2012
HALF YEARLY
Examination

Number: _____

Teacher: _____

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided at the back of the paper
- Show all necessary working in questions 11-16

Section I ~ Pages 1-3

- 10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II ~ Pages 4-10

- 90 marks
- Attempt Questions 11-16
- Allow about 2 hours 45 minutes for this section

Teachers:

Mr Bradford (Examiner)
Mr Sedgman

Write your Board of Studies Student Number on the front cover of each answer booklet

This paper MUST NOT be removed from the examination room.

Number of Students in Course: xx

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Section I

10 marks
Attempt questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which of the following equations have the same graph?

$$\text{I } y = x - 2 \quad \text{II } y = \frac{x^2-4}{x+2} \quad \text{III } (x+2)y = x^2 - 4$$

(A) I and II only

(B) I and III only

(C) None. All equations have different graphs.

(D) I, II and III

2 Which of the following is an expression for $\int xe^{2x}dx$?

(A) $e^{2x}\left(\frac{x}{2} - \frac{1}{4}\right) + C$

(B) $e^{2x}\left(\frac{x}{2} - 1\right) + C$

(C) $e^{2x}\left(x - \frac{1}{4}\right) + C$

(D) $e^{2x}(2x - 1) + C$

3 The polynomial $P(z)$ has real coefficients. The roots of $P(z) = 0$ include $z = 1 - i$ and $z = 2$.

What is the lowest possible degree of $P(z)$?

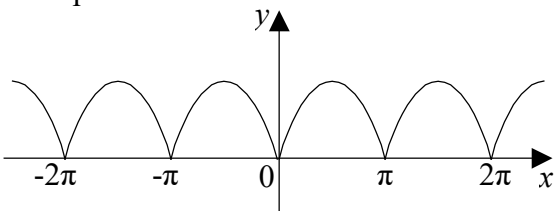
(A) One

(B) Two

(C) Three

(D) Four

4 The graph below could represent



(A) $y = \sin|x|$

(B) $y = 1 - |\cos x|$

(C) $|y| = |\sin x|$

(D) $y = |\sin x|$

5 Given $\sin x = k$ and $\frac{\pi}{2} < x < \pi$ then $\tan\left(\frac{\pi}{2} + x\right)$ equals

(A) $-\frac{k}{\sqrt{1-k^2}}$

(B) $\frac{k}{\sqrt{1-k^2}}$

(C) $-\frac{\sqrt{1-k^2}}{k}$

(D) $\frac{\sqrt{1-k^2}}{k}$

6 If the line y = 2x + 3 is reflected in the line $y = x + 1$, its equation becomes:

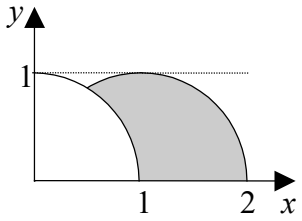
(A) $2x - y = 0$

(B) $x - 2y + 1 = 0$

(C) $x - 2y = 0$

(D) $x - 2y - 2 = 0$

7 Consider the diagram below.



The diagram shows parts of two circles, each of radius 1 unit, and having centre on the x-axis. The area shaded, in square units, is:

(A) 1

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{6}-\frac{\sqrt{3}}{4}$

(D) $\frac{\pi}{6}+\frac{\sqrt{3}}{4}$

8 For what values of x is $|x|+|x-1|=1$?

(A) 0 and 1 only

(B) 0 and -1 only

(C) $0\le x\le 1$

(D) $-1<x\le 1$

9 If $0\le x\le 1$, then the maximum value of $\left(\log_{10}(\frac{99999x+1}{1000})\right)^2$ is:

(A) 4

(B) 9

(C) 25

(D) 100

10 Find the number of ordered pairs of real numbers (a, b) such that $(a+bi)^{2002}=a-bi$.

(A) 1001

(B) 1002

(C) 2003

(D) 2004

Section II

90 marks
Attempt questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a separate writing booklet. Extra writing booklets are available.

All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $w = i$ and $z = 1 + 2i$. Express the following in the form $a + ib$, where appropriate, with a and b real numbers.

(i) $w - z$ **1**

(ii) $|wz|$ **1**

(iii) w^{2011} **1**

(iv) $\dfrac{\overline{w}}{z}$ **2**

(b) (i) Find the square roots of $-15 - 8i$. Express your answers in the form $a + ib$, where a and b are real numbers. **3**

(ii) Hence, solve the equation $z^2 + (2i - 3)z + 5 - i = 0$. **3**
Again, express your answers in the form $a + ib$, where a and b are real numbers.

(c) (i) Write $-2 + 2i$ in modulus-argument form. **1**

(ii) Use de Moivre's theorem and your result from part (i) to prove by mathematical induction that $(-2 + 2i)^n$ is purely real for $n = 4, 8, 12, 16, \ldots$, that is, for all positive multiples of four. **3**

Question 12 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, **3**
find the quadratic equation whose roots are $\frac{\alpha}{1-\alpha}$ and $\frac{\beta}{1-\beta}$.

Write your answer without any fractions for the coefficients.

(b) If α, β and γ are the roots of the equation $x^3 - 3x^2 - 3x - 4 = 0$, form a polynomial equation with integral coefficients whose roots are:-

(i) 2α, 2β and 2γ **1**

(ii) α^2, β^2 and γ^2. **3**

(c) The polynomial equation $P(x) = 0$ has a triple root with $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$.
Find all four roots of $P(x) = 0$. **3**

(d) Let $f(x) = x^n + px - q$ be a polynomial function with real, non-zero coefficients and of arbitrary degree n with $n \neq 1$.

(i) Show that the value of x for which $f(x) = 0$ has a double root is $x = \frac{nq}{p(n-1)}$. **2**

(ii) Hence, or otherwise, show that:- **3**

$$\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$$

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

A relation R is defined as follows: $R = \left\{(x, y) : 9y^2 = x(x-3)^2\right\}$.

(a) What is its domain? **1**

(b) List its x and y intercepts. **1**

(c) The graph of the relation has a relative maximum. Explain, *without using calculus*, why the relation must also have a relative minimum. **1**

(d) Use implicit differentiation to show that $(y')^2 = \dfrac{(x-1)^2}{4x}$. **3**

(e) A critical point of a relation is defined as any point (x, y) where x belongs to the domain of R and either $y'(x) = 0$ or $y'(x)$ is undefined. **2**
Show, by listing their coordinates, that the relation has *three* critical points.

(f) What is the equation of the tangent line to the relation at the point where $y'(x)$ is undefined? Justify your answer. **2**

(g) Sketch the relation $R = \left\{(x, y) : 9y^2 = x(x-3)^2\right\}$ on a number plane noting intercepts. **2**

(h) The area enclosed between the x-axis and that part of the curve which lies in the first quadrant from $x = 0$ to $x = 3$ is $a\sqrt{3}$ square units. Find the value of a. **3**

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A general polynomial $P(z)$ of degree n is defined by:-

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0 \text{ where } a_i \in \Re; i = 1, 2, ..., n.$$

(i) If a complex number ω satisfies $P(\omega) = 0$, prove that $P(\overline{\omega}) = 0$. **3**

Justify each step of your proof.

(ii) Hence, or otherwise, solve the equation $z^4 - 2z^3 - 7z^2 + 26z - 20 = 0$ given that **3**
$z = 2 + i$ is one of the roots.

(b) (i) Sketch the graph of $y = \dfrac{x^2 + 4x + 5}{(x+1)(x+3)}$, noting asymptotes and intercepts. **4**

(ii) By considering the intersection of this graph with the graph $y = \dfrac{-1}{x}$, show **2**
that the equation $x^3 + 5x^2 + 9x + 3 = 0$ has only one real root.

(iii) Let α be this root. If α satisfies $k < \alpha < k + 1$ with k an integer, find k. **1**

(iv) From your answer to part (iii) use the method of halving the interval **2**
to approximate α correct to 1 decimal place.

Question 15 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. Prove that $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$. **3**

(b) Use the above result to describe and sketch the locus of all complex numbers z such that $\left|\frac{z-i}{z-1}\right| = 1$. What is the cartesian equation of the locus of z? **3**

(c) More generally, consider $\left|\frac{z-i}{z-1}\right| = k$ where k is a constant.

Describe and write down where possible the equation of the locus of all points z when:

(α) $k < 0$ **1**

(β) $k = 0$ **1**

(γ) $k > 0$, *but* $k \neq 1$. **3**

(d) Sketch the locus of all points z where $\left|\frac{z-z_1}{\bar{z}_1 z - 1}\right| = 1$ with $z_1 = x_1 + iy_1$ and $|z_1| \neq 1$. **3**

Note that z_1 is a fixed complex number, that is, it is a constant.

(e) Explain how the locus would change in part (d), if additionally $|z_1| = 1$. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Prove that $\sin(A+B)-\sin(A-B)=2\sin B\cos A$. **1**

(b) (i) If $\sin\theta \neq 0$ show that $\dfrac{\sin 4\theta-\sin 3\theta}{\sin\theta}=8\cos^3\theta-4\cos^2\theta-4\cos\theta+1$. **2**

You may assume the binomial coefficients 1 3 3 1 for the expansion of $(x+y)^3$ *and 1 4 6 4 1 for* $(x+y)^4$.

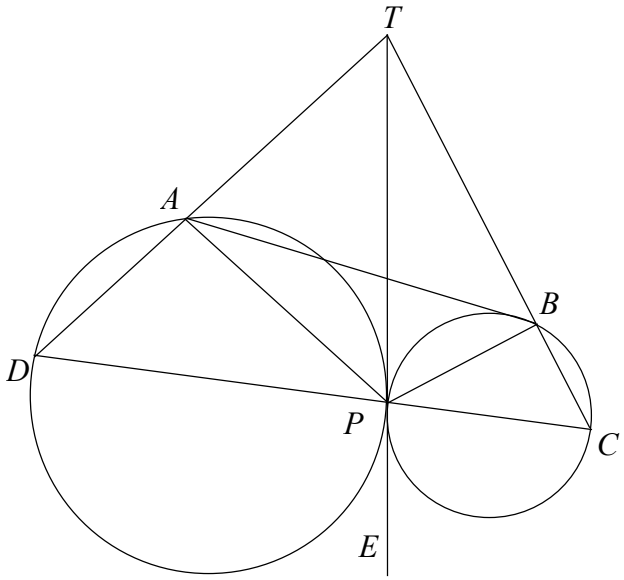
(ii) Hence solve the equation $x^3-x^2-2x+1=0$. **4**

Leave your answers as trigonometric expressions with acute angles.

(iii) Deduce that $\cos\dfrac{\pi}{7}\cos\dfrac{2\pi}{7}\cos\dfrac{3\pi}{7}=\dfrac{1}{8}$. **1**

Question 16 continues on page

(c)



Circles APD and BPC touch at P. D, P and C are collinear. TP is the common tangent at P with T, P and E collinear. TC cuts circle BPC in B while TD cuts circle APD in A.

(i) Copy or trace the diagram into your writing booklet.

(ii) Show that $ATBP$ is a cyclic quadrilateral. **4**

(iii) Show that $ABCD$ is a cyclic quadrilateral. **3**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$