k+S(a_k)$ for $k\geq 0$. For what positive integers A is this sequence eventually constant?

B–2 Suppose f and g are non-constant, differentiable, real-valued functions defined on $(-\infty,\infty)$. Furthermore, suppose that for each pair of real numbers x and y,

$$f(x+y)=f(x)f(y)-g(x)g(y),$$
$$g(x+y)=f(x)g(y)+g(x)f(y).$$

If $f'(0)=0$, prove that $(f(x))^2+(g(x))^2=1$ for all x.

B–3 Does there exist a real number L such that, if m and n are integers greater than L, then an $m\times n$ rectangle may be expressed as a union of 4×6 and 5×7 rectangles, any two of which intersect at most along their boundaries?

B–4 Suppose p is an odd prime. Prove that

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}\equiv 2^p+1 \pmod{p^2}.$$

B–5 Let p be an odd prime and let $\mathbb{Z}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{x^2 : x\in\mathbb{Z}_p\}\cap\{y^2+1 : y\in\mathbb{Z}_p\}?$$

B–6 Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x}\leq a\frac{\sinh ux}{\sinh u}+b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0<|u|\leq c$ and for all x, $0<x<1$. (Note: $\sinh u=(e^u-e^{-u})/2$.)

The 53rd William Lowell Putnam Mathematical Competition
Saturday, December 5, 1992

A–1 Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers n;

(ii) $f(f(n+2)+2) = n$ for all integers n;

(iii) $f(0) = 1$.

A–2 Define $C(\alpha)$ to be the coefficient of x^{1992} in the power series about $x = 0$ of $(1+x)^\alpha$. Evaluate

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

A–3 For a given positive integer m, find all triples (n,x,y) of positive integers, with n relatively prime to m, which satisfy

$$(x^2+y^2)^m = (xy)^n.$$

A–4 Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1,2,3,\ldots,$$

compute the values of the derivatives $f^{(k)}(0), k = 1,2,3,\ldots$.

A–5 For each positive integer n, let $a_n = 0$ (or 1) if the number of 1's in the binary representation of n is even (or odd), respectively. Show that there do not exist positive integers k and m such that

$$a_{k+j} = a_{k+m+j} = a_{k+2m+j},$$

for $0 \le j \le m-1$.

A–6 Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

B–1 Let S be a set of n distinct real numbers. Let A_S be the set of numbers that occur as averages of two distinct elements of S. For a given $n \ge 2$, what is the smallest possible number of elements in A_S?

B–2 For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \binom{n}{k-2j},$$

where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \ge 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \le b \le a$, with $\binom{a}{b} = 0$ otherwise.)

B–3 For any pair (x,y) of real numbers, a sequence $(a_n(x,y))_{n\ge 0}$ is defined as follows:

$$a_0(x,y) = x,$$
$$a_{n+1}(x,y) = \frac{(a_n(x,y))^2 + y^2}{2}, \qquad \text{for } n \ge 0.$$

Find the area of the region

$$\{(x,y) | (a_n(x,y))_{n\ge 0} \text{ converges}\}.$$

B–4 Let $p(x)$ be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with $x^3 - x$. Let

$$\frac{d^{1992}}{dx^{1992}} \left(\frac{p(x)}{x^3-x} \right) = \frac{f(x)}{g(x)}$$

for polynomials $f(x)$ and $g(x)$. Find the smallest possible degree of $f(x)$.

B–5 Let D_n denote the value of the $(n-1) \times (n-1)$ determinant

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}.$$

Is the set $\left\{ \frac{D_n}{n!} \right\}_{n\ge 2}$ bounded?

B–6 Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(i) $I \in \mathcal{M}$, where I is the $n \times n$ identity matrix;

(ii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(iii) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(iv) if $A \in \mathcal{M}$ and $A \ne I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ contains at most n^2 matrices.

The 54th William Lowell Putnam Mathematical Competition
Saturday, December 4, 1993

A–1 The horizontal line $y = c$ intersects the curve $y = 2x - 3x^3$ in the first quadrant as in the figure. Find c so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line $y = c$ and the curve; the other lies under the curve and above the line $y = c$ between their two points of intersection.]

A–2 Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that $x_n^2 - x_{n-1}x_{n+1} = 1$ for $n = 1, 2, 3, \ldots$. Prove there exists a real number a such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

A–3 Let $\mathscr{P}_n$ be the set of subsets of $\{1, 2, \ldots, n\}$. Let $c(n,m)$ be the number of functions $f : \mathscr{P}_n \to \{1, 2, \ldots, m\}$ such that $f(A \cap B) = \min\{f(A), f(B)\}$. Prove that

$$c(n,m) = \sum_{j=1}^{m} j^n.$$

A–4 Let $x_1, x_2, \ldots, x_{19}$ be positive integers each of which is less than or equal to 93. Let $y_1, y_2, \ldots, y_{93}$ be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

A–5 Show that

$$\int_{-100}^{-10} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{1}{101}}^{\frac{1}{11}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx + \int_{\frac{101}{100}}^{\frac{11}{10}} \left(\frac{x^2 - x}{x^3 - 3x + 1}\right)^2 dx$$

is a rational number.

A–6 The infinite sequence of 2's and 3's

$$2,3,3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,$$
$$3,2,3,3,3,2,3,3,3,2,3,3,2,3,3,3,2,\ldots$$

has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the given sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m. (Note: $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

B–1 Find the smallest positive integer n such that for every integer m with $0 < m < 1993$, there exists an integer k for which

$$\frac{m}{1993} < \frac{k}{n} < \frac{m+1}{1994}.$$

B–2 Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and n cards are dealt to each of two players. Beginning with A, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n+1$. The last person to discard wins the game. Assuming optimal strategy by both A and B, what is the probability that A wins?

B–3 Two real numbers x and y are chosen at random in the interval (0,1) with respect to the uniform distribution. What is the probability that the closest integer to x/y is even? Express the answer in the form $r + s\pi$, where r and s are rational numbers.

B–4 The function $K(x,y)$ is positive and continuous for $0 \leq x \leq 1, 0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all x, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x,y)\,dy = g(x)$$

and

$$\int_0^1 g(y)K(x,y)\,dy = f(x).$$

Show that $f(x) = g(x)$ for $9 \leq x \leq 1$.

B–5 Show there do not exist four points in the Euclidean plane such that the pairwise distances between the points are all odd integers.

B–6 Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$ and replace them with $2x$ and $y - x$.

The 55th William Lowell Putnam Mathematical Competition
Saturday, December 3, 1994

A–1 Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies $0 < a_n \leq a_{2n} + a_{2n+1}$ for all $n \geq 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

A–2 Let A be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the x-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$. Find the positive number m such that A is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the y-axis, and the ellipse $\frac{1}{9}x^2 + y^2 = 1$.

A–3 Show that if the points of an isosceles right triangle of side length 1 are each colored with one of four colors, then there must be two points of the same color whch are at least a distance $2 - \sqrt{2}$ apart.

A–4 Let A and B be 2×2 matrices with integer entries such that $A, A+B, A+2B, A+3B$, and $A+4B$ are all invertible matrices whose inverses have integer entries. Show that $A+5B$ is invertible and that its inverse has integer entries.

A–5 Let $(r_n)_{n \geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. Let S be the set of numbers representable as a sum

$$r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}},$$

with $i_1 < i_2 < \cdots < i_{1994}$. Show that every nonempty interval (a,b) contains a nonempty subinterval (c,d) that does not intersect S.

A–6 Let $f_1, \ldots, f_{10}$ be bijections of the set of integers such that for each integer n, there is some composition $f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$ of these functions (allowing repetitions) which maps 0 to n. Consider the set of 1024 functions

$$\mathscr{F} = \{f_1^{e_1} \circ f_2^{e_2} \circ \cdots \circ f_{10}^{e_{10}}\},$$

$e_i = 0$ or 1 for $1 \leq i \leq 10$. (f_i^0 is the identity function and $f_i^1 = f_i$.) Show that if A is any nonempty finite set of integers, then at most 512 of the functions in $\mathscr{F}$ map A to itself.

B–1 Find all positive integers n that are within 250 of exactly 15 perfect squares.

B–2 For which real numbers c is there a straight line that intersects the curve

$$x^4 + 9x^3 + cx^2 + 9x + 4$$

in four distinct points?

B–3 Find the set of all real numbers k with the following property: For any positive, differentiable function f that satisfies $f'(x) > f(x)$ for all x, there is some number N such that $f(x) > e^{kx}$ for all $x > N$.

B–4 For $n \geq 1$, let d_n be the greatest common divisor of the entries of $A^n - I$, where

$$A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Show that $\lim_{n\to\infty} d_n = \infty$.

B–5 For any real number α, define the function $f_\alpha(x) = \lfloor \alpha x \rfloor$. Let n be a positive integer. Show that there exists an α such that for $1 \leq k \leq n$,

$$f_\alpha^k(n^2) = n^2 - k = f_{\alpha^k}(n^2).$$

B–6 For any integer n, set

$$n_a = 101a - 100 \cdot 2^a.$$

Show that for $0 \leq a, b, c, d \leq 99$, $n_a + n_b \equiv n_c + n_d \pmod{10100}$ implies $\{a,b\} = \{c,d\}$.