THE 1989 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let

$$S = x_1 + x_2 + \cdots + x_n.$$

Prove that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \cdots + \frac{S^n}{n!} .$$

Question 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solutions in integers except $a = b = c = n = 0$.

Question 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2, and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

Question 4

Let S be a set consisting of m pairs (a, b) of positive integers with the property that $1 \leq a < b \leq n$. Show that there are at least

$$4m \cdot \frac{(m - \frac{n^2}{4})}{3n}$$

triples (a, b, c) such that (a, b), (a, c), and (b, c) belong to S.

Question 5

Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,
(2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

THE 1990 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Given triagnle ABC, let D, E, F be the midpoints of BC, AC, AB respectively and let G be the centroid of the triangle.
For each value of $\angle BAC$, how many non-similar triangles are there in which $AEGF$ is a cyclic quadrilateral?

Question 2

Let a_1, a_2, ..., a_n be positive real numbers, and let S_k be the sum of the products of a_1, a_2, ..., a_n taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n$$

for $k = 1, 2, \ldots, n-1$.

Question 3

Consider all the triangles ABC which have a fixed base AB and whose altitude from C is a constant h. For which of these triangles is the product of its altitudes a maximum?

Question 4

A set of 1990 persons is divided into non-intersecting subsets in such a way that

1. No one in a subset knows all the others in the subset,

2. Among any three persons in a subset, there are always at least two who do not know each other, and

3. For any two persons in a subset who do not know each other, there is exactly one person in the same subset knowing both of them.

(a) Prove that within each subset, every person has the same number of acquaintances.

(b) Determine the maximum possible number of subsets.
Note: It is understood that if a person A knows person B, then person B will know person A; an acquaintance is someone who is known. Every person is assumed to know one's self.

Question 5

Show that for every integer $n \geq 6$, there exists a convex hexagon which can be dissected into exactly n congruent triangles.

THE 1991 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let G be the centroid of triangle ABC and M be the midpoint of BC. Let X be on AB and Y on AC such that the points X, Y, and G are collinear and XY and BC are parallel. Suppose that XC and GB intersect at Q and YB and GC intersect at P. Show that triangle MPQ is similar to triangle ABC.

Question 2

Suppose there are 997 points given in a plane. If every two points are joined by a line segment with its midpoint coloured in red, show that there are at least 1991 red points in the plane. Can you find a special case with exactly 1991 red points?

Question 3

Let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$. Show that

$$\frac{a_1^2}{a_1 + b_1} + \frac{a_2^2}{a_2 + b_2} + \cdots + \frac{a_n^2}{a_n + b_n} \geq \frac{a_1 + a_2 + \cdots + a_n}{2} .$$

Question 4

During a break, n children at school sit in a circle around their teacher to play a game. The teacher walks clockwise close to the children and hands out candies to some of them according to the following rule. He selects one child and gives him a candy, then he skips the next child and gives a candy to the next one, then he skips 2 and gives a candy to the next one, then he skips 3, and so on. Determine the values of n for which eventually, perhaps after many rounds, all children will have at least one candy each.

Question 5

Given are two tangent circles and a point P on their common tangent perpendicular to the lines joining their centres. Construct with ruler and compass all the circles that are tangent to these two circles and pass through the point P.

THE 1992 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

A triangle with sides a, b, and c is given. Denote by s the semiperimeter, that is $s = (a+b+c)/2$. Construct a triangle with sides $s-a$, $s-b$, and $s-c$. This process is repeated until a triangle can no longer be constructed with the side lengths given.
For which original triangles can this process be repeated indefinitely?

Question 2

In a circle C with centre O and radius r, let C_1, C_2 be two circles with centres O_1, O_2 and radii r_1, r_2 respectively, so that each circle C_i is internally tangent to C at A_i and so that C_1, C_2 are externally tangent to each other at A.
Prove that the three lines OA, O_1A_2, and O_2A_1 are concurrent.

Question 3

Let n be an integer such that $n > 3$. Suppose that we choose three numbers from the set $\{1, 2, \ldots, n\}$. Using each of these three numbers only once and using addition, multiplication, and parenthesis, let us form all possible combinations.

(a) Show that if we choose all three numbers greater than $n/2$, then the values of these combinations are all distinct.

(b) Let p be a prime number such that $p \leq \sqrt{n}$. Show that the number of ways of choosing three numbers so that the smallest one is p and the values of the combinations are not all distinct is precisely the number of positive divisors of $p-1$.

Question 4

Determine all pairs (h, s) of positive integers with the following property:

If one draws h horizontal lines and another s lines which satisfy

(i) they are not horizontal,

(ii) no two of them are parallel,

(iii) no three of the $h+s$ lines are concurrent,

then the number of regions formed by these $h+s$ lines is 1992.

Question 5

Find a sequence of maximal length consisting of non-zero integers in which the sum of any

seven consecutive terms is positive and that of any eleven consecutive terms is negative.

THE 1993 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral such that all sides have equal length and angle ABC is 60 deg. Let l be a line passing through D and not intersecting the quadrilateral (except at D). Let E and F be the points of intersection of l with AB and BC respectively. Let M be the point of intersection of CE and AF.
Prove that $CA^2 = CM \times CE$.

Question 2

Find the total number of different integer values the function

$$f(x) = [x] + [2x] + [\frac{5x}{3}] + [3x] + [4x]$$

takes for real numbers x with $0 \leq x \leq 100$.

Question 3

Let

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \quad \text{and} \\ g(x) &= c_{n+1}x^{n+1} + c_n x^n + \cdots + c_0 \end{aligned}$$

be non-zero polynomials with real coefficients such that $g(x) = (x + r)f(x)$ for some real number r. If $a = \max(|a_n|, \ldots, |a_0|)$ and $c = \max(|c_{n+1}|, \ldots, |c_0|)$, prove that $\frac{a}{c} \leq n + 1$.

Question 4

Determine all positive integers n for which the equation

$$x^n + (2 + x)^n + (2 - x)^n = 0$$

has an integer as a solution.

Question 5

Let P_1, P_2, ..., $P_{1993} = P_0$ be distinct points in the xy-plane with the following properties:

(i) both coordinates of P_i are integers, for $i = 1, 2, \ldots, 1993$;

(ii) there is no point other than P_i and P_{i+1} on the line segment joining P_i with P_{i+1} whose coordinates are both integers, for $i = 0, 1, \ldots, 1992$.

Prove that for some i, $0 \leq i \leq 1992$, there exists a point Q with coordinates (q_x, q_y) on the line segment joining P_i with P_{i+1} such that both $2q_x$ and $2q_y$ are odd integers.

THE 1994 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

(i) For all $x, y \in \mathbb{R}$,

$$f(x) + f(y) + 1 \geq f(x + y) \geq f(x) + f(y),$$

(ii) For all $x \in [0, 1)$, $f(0) \geq f(x)$,

(iii) $-f(-1) = f(1) = 1$.

Find all such functions f.

Question 2

Given a nondegenerate triangle ABC, with circumcentre O, orthocentre H, and circumradius R, prove that $|OH| < 3R$.

Question 3

Let n be an integer of the form $a^2 + b^2$, where a and b are relatively prime integers and such that if p is a prime, $p \leq \sqrt{n}$, then p divides ab. Determine all such n.

Question 4

Is there an infinite set of points in the plane such that no three points are collinear, and the distance between any two points is rational?

Question 5

You are given three lists A, B, and C. List A contains the numbers of the form 10^k in base 10, with k any integer greater than or equal to 1. Lists B and C contain the same numbers translated into base 2 and 5 respectively:

A	B	C
10	1010	20
100	1100100	400
1000	1111101000	13000
$\vdots$	$\vdots$	$\vdots$

Prove that for every integer $n > 1$, there is exactly one number in exactly one of the lists B or C that has exactly n digits.

THE 1995 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Determine all sequences of real numbers $a_1, a_2, \ldots, a_{1995}$ which satisfy:

$$2\sqrt{a_n - (n-1)} \geq a_{n+1} - (n-1), \text{ for } n = 1, 2, \ldots 1994,$$

and

$$2\sqrt{a_{1995} - 1994} \geq a_1 + 1.$$

Question 2

Let $a_1, a_2, \ldots, a_n$ be a sequence of integers with values between 2 and 1995 such that:

(i) Any two of the a_i's are realtively prime,
(ii) Each a_i is either a prime or a product of primes.

Determine the smallest possible values of n to make sure that the sequence will contain a prime number.

Question 3

Let $PQRS$ be a cyclic quadrilateral such that the segments PQ and RS are not parallel. Consider the set of circles through P and Q, and the set of circles through R and S. Determine the set A of points of tangency of circles in these two sets.

Question 4

Let C be a circle with radius R and centre O, and S a fixed point in the interior of C. Let AA' and BB' be perpendicular chords through S. Consider the rectangles $SAMB$, $SBN'A'$, $SA'M'B'$, and $SB'NA$. Find the set of all points M, N', M', and N when A moves around the whole circle.

Question 5

Find the minimum positive integer k such that there exists a function f from the set $\mathbb{Z}$ of all integers to $\{1, 2, \ldots k\}$ with the property that $f(x) \neq f(y)$ whenever $|x - y| \in \{5, 7, 12\}$.

THE 1996 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Let $ABCD$ be a quadrilateral $AB = BC = CD = DA$. Let MN and PQ be two segments perpendicular to the diagonal BD and such that the distance between them is $d > BD/2$, with $M \in AD$, $N \in DC$, $P \in AB$, and $Q \in BC$. Show that the perimeter of hexagon $AMNCQP$ does not depend on the position of MN and PQ so long as the distance between them remains constant.

Question 2

Let m and n be positive integers such that $n \leq m$. Prove that

$$2^n n! \leq \frac{(m+n)!}{(m-n)!} \leq (m^2+m)^n \ .$$

Question 3

Let P_1, P_2, P_3, P_4 be four points on a circle, and let I_1 be the incentre of the triangle $P_2P_3P_4$; I_2 be the incentre of the triangle $P_1P_3P_4$; I_3 be the incentre of the triangle $P_1P_2P_4$; I_4 be the incentre of the triangle $P_1P_2P_3$. Prove that I_1, I_2, I_3, I_4 are the vertices of a rectangle.

Question 4

The National Marriage Council wishes to invite n couples to form 17 discussion groups under the following conditions:

1. All members of a group must be of the same sex; i.e. they are either all male or all female.
2. The difference in the size of any two groups is 0 or 1.
3. All groups have at least 1 member.
4. Each person must belong to one and only one group.

Find all values of n, $n \leq 1996$, for which this is possible. Justify your answer.

Question 5

Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c} + \sqrt{b+c-a} + \sqrt{c+a-b} \leq \sqrt{a} + \sqrt{b} + \sqrt{c} \ ,$$

and determine when equality occurs.

THE 1997 ASIAN PACIFIC MATHEMATICAL OLYMPIAD

Time allowed: 4 hours
NO calculators are to be used.
Each question is worth seven points.

Question 1

Given

$$S = 1 + \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}+\frac{1}{6}} + \cdots + \frac{1}{1+\frac{1}{3}+\frac{1}{6}+\cdots+\frac{1}{1993006}} ,$$

where the denominators contain partial sums of the sequence of reciprocals of triangular numbers (i.e. $k = n(n+1)/2$ for $n = 1, 2, \ldots, 1996$). Prove that $S > 1001$.

Question 2

Find an integer n, where $100 \leq n \leq 1997$, such that

$$\frac{2^n + 2}{n}$$

is also an integer.

Question 3

Let ABC be a triangle inscribed in a circle and let

$$l_a = \frac{m_a}{M_a} , \quad l_b = \frac{m_b}{M_b} , \quad l_c = \frac{m_c}{M_c} ,$$

where m_a, m_b, m_c are the lengths of the angle bisectors (internal to the triangle) and M_a, M_b, M_c are the lengths of the angle bisectors extended until they meet the circle. Prove that

$$\frac{l_a}{\sin^2 A} + \frac{l_b}{\sin^2 B} + \frac{l_c}{\sin^2 C} \geq 3,$$

and that equality holds iff ABC is an equilateral triangle.

Question 4

Triangle $A_1A_2A_3$ has a right angle at A_3. A sequence of points is now defined by the following iterative process, where n is a positive integer. From A_n ($n \geq 3$), a perpendicular line is drawn to meet $A_{n-2}A_{n-1}$ at A_{n+1}.

(a) Prove that if this process is continued indefinitely, then one and only one point P is interior to every triangle $A_{n-2}A_{n-1}A_n$, $n \geq 3$.

(b) Let A_1 and A_3 be fixed points. By considering all possible locations of A_2 on the plane, find the locus of P.

Question 5

Suppose that n people A_1, A_2, …, A_n, ($n \geq 3$) are seated in a circle and that A_i has a_i

objects such that

$$a_1 + a_2 + \cdots + a_n = nN,$$

where N is a positive integer. In order that each person has the same number of objects, each person A_i is to give or to receive a certain number of objects to or from its two neighbours A_{i-1} and A_{i+1}. (Here A_{n+1} means A_1 and A_n means A_0.) How should this redistribution be performed so that the total number of objects transferred is minimum?