Canadian Mathematical Olympiad
1969

PROBLEM 1

Show that if $a_1/b_1 = a_2/b_2 = a_3/b_3$ and p_1, p_2, p_3 are not all zero, then

$$\left(\frac{a_1}{b_1}\right)^n = \frac{p_1 a_1^n + p_2 a_2^n + p_3 a_3^n}{p_1 b_1^n + p_2 b_2^n + p_3 b_3^n}$$

for every positive integer n.

PROBLEM 2

Determine which of the two numbers $\sqrt{c+1} - \sqrt{c}$, $\sqrt{c} - \sqrt{c-1}$ is greater for any $c \geq 1$.

PROBLEM 3

Let c be the length of the hypotenuse of a right angle triangle whose other two sides have lengths a and b. Prove that $a + b \leq \sqrt{2}c$. When does the equality hold?

PROBLEM 4

Let ABC be an equilateral triangle, and P be an arbitrary point within the triangle. Perpendiculars PD, PE, PF are drawn to the three sides of the triangle. Show that, no matter where P is chosen,

$$\frac{PD + PE + PF}{AB + BC + CA} = \frac{1}{2\sqrt{3}}.$$

PROBLEM 5

Let ABC be a triangle with sides of lengths a, b and c. Let the bisector of the angle C cut AB in D. Prove that the length of CD is

$$\frac{2ab\cos\frac{C}{2}}{a+b}.$$

PROBLEM 6

Find the sum of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (n-1)(n-1)! + n \cdot n!$, where $n! = n(n-1)(n-2)\cdots 2 \cdot 1$.

PROBLEM 7

Show that there are no integers a, b, c for which $a^2 + b^2 - 8c = 6$.

Problem 8

Let f be a function with the following properties:

1) $f(n)$ is defined for every positive integer n;
2) $f(n)$ is an integer;
3) $f(2) = 2$;
4) $f(mn) = f(m)f(n)$ for all m and n;
5) $f(m) > f(n)$ whenever $m > n$.

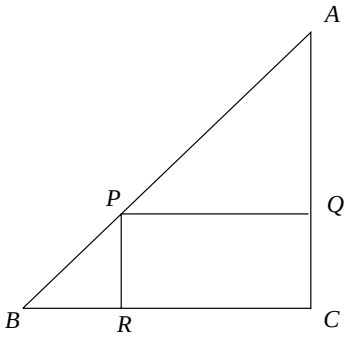
Prove that $f(n) = n$.

Problem 9

Show that for any quadrilateral inscribed in a circle of radius 1, the length of the shortest side is less than or equal to $\sqrt{2}$.

Problem 10

Let ABC be the right-angled isosceles triangle whose equal sides have length 1. P is a point on the hypotenuse, and the feet of the perpendiculars from P to the other sides are Q and R. Consider the areas of the triangles APQ and PBR, and the area of the rectangle $QCRP$. Prove that regardless of how P is chosen, the largest of these three areas is at least $2/9$.



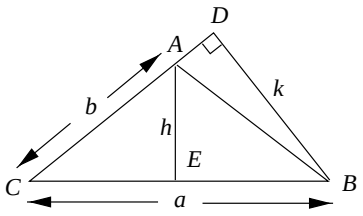
Canadian Mathematical Olympiad
1970

PROBLEM 1

Find all number triples (x, y, z) such that when any one of these numbers is added to the product of the other two, the result is 2.

PROBLEM 2

Given a triangle ABC with angle A obtuse and with altitudes of length h and k as shown in the diagram, prove that $a + h \geq b + k$. Find under what conditions $a + h = b + k$.



PROBLEM 3

A set of balls is given. Each ball is coloured red or blue, and there is at least one of each colour. Each ball weighs either 1 pound or 2 pounds, and there is at least one of each weight. Prove that there are 2 balls having different weights and different colours.

PROBLEM 4

a) Find all positive integers with initial digit 6 such that the integer formed by deleting this 6 is 1/25 of the original integer.
b) Show that there is no integer such that deletion of the first digit produces a result which is 1/35 of the original integer.

PROBLEM 5

A quadrilateral has one vertex on each side of a square of side-length 1. Show that the lengths a, b, c and d of the sides of the quadrilateral satisfy the inequalities

$$2 \leq a^2 + b^2 + c^2 + d^2 \leq 4.$$

PROBLEM 6

Given three non-collinear points A, B, C, construct a circle with centre C such that the tangents from A and B to the circle are parallel.

PROBLEM 7

Show that from any five integers, not necessarily distinct, one can always choose three of these integers whose sum is divisible by 3.

PROBLEM 8

Consider all line segments of length 4 with one end-point on the line $y = x$ and the other end-point on the line $y = 2x$. Find the equation of the locus of the midpoints of these line segments.

PROBLEM 9

Let $f(n)$ be the sum of the first n terms of the sequence

$$0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, \ldots .$$

a) Give a formula for $f(n)$.
b) Prove that $f(s+t) - f(s-t) = st$ where s and t are positive integers and $s > t$.

PROBLEM 10

Given the polynomial

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

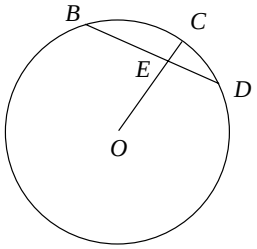
with integral coefficients $a_1, a_2, \ldots, a_n$, and given also that there exist four distinct integers a, b, c and d such that

$$f(a) = f(b) = f(c) = f(d) = 5,$$

show that there is no integer k such that $f(k) = 8$.

Canadian Mathematical Olympiad
1971

PROBLEM 1



DEB is a chord of a circle such that $DE = 3$ and $EB = 5$. Let O be the centre of the circle. Join OE and extend OE to cut the circle at C. (See diagram). Given $EC = 1$, find the radius of the circle.

PROBLEM 2

Let x and y be positive real numbers such that $x + y = 1$. Show that

$$\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right) \geq 9.$$

PROBLEM 3

$ABCD$ is a quadrilateral with $AD = BC$. If $\angle ADC$ is greater than $\angle BCD$, prove that $AC > BD$.

PROBLEM 4

Determine all real numbers a such that the two polynomials x^2+ax+1 and x^2+x+a have at least one root in common.

PROBLEM 5

Let

$$p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

where the coefficients a_i are integers. If $p(0)$ and $p(1)$ are both odd, show that $p(x)$ has no integral roots.

PROBLEM 6

Show that, for all integers n, $n^2 + 2n + 12$ is not a multiple of 121.

PROBLEM 7

Let n be a five digit number (whose first digit is non-zero) and let m be the four digit number formed from n by deleting its middle digit. Determine all n such that n/m is an integer.

PROBLEM 8

A regular pentagon is inscribed in a circle of radius r. P is any point inside the pentagon. Perpendiculars are dropped from P to the sides, or the sides produced, of the pentagon.

a) Prove that the sum of the lengths of these perpendiculars is constant.
b) Express this constant in terms of the radius r.

PROBLEM 9

Two flag poles of heights h and k are situated $2a$ units apart on a level surface. Find the set of all points on the surface which are so situated that the angles of elevation of the tops of the poles are equal.

PROBLEM 10

Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything? Prove your answer is a minimum.

Canadian Mathematical Olympiad
1972

PROBLEM 1

Given three distinct unit circles, each of which is tangent to the other two, find the radii of the circles which are tangent to all three circles.

PROBLEM 2

Let $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Define M to be the sum of all products of pairs $a_i a_j$ $(i < j)$, *i.e.*,

$$M = a_1(a_2 + a_3 + \cdots + a_n) + a_2(a_3 + a_4 + \cdots + a_n) + \cdots + a_{n-1}a_n.$$

Prove that the square of at least one of the numbers $a_1, a_2, \ldots, a_n$ does not exceed $2M/n(n-1)$.

PROBLEM 3

a) Prove that 10201 is composite in any base greater than 2.
b) Prove that 10101 is composite in any base.

PROBLEM 4

Describe a construction of a quadrilateral $ABCD$ given:
(i) the lengths of all four sides;
(ii) that AB and CD are parallel;
(iii) that BC and DA do not intersect.

PROBLEM 5

Prove that the equation $x^3 + 11^3 = y^3$ has no solution in positive integers x and y.

PROBLEM 6

Let a and b be distinct real numbers. Prove that there exist integers m and n such that $am + bn < 0$, $bm + an > 0$.

PROBLEM 7

a) Prove that the values of x for which $x = (x^2 + 1)/198$ lie between $1/198$ and $197.99494949\cdots$.
b) Use the result of a) to prove that $\sqrt{2} < 1.41\overline{421356}$.
c) Is it true that $\sqrt{2} < 1.41421356$?

PROBLEM 8

During a certain election campaign, p different kinds of promises are made by the various political parties $(p > 0)$. While several parties may make the same promise, any two parties have at least one promise in common; no two parties have exactly the same set of promises. Prove that there are no more than 2^{p-1} parties.

Problem 9

Four distinct lines L_1, L_2, L_3, L_4 are given in the plane: L_1 and L_2 are respectively parallel to L_3 and L_4. Find the locus of a point moving so that the sum of its perpendicular distances from the four lines is constant.

Problem 10

What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive?

Canadian Mathematical Olympiad
1973

PROBLEM 1

(i) Solve the simultaneous inequalities, $x < \frac{1}{4x}$ and $x < 0$; *i.e.*, find a single inequality equivalent to the two given simultaneous inequalities.

(ii) What is the greatest integer which satisfies both inequalities $4x + 13 < 0$ and $x^2 + 3x > 16$?

(iii) Give a rational number between 11/24 and 6/13.

(iv) Express 100000 as a product of two integers neither of which is an integral multiple of 10.

(v) Without the use of logarithm tables evaluate

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36}.$$

PROBLEM 2

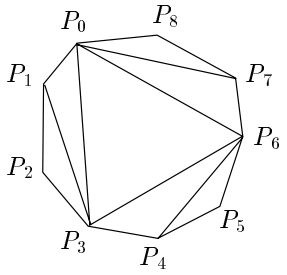
Find all the real numbers which satisfy the equation $|x+3| - |x-1| = x+1$. (Note: $|a| = a$ if $a \geq 0$; $|a| = -a$ if $a < 0$.)

PROBLEM 3

Prove that if p and $p+2$ are both prime integers greater than 3, then 6 is a factor of $p+1$.

PROBLEM 4

The figure shows a (convex) polygon with nine vertices. The six diagonals which have been drawn dissect the polygon into the seven triangles: $P_0P_1P_3$, $P_0P_3P_6$, $P_0P_6P_7$, $P_0P_7P_8$, $P_1P_2P_3$, $P_3P_4P_6$, $P_4P_5P_6$. In how many ways can these triangles be labelled with the names $\triangle_1$, $\triangle_2$, $\triangle_3$, $\triangle_4$, $\triangle_5$, $\triangle_6$, $\triangle_7$ so that P_i is a vertex of triangle $\triangle_i$ for $i = 1, 2, 3, 4, 5, 6, 7$? Justify your answer.



PROBLEM 5

For every positive integer n, let

$$h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

For example, $h(1) = 1$, $h(2) = 1 + \frac{1}{2}$, $h(3) = 1 + \frac{1}{2} + \frac{1}{3}$. Prove that for $n = 2, 3, 4, \ldots$

$$n + h(1) + h(2) + h(3) + \cdots + h(n-1) = nh(n).$$

PROBLEM 6

If A and B are fixed points on a given circle not collinear with centre O of the circle, and if XY is a variable diameter, find the locus of P (the intersection of the line through A and X and the line through B and Y).

PROBLEM 7

Observe that

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{2}; \quad \frac{1}{2} = \frac{1}{3} + \frac{1}{6}; \quad \frac{1}{3} = \frac{1}{4} + \frac{1}{12}; \quad \frac{1}{4} = \frac{1}{5} + \frac{1}{20}.$$

State a general law suggested by these examples, and prove it.
Prove that for any integer n greater than 1 there exist positive integers i and j such that

$$\frac{1}{n} = \frac{1}{i(i+1)} + \frac{1}{(i+1)(i+2)} + \frac{1}{(i+2)(i+3)} + \cdots + \frac{1}{j(j+1)}.$$

Canadian Mathematical Olympiad
1974

PART A

PROBLEM 1

i) If $x = (1 + \frac{1}{n})^n$ and $y = (1 + \frac{1}{n})^{n+1}$, show that $y^x = x^y$.
ii) Show that, for all positive integers n,

$$1^2 - 2^2 + 3^2 - 4^2 + \cdots + (-1)^n(n-1)^2 + (-1)^{n+1}n^2 = (-1)^{n+1}(1 + 2 + \cdots + n).$$

PROBLEM 2

Let $ABCD$ be a rectangle with $BC = 3AB$. Show that if P, Q are the points on side BC with $BP = PQ = QC$, then

$$\angle DBC + \angle DPC = \angle DQC.$$

PART B

PROBLEM 3

Let

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

be a polynomial with coefficients satisfying the conditions:

$$0 \le a_i \le a_0, \quad i = 1, 2, \ldots, n.$$

Let $b_0, b_1, \ldots, b_{2n}$ be the coefficients of the polynomial

$$\begin{aligned}\big(f(x)\big)^2 &= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)^2 \\ &= b_0 + b_1x + b_2x^2 + \cdots + b_{n+1}x^{n+1} + \cdots + b_{2n}x^{2n}.\end{aligned}$$

Prove that

$$b_{n+1} \le \frac{1}{2}\big(f(1)\big)^2.$$

PROBLEM 4

Let n be a fixed positive integer. To any choice of n real numbers satisfying

$$0 \le x_i \le 1, \quad i = 1, 2, \ldots, n,$$

there corresponds the sum

$(*)$

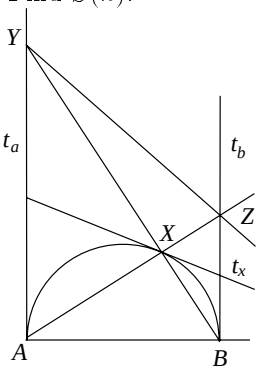
$$\sum_{1\le i<j\le n} |x_i - x_j|$$

$$\begin{aligned} &= |x_1 - x_2| + |x_1 - x_3| + |x_1 - x_4| + \cdots + |x_1 - x_{n-1}| + |x_1 - x_n| \\ &\quad + |x_2 - x_3| + |x_2 - x_4| + \cdots + |x_2 - x_{n-1}| + |x_2 - x_n| \\ &\quad + |x_3 - x_4| + \cdots + |x_3 - x_{n-1}| + |x_3 - x_n| \\ &\quad + \cdots + |x_{n-2} - x_{n-1}| + |x_{n-2} - x_n| \\ &\quad + |x_{n-1} - x_n|. \end{aligned}$$

Let $S(n)$ denote the largest possible value of the sum $(*)$. Find $S(n)$.

PROBLEM 5

Given a circle with diameter AB and a point X on the circle different from A and B, let t_a, t_b and t_x be the tangents to the circle at A, B and X respectively. Let Z be the point where line AX meets t_b and Y the point where line BX meets t_a. Show that the three lines YZ, t_x and AB are either concurrent (*i.e.*, all pass through the same point) or parallel.

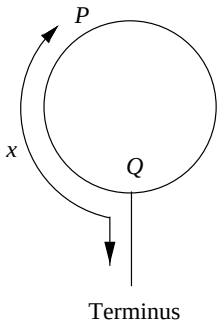


PROBLEM 6

An unlimited supply of 8-cent and 15-cent stamps is available. Some amounts of postage cannot be made up exactly, *e.g.*, 7 cents, 29 cents. What is the largest unattainable amount, *i.e.*, the amount, say n, of postage which is unattainable while all amounts larger than n are attainable? (Justify your answer.)

PROBLEM 7

A bus route consists of a circular road of circumference 10 miles and a straight road of length 1 mile which runs from a terminus to the point Q on the circular road (see diagram). It is served by two buses, each of which requires 20 minutes for the round trip. Bus No. 1, upon leaving the terminus, travels along the straight road, once around the circle clockwise and returns along the straight road to the terminus. Bus No. 2, reaching the terminus 10 minutes after Bus No. 1, has a similar route except that it proceeds counterclockwise around the circle. Both buses run continuously and do not wait at any point on the route except for a negligible amount of time to pick up and discharge passengers.



A man plans to wait at a point P which is x miles ($0 \le x < 12$) from the terminus along the route of Bus No. 1 and travel to the terminus on one of the buses.

Assuming that he chooses to board that bus which will bring him to his destination at the earliest moment, there is a maximum time $w(x)$ that his journey (waiting plus travel time) could take.
Find $w(2)$; find $w(4)$.
For what value of x will the time $w(x)$ be the longest?
Sketch a graph of $y = w(x)$ for $0 \leq x < 12$.

Canadian Mathematical Olympiad
1975

PROBLEM 1

Simplify

$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+\cdots+n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+\cdots+n\cdot 3n\cdot 9n}\right)^{1/3}.$$

PROBLEM 2

A sequence of numbers $a_1, a_2, a_3, \ldots$ satisfies

(i) $a_1 = \frac{1}{2}$,

(ii) $a_1 + a_2 + \cdots + a_n = n^2 a_n \quad (n \geq 1)$.

Determine the value of $a_n \quad (n \geq 1)$.

PROBLEM 3

For each real number r, $[r]$ denotes the largest integer less than or equal to r, *e.g.*, $[6] = 6$, $[\pi] = 3$, $[-1.5] = -2$. Indicate on the (x, y)-plane the set of all points (x, y) for which $[x]^2 + [y]^2 = 4$.

PROBLEM 4

For a positive number such as 3.27, 3 is referred to as the integral part of the number and .27 as the decimal part. Find a positive number such that its decimal part, its integral part, and the number itself form a geometric progression.

PROBLEM 5

A, B, C, D are four "consecutive" points on the circumference of a circle and P, Q, R, S are points on the circumference which are respectively the midpoints of the arcs AB, BC, CD, DA. Prove that PR is perpendicular to QS.

PROBLEM 6

(i) 15 chairs are equally placed around a circular table on which are name cards for 15 guests. The guests fail to notice these cards until after they have sat down, and it turns out that no one is sitting in the correct seat. Prove that the table can be rotated so that at least two of the guests are simultaneously correctly seated.

(ii) Give an example of an arrangement in which just one of the 15 guests is correctly seated and for which no rotation correctly places more than one person.

PROBLEM 7

A function $f(x)$ is *periodic* if there is a positive number p such that $f(x+p) = f(x)$ for all x. For example, $\sin x$ is periodic with period 2π. Is the function $\sin(x^2)$ periodic? Prove your assertion.

Problem 8

Let k be a positive integer. Find all polynomials

$$P(x) = a_0 + a_1 x + \cdots + a_n x^n,$$

where the a_i are real, which satisfy the equation

$$P\big(P(x)\big) = \{P(x)\}^k.$$

Canadian Mathematical Olympiad
1976

PROBLEM 1

Given four weights in geometric progression and an equal arm balance, show how to find the heaviest weight using the balance only twice.

PROBLEM 2

Suppose

$$n(n+1)a_{n+1} = n(n-1)a_n - (n-2)a_{n-1}$$

for every positive integer $n \geq 1$.
Given that $a_0 = 1$, $a_1 = 2$, find

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{50}}{a_{51}}.$$

PROBLEM 3

Two grade seven students were allowed to enter a chess tournament otherwise composed of grade eight students. Each contestant played once with each other contestant and received one point for a win, one half point for a tie and zero for a loss. The two grade seven students together gained a total of eight points and each grade eight student scored the same number of points as his classmates. How many students from grade eight participated in the chess tournament? Is the solution unique?

PROBLEM 4

Let AB be a diameter of a circle, C be any fixed point between A and B on this diameter, and Q be a variable point on the circumference of the circle. Let P be the point on the line determined by Q and C for which $\frac{AC}{CB} = \frac{QC}{CP}$. Describe, with proof, the locus of the point P.

PROBLEM 5

Prove that a positive integer is a sum of at least two consecutive positive integers if and only if it is not a power of two.

PROBLEM 6

If A, B, C, D are four points in space, such that

$$\angle ABC = \angle BCD = \angle CDA = \angle DAB = \pi/2,$$

prove that A, B, C, D lie in a plane.

Problem 7

Let $P(x, y)$ be a polynomial in two variables x, y such that $P(x, y) = P(y, x)$ for every x, y (for example, the polynomial $x^2 - 2xy + y^2$ satisfies this condition). Given that $(x - y)$ is a factor of $P(x, y)$, show that $(x - y)^2$ is a factor of $P(x, y)$.

Problem 8

Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are all red.

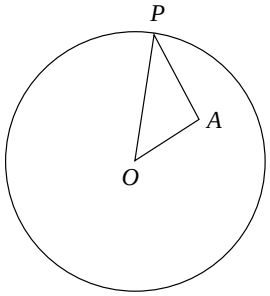
Canadian Mathematical Olympiad
1977

PROBLEM 1

If $f(x) = x^2 + x$, prove that the equation $4f(a) = f(b)$ has no solutions in positive integers a and b.

PROBLEM 2

Let O be the centre of a circle and A a fixed interior point of the circle different from O. Determine all points P on the circumference of the circle such that the angle OPA is a maximum.



PROBLEM 3

N is an integer whose representation in base b is 777. Find the smallest positive integer b for which N is the fourth power of an integer.

PROBLEM 4

Let

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

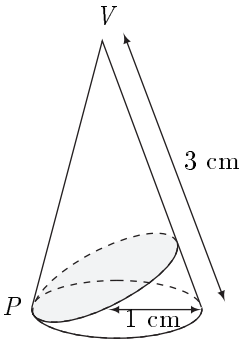
$$q(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

be two polynomials with integer coefficients. Suppose that all the coefficients of the product $p(x) \cdot q(x)$ are even but not all of them are divisible by 4. Show that one of $p(x)$ and $q(x)$ has all even coefficients and the other has at least one odd coefficient.

PROBLEM 5

A right circular cone of base radius 1 cm and slant height 3 cm is given. P is a point on the circumference of the base and the shortest path from P around the cone and back to P is drawn (see diagram). What is the minimum distance from

the vertex V to this path?



PROBLEM 6

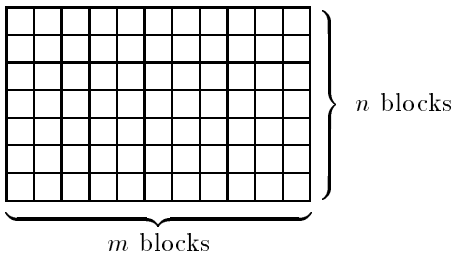
Let $0 < u < 1$ and define

$$u_1 = 1 + u, \quad u_2 = \frac{1}{u_1} + u, \quad \ldots, \quad u_{n+1} = \frac{1}{u_n} + u, \quad n \geq 1.$$

Show that $u_n > 1$ for all values of $n = 1, 2, 3, \ldots$.

PROBLEM 7

A rectangular city is exactly m blocks long and n blocks wide (see diagram). A woman lives in the southwest corner of the city and works in the northeast corner. She walks to work each day but, on any given trip, she makes sure that her path does not include any intersection twice. Show that the number $f(m, n)$ of different paths she can take to work satisfies $f(m, n) \leq 2^{mn}$.



Canadian Mathematical Olympiad
1978

PROBLEM 1

Let n be an integer. If the tens digit of n^2 is 7, what is the units digit of n^2?

PROBLEM 2

Find all pairs a, b of positive integers satisfying the equation $2a^2 = 3b^3$.

PROBLEM 3

Determine the largest real number z such that

$$x + y + z = 5$$
$$xy + yz + xz = 3$$

and x, y are also real.

PROBLEM 4

The sides AD and BC of a convex quadrilateral $ABCD$ are extended to meet at E. Let H and G be the midpoints of BD and AC, respectively. Find the ratio of the area of the triangle EHG to that of the quadrilateral $ABCD$.

PROBLEM 5

Eve and Odette play a game on a 3×3 checkerboard, with black checkers and white checkers. The rules are as follows:

I. They play alternately.
II. A turn consists of placing one checker on an unoccupied square of the board.
III. In her turn, a player may select either a white checker or a black checker and need not always use the same colour.
IV. When the board is full, Eve obtains one point for every row, column or diagonal that has an even number of black checkers, and Odette obtains one point for every row, column or diagonal that has an odd number of black checkers.
V. The player obtaining at least five of the eight points WINS.

(a) Is a 4–4 tie possible? Explain.
(b) Describe a winning strategy for the girl who is first to play.

PROBLEM 6

Sketch the graph of $x^3 + xy + y^3 = 3$.

Canadian Mathematical Olympiad
1979

PROBLEM 1

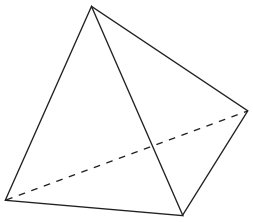
Given: (i) $a, b > 0$; (ii) a, A_1, A_2, b is an arithmetic progression; (iii) a, G_1, G_2, b is a geometric progression. Show that

$$A_1 A_2 \geq G_1 G_2$$

PROBLEM 2

It is known in Euclidean geometry that the sum of the angles of a triangle is constant. Prove, however, that the sum of the dihedral angles of a tetrahedron is *not* constant.

NOTE. (i) A tetrahedron is a triangular pyramid, and (ii) a dihedral angle is the interior angle between a pair of faces.



PROBLEM 3

Let a, b, c, d, e be integers such that $1 \leq a < b < c < d < e$. Prove that

$$\frac{1}{[a,b]} + \frac{1}{[b,c]} + \frac{1}{[c,d]} + \frac{1}{[d,e]} \leq \frac{15}{16},$$

where $[m, n]$ denotes the least common multiple of m and n (*e.g.* $[4, 6] = 12$).

PROBLEM 4

A dog standing at the centre of a circular arena sees a rabbit at the wall. The rabbit runs around the wall and the dog pursues it along a unique path which is determined by running at the same speed and staying on the radial line joining the centre of the arena to the rabbit. Show that the dog overtakes the rabbit just as it reaches a point one-quarter of the way around the arena.

PROBLEM 5

A walk consists of a sequence of steps of length 1 taken in directions north, south, east or west. A walk is *self-avoiding* if it never passes through the same point twice. Let $f(n)$ denote the number of n-step self-avoiding walks which begin at the origin. Compute $f(1)$, $f(2)$, $f(3)$, $f(4)$, and show that

$$2^n < f(n) \leq 4 \cdot 3^{n-1}.$$

Canadian Mathematical Olympiad
1980

PROBLEM 1

If $a679b$ is a five digit number (in base 10) which is divisible by 72, determine a and b.

PROBLEM 2

The numbers from 1 to 50 are printed on cards. The cards are shuffled and then laid out face up in 5 rows of 10 cards each. The cards in each row are rearranged to make them increase from left to right. The cards in each column are then rearranged to make them increase from top to bottom. In the final arrangement, do the cards in the rows still increase from left to right?

PROBLEM 3

Among all triangles having (i) a fixed angle A and (ii) an inscribed circle of fixed radius r, determine which triangle has the least perimeter.

PROBLEM 4

A gambling student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring *exactly* n points is $\frac{1}{3}[2 + (-\frac{1}{2})^n]$.

PROBLEM 5

A parallelepiped has the property that all cross sections which are parallel to any fixed face F, have the same perimeter as F. Determine whether or not any other polyhedron has this property.

Canadian Mathematical Olympiad
1981

PROBLEM 1

For any real number t, denote by $[t]$ the greatest integer which is less than or equal to t. For example: $[8] = 8$, $[\pi] = 3$ and $[-5/2] = -3$. Show that the equation

$$[x] + [2x] + [4x] + [8x] + [16x] + [32x] = 12345$$

has no real solution.

PROBLEM 2

Given a circle of radius r and a tangent line ℓ to the circle through a given point P on the circle. From a variable point R on the circle, a perpendicular RQ is drawn to ℓ with Q on ℓ. Determine the maximum of the area of triangle PQR.

PROBLEM 3

Given a finite collection of lines in a plane P, show that it is possible to draw an arbitrarily large circle in P which does not meet any of them. On the other hand, show that it is possible to arrange an infinite sequence of lines (first line, second line, third line, *etc.*) in P so that every circle in P meets at least one of the lines. (A point is not considered to be a circle.)

PROBLEM 4

$P(x)$ and $Q(x)$ are two polynomials that satisfy the identity $P\big(Q(x)\big) \equiv Q\big(P(x)\big)$ for real numbers x. If the equation $P(x) = Q(x)$ has no real solution, show that the equation $P\big(P(x)\big) = Q\big(Q(x)\big)$ also has no real solution.

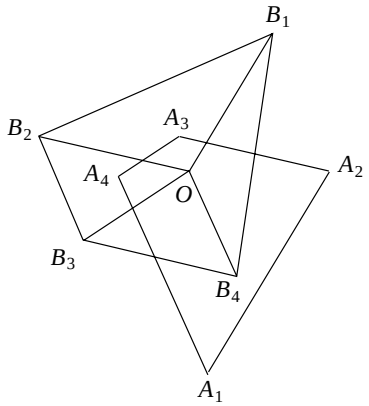
PROBLEM 5

Eleven theatrical groups participated in a festival. Each day, some of the groups were scheduled to perform while the remaining groups joined the general audience. At the conclusion of the festival, each group had seen, during its days off, at least one performance of every other group. At least how many days did the festival last?

Canadian Mathematical Olympiad
1982

PROBLEM 1

In the diagram, OB_i is parallel and equal in length to A_iA_{i+1} for $i = 1, 2, 3$ and $4(A_5 = A_1)$. Show that the area of $B_1B_2B_3B_4$ is twice that of $A_1A_2A_3A_4$.



PROBLEM 2

If a, b and c are the roots of the equation $x^3 - x^2 - x - 1 = 0$,

(i) show that a, b and c are distinct:

(ii) show that

$$\frac{a^{1982} - b^{1982}}{a - b} + \frac{b^{1982} - c^{1982}}{b - c} + \frac{c^{1982} - a^{1982}}{c - a}$$

is an integer.

PROBLEM 3

Let R^n be the n-dimensional Euclidean space. Determine the smallest number $g(n)$ of points of a set in R^n such that every point in R^n is at irrational distance from at least one point in that set.

PROBLEM 4

Let p be a permutation of the set $S_n = \{1, 2, \ldots, n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

PROBLEM 5

The altitudes of a tetrahedron $ABCD$ are extended externally to points A', B', C' and D' respectively, where $AA' = k/h_a$, $BB' = k/h_b$, $CC' = k/h_c$ and $DD' =$

k/h_d. Here, k is a constant and h_a denotes the length of the altitude of $ABCD$ from vertex A, *etc.* Prove that the centroid of the tetrahedron $A'B'C'D'$ coincides with the centroid of $ABCD$.

Canadian Mathematical Olympiad
1983

PROBLEM 1

Find all positive integers w, x, y and z which satisfy $w! = x! + y! + z!$.

PROBLEM 2

For each real number r let T_r be the transformation of the plane that takes the point (x, y) into the point $(2^r x, r2^r x + 2^r y)$. Let F be the family of all such transformations *i.e.* $F = \{T_r : r \text{ a real number}\}$. Find all curves $y = f(x)$ whose graphs remain unchanged by every transformation in F.

PROBLEM 3

The area of a triangle is determined by the lengths of its sides. Is the volume of a tetrahedron determined by the areas of its faces?

PROBLEM 4

Prove that for every prime number p, there are infinitely many positive integers n such that p divides $2^n - n$.

PROBLEM 5

The geometric mean (G.M.) of a k positive numbers $a_1, a_2, \ldots, a_k$ is defined to be the (positive) k-th root of their product. For example, the G.M. of 3, 4, 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all non-empty subsets of S.

Canadian Mathematical Olympiad
1984

PROBLEM 1

Prove that the sum of the squares of 1984 consecutive positive integers cannot be the square of an integer.

PROBLEM 2

Alice and Bob are in a hardware store. The store sells coloured sleeves that fit over keys to distinguish them. The following conversation takes place:

Alice: Are you going to cover your keys?

Bob: I would like to, but there are only 7 colours and I have 8 keys.

Alice: Yes, but you could always distinguish a key by noticing that the red key next to the green key was different from the red key next to the blue key.

Bob: You must be careful what you mean by "next to" or "three keys over from" since you can turn the key ring over and the keys are arranged in a circle.

Alice: Even so, you don't need 8 colours.

Problem: What is the smallest number of colours needed to distinguish n keys if all the keys are to be covered.

PROBLEM 3

An integer is *digitally divisible* if

(a) none of its digits is zero;

(b) it is divisible by the sum of its digits (*e.g.*, 322 is digitally divisible).

Show that there are infinitely many digitally divisible integers.

PROBLEM 4

An acute-angled triangle has unit area. Show that there is a point inside the triangle whose distance from each of the vertices is at least $\frac{2}{\sqrt[4]{27}}$.

PROBLEM 5

Given any 7 real numbers, prove that there are two of them, say x and y, such that

$$0 \leq \frac{x-y}{1+xy} \leq \frac{1}{\sqrt{3}}.$$

Canadian Mathematical Olympiad
1985

PROBLEM 1

The lengths of the sides of a triangle are 6, 8 and 10 units. Prove that there is exactly one straight line which simultaneously bisects the area and perimeter of the triangle.

PROBLEM 2

Prove or disprove that there exists an integer which is doubled when the initial digit is transferred to the end.

PROBLEM 3

Let P_1 and P_2 be regular polygons of 1985 sides and perimeters x and y respectively. Each side of P_1 is tangent to a given circle of circumference c and this circle passes through each vertex of P_2. Prove $x + y \geq 2c$. (You may assume that $\tan\theta \geq \theta$ for $0 \leq \theta < \frac{\pi}{2}$).

PROBLEM 4

Prove that 2^{n-1} divides $n!$ if and only if $n = 2^{k-1}$ for some positive integer k.

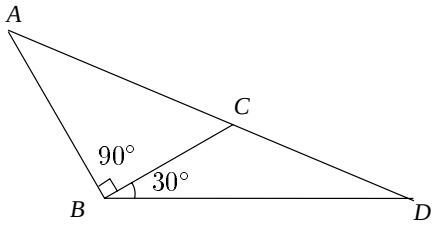
PROBLEM 5

Let $1 < x_1 < 2$ and, for $n = 1, 2, \ldots$, define $x_{n+1} = 1 + x_n - \frac{1}{2}x_n^2$. Prove that, for $n \geq 3$, $|x_n - \sqrt{2}| < 2^{-n}$.

Canadian Mathematical Olympiad
1986

PROBLEM 1

In the diagram line segments AB and CD are of length 1 while angles ABC and CBD are 90° and 30° respectively. Find AC.



PROBLEM 2

A Mathlon is a competition in which there are M athletic events. Such a competition was held in which only A, B, and C participated. In each event p_1 points were awarded for first place, p_2 for second and p_3 for third, where $p_1 > p_2 > p_3 > 0$ and p_1, p_2, p_3 are integers. The final score for A was 22, for B was 9 and for C was also 9. B won the 100 metres. What is the value of M and who was second in the high jump?

PROBLEM 3

A chord ST of constant length slides around a semicircle with diameter AB. M is the mid-point of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

PROBLEM 4

For positive integers n and k, define $F(n,k) = \sum_{r=1}^{n} r^{2k-1}$. Prove that $F(n,1)$ divides $F(n,k)$.

PROBLEM 5

Let $u_1, u_2, u_3, \ldots$ be a sequence of integers satisfying the recurrence relation $u_{n+2} = u_{n+1}^2 - u_n$. Suppose $u_1 = 39$ and $u_2 = 45$. Prove that 1986 divides infinitely many terms of the sequence.

Canadian Mathematical Olympiad
1987

PROBLEM 1

Find all solutions of $a^2 + b^2 = n!$ for positive integers a, b, n with $a \leq b$ and $n < 14$.

PROBLEM 2

The number 1987 can be written as a three digit number xyz in some base b. If $x + y + z = 1 + 9 + 8 + 7$, determine all possible values of x, y, z, b.

PROBLEM 3

Suppose $ABCD$ is a parallelogram and E is a point between B and C on the line BC. If the triangles DEC, BED and BAD are isosceles what are the possible values for the angle DAB?

PROBLEM 4

On a large, flat field n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd show that there is at least one person left dry. Is this always true when n is even?

PROBLEM 5

For every positive integer n show that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}] = [\sqrt{4n+2}] = [\sqrt{4n+3}]$$

where $[x]$ is the greatest integer less than or equal to x (for example $[2.3] = 2$, $[\pi] = 3$, $[5] = 5$).

Canadian Mathematical Olympiad
1988

PROBLEM 1

For what values of b do the equations: $1988x^2 + bx + 8891 = 0$ and $8891x^2 + bx + 1988 = 0$ have a common root?

PROBLEM 2

A house is in the shape of a triangle, perimeter P metres and area A square metres. The garden consists of all the land within 5 metres of the house. How much land do the garden and house together occupy?

PROBLEM 3

Suppose that S is a finite set of at least five points in the plane; some are coloured red, the others are coloured blue. No subset of three or more similarly coloured points is collinear. Show that there is a triangle

(i) whose vertices are all the same colour, and

(ii) at least one side of the triangle does not contain a point of the opposite colour.

PROBLEM 4

Let $x_{n+1} = 4x_n - x_{n-1}$, $x_0 = 0$, $x_1 = 1$, and $y_{n+1} = 4y_n - y_{n-1}$, $y_0 = 1$, $y_1 = 2$. Show for all $n \geq 0$ that $y_n^2 = 3x_n^2 + 1$.

PROBLEM 5

Let $S = \{a_1, a_2, \ldots, a_r\}$ denote a sequence of integers. For each non-empty subsequence A of S, we define $p(A)$ to be the product of all the integers in A. Let $m(S)$ be the arithmetic average of $p(A)$ over all non-empty subsets A of S. If $m(S) = 13$ and if $m(S \cup \{a_{r+1}\}) = 49$ for some positive integer a_{r+1}, determine the values of $a_1, a_2, \ldots, a_r$ and a_{r+1}.

Canadian Mathematical Olympiad
1989

PROBLEM 1

The integers $1, 2, \ldots, n$ are placed in order so that each value is either strictly bigger than all the preceding values or is strictly smaller than all preceding values. In how many ways can this be done?

PROBLEM 2

Let ABC be a right angled triangle of area 1. Let $A'B'C'$ be the points obtained by reflecting A, B, C respectively, in their opposite sides. Find the area of $\triangle A'B'C'$.

PROBLEM 3

Define $\{a_n\}_{n=1}$ as follows: $a_1 = 1989^{1989}$; a_n, $n > 1$, is the sum of the digits of a_{n-1}. What is the value of a_5?

PROBLEM 4

There are 5 monkeys and 5 ladders and at the top of each ladder there is a banana. A number of ropes connect the ladders, each rope connects two ladders. No two ropes are attached to the same rung of the same ladder. Each monkey starts at the bottom of a different ladder. The monkeys climb up the ladders but each time they encounter a rope they climb along it to the ladder at the other end of the rope and then continue climbing upwards. Show that, no matter how many ropes there are, each monkey gets a banana.

PROBLEM 5

Given the numbers $1, 2, 2^2, \ldots, 2^{n-1}$. For a specific permutation $\sigma = X_1, X_2, \ldots, X_n$ of these numbers we define $S_1(\sigma) = X_1$, $S_2(\sigma) = X_1 + X_2$, $S_3(\sigma) = X_1 + X_2 + X_3, \ldots$ and $Q(\sigma) = S_1(\sigma)S_2(\sigma)\cdots S_n(\sigma)$. Evaluate $\sum 1/Q(\sigma)$ where the sum is taken over all possible permutations.

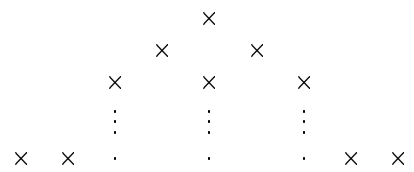
Canadian Mathematical Olympiad
1990

PROBLEM 1

A competition involving $n \geq 2$ players was held over k days. On each day, the players received scores of $1, 2, 3, \ldots, n$ points with no two players receiving the same score. At the end of the k days, it was found that each player had exactly 26 points in total. Determine all pairs (n, k) for which this is possible.

PROBLEM 2

A set of $\frac{1}{2}n(n+1)$ distinct numbers is arranged at random in a triangular array:



Let M_k be the largest number in the k-th row from the top. Find the probability that

$$M_1 < M_2 < M_3 < \cdots < M_n.$$

PROBLEM 3

Let $ABCD$ be a convex quadrilateral inscribed in a circle, and let diagonals AC and BD meet at X. The perpendiculars from X meet the sides AB, BC, CD, DA at A', B', C', D' respectively. Prove that

$$|A'B'| + |C'D'| = |A'D'| + |B'C'|.$$

($|A'B'|$ is the length of line segment $A'B'$, *etc.*)

PROBLEM 4

A particle can travel at speeds up to 2 metres per second along the x-axis, and up to 1 metre per second elsewhere in the plane. Provide a labelled sketch of the region which can be reached within one second by the particle starting at the origin.

PROBLEM 5

Suppose that a function f defined on the positive integers satisfies

$$f(1) = 1, \quad f(2) = 2,$$
$$f(n+2) = f\big(n + 2 - f(n+1)\big) + f\big(n + 1 - f(n)\big) \quad (n \geq 1).$$

(a) Show that
  (i) $0 \leq f(n+1) - f(n) \leq 1$
  (ii) if $f(n)$ is odd, then $f(n+1) = f(n) + 1$.

(b) Determine, with justification, all values of n for which

$$f(n) = 2^{10} + 1.$$

Canadian Mathematical Olympiad
1991

PROBLEM 1

Show that the equation $x^2 + y^5 = z^3$ has infinitely many solutions in integers x, y, z for which $xyz \neq 0$.

PROBLEM 2

Let n be a fixed positive integer. Find the sum of all positive integers with the following property: In base 2, it has exactly $2n$ digits consisting of n 1's and n 0's. (The first digit cannot be 0.)

PROBLEM 3

Let C be a circle and P a given point in the plane. Each line through P which intersects C determines a chord of C. Show that the midpoints of these chords lie on a circle.

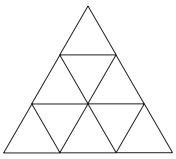
PROBLEM 4

Ten distinct numbers from the set $\{0, 1, 2, \ldots, 13, 14\}$ are to be chosen to fill in the ten circles in the diagram. The absolute values of the differences of the two numbers joined by each segment must be different from the values for all other segments. Is it possible to do this? Justify your answer.



PROBLEM 5

In the figure, the side length of the large equilateral triangle is 3 and $f(3)$, the number of parallelograms bounded by sides in the grid, is 15. For the general analogous situation, find a formula for $f(n)$, the number of parallelograms, for a triangle of side length n.



Canadian Mathematical Olympiad
1992

PROBLEM 1

Prove that the product of the first n natural numbers is divisible by the sum of the first n natural numbers if and only if $n+1$ is not an odd prime.

PROBLEM 2

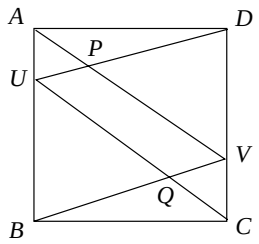
For $x, y, z \geq 0$, establish the inequality

$$x(x-z)^2 + y(y-z)^2 \geq (x-z)(y-z)(x+y-z)$$

and determine when equality holds.

PROBLEM 3

In the diagram, $ABCD$ is a square, with U and V interior points of the sides AB and CD respectively. Determine all the possible ways of selecting U and V so as to maximize the area of the quadrilateral $PUQV$.



PROBLEM 4

Solve the equation

$$x^2 + \frac{x^2}{(x+1)^2} = 3.$$

PROBLEM 5

A deck of $2n+1$ cards consists of a joker and, for each number between 1 and n inclusive, two cards marked with that number. The $2n+1$ cards are placed in a row, with the joker in the middle. For each k with $1 \leq k \leq n$, the two cards numbered k have exactly $k-1$ cards between them. Determine all the values of n not exceeding 10 for which this arrangement is possible. For which values of n is it impossible?

Canadian Mathematical Olympiad
1993

PROBLEM 1

Determine a triangle for which the three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

PROBLEM 2

Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3,\ldots$$

PROBLEM 3

In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

PROBLEM 4

A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number of boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

PROBLEM 5

Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even} \end{cases}$$

Show that the sequence $y_1, y_2, y_3, \ldots$ takes on every positive integer value exactly once.