VIRILE AGITUR

2011
HALF YEARLY
Examination

Number: _____

Teacher: _____

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value
- Use a SEPARATE writing booklet for each question
- **Write your Board of Studies Student Number on the front cover of each answer booklet**

Teachers:

Mr Bradford (Examiner)
Mr Vuletich

This paper MUST NOT be removed from the examination room.

BOS Number: _____

Number of Students in Course: 34

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The complex number z is given by $z = 5 + i$.

(i) Express z^2 in the form $x + iy$, where x and y are real numbers. **1**

(ii) Find $|z|^2 - z\bar{z}$ in the form $x + iy$, where x and y are real numbers. **1**

(b) (i) Express $\sqrt{3} + i$ in modulus-argument form. **2**

(ii) Show that $\left(\sqrt{3} + i\right)^6$ is a real number. **2**

(c) Sketch the region in the complex plane satisfied by z where $\bar{\alpha}z + \alpha\bar{z} + 2 \le 0$ and $\alpha = 1 + i$. **2**

(d) The point A represents the complex number $z = x + iy$.

(i) Assume that the point A is moving on the complex plane such that $|z - (1 + i)| = 1$. **3**
Find the cartesian equation and sketch the locus of A, noting intercepts.

(ii) Find the maximum and minimum of the principal argument of z on its locus. **2**

(iii) For t a real variable, the point B represents the complex number $w = 1 + it$. **2**
Describe the locus of B and find the value(s) of t for which $A = B$.

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Sketch the graph of $y = x^2 - 2x$. **1**

(ii) On a separate number plane, draw a neat sketch of $y = \dfrac{1}{x^2 - 2x}$. **2**

(b) Use implicit differentiation to find the gradient of the tangent line to the curve $x^2 + 2xy - y^2 + x = 2$ at the point (1, 2). **2**

(c) Let the function f be defined by $f(x) = \dfrac{1 - x^2}{x}$.

(i) What is the domain of f? **1**

(ii) What is the range of f? **1**

(iii) The function possesses one vertical asymptote. What is its equation? **1**

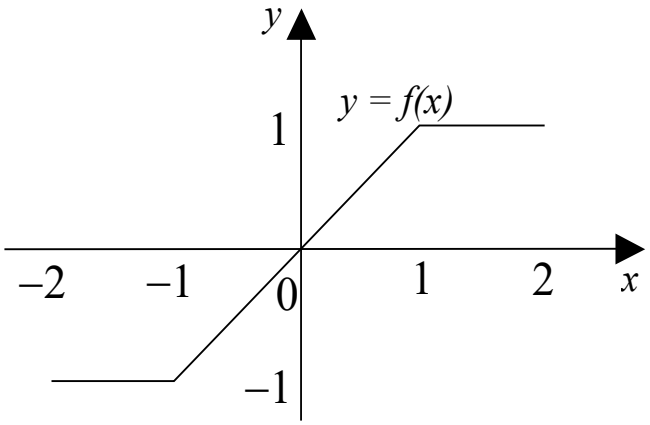
(iv) The function possesses one oblique asymptote. What is its equation? **1**

(v) Show that its derivative $f'(x) = -\dfrac{1}{x^2} - 1$. Hence, explain why f is invertible. **2**

(vi) Sketch the graph of f noting asymptotes and intercepts. **2**

(vii) Hence, or otherwise, sketch $y = \dfrac{\sqrt{x^2 + 4} - x}{2}$. **2**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**



The graph of a certain *odd* function $y = f(x)$ defined on $-2 \le x \le 2$ is shown above.

Provide separate half-page sketches of the graphs of the following, including intercepts, asymptotes and discontinuities, where appropriate.

(a) $y = f(-x)$ **1**

(b) $y = |f(x)|$ **1**

(c) $y = \dfrac{1}{|f(x)|}$ **2**

(d) $y^2 = f(x)$ **2**

(e) $y = \dfrac{d}{dx}(f(x))$ **2**

(f) $y = \int f(x)\,dx$ **2**

(g) $y = e^{f(x)}$ **2**

(h) $y = g(x)$ where the domain of g is all real numbers with: **3**

$g(x) = f(x)$ for $-2 \le x < 2$ and $g(x+4) = g(x)$.

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) If $x-5$ is a factor of $x^3-k^2x^2-2kx-5$ find k, given that k is an integer. **2**

(ii) For this value of k solve $x^3-k^2x^2-2kx-5=0$. **2**

(b) If α and β are the roots of the equation $3x^2+5x+4=0$, without solving the equation, find the value of $\alpha^3+\beta^3$, leaving your answer as a fraction in lowest terms. **3**

(c) The polynomial $P(x)=x^4+3x^3+6x^2+12x+8$ has one root equal to $2i$. **2**

Find all four roots of $P(x)=0$.

(d) (i) Let $p(z)$ be a polynomial of arbitrary degree n with $n \geq 2$. Show that if $p(z)$ has a double zero α, then α is a zero of the derivative $p'(z)$. **3**

(ii) Hence, show that for any given positive integer $n \geq 2$, the polynomial: **3**

$$P(z)=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots+\frac{z^n}{n!}$$

has **no** double zeros, where the factorial $n!$ is defined by $n!=1\times 2\times 3\times \ldots \times n$.

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The cubic equation $2x^3 + 3x + 1 = 0$ has roots α, β and γ.

Find a cubic equation in the form $ay^3 + by^2 + cy + d = 0$ where a, b, c and d are *integers* and whose roots are:

(i) 2α, 2β and 2γ **2**

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

(b) A function f is defined by $f(x) = \dfrac{\sqrt{1+16x^2}-1}{2x}$ for every real $x \neq 0$. **2**

Find the value of $f(0)$ where $f(0) = \lim\limits_{x \to 0} \dfrac{\sqrt{1+16x^2}-1}{2x}$.

(c) Let $f(z) = 1 + z^2 + z^4 + z^6 + ... + z^{2(n-1)}$.

(i) By using a formula for geometric series, simplify the expression for $f(z)$. **1**

(ii) If $z = \cos\theta + i\sin\theta$, prove that $z^n - z^{-n} = 2i\sin n\theta$ and hence show that: **3**

$$f(z) = z^{n-1}\left(\frac{\sin n\theta}{\sin\theta}\right).$$

(iii) By using the above results, prove that: **2**

$$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + ... + \cos 2(n-1)\theta = \frac{\cos(n-1)\theta \times \sin n\theta}{\sin\theta}.$$

(d) A sequence a_n is defined by $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 2$ with $a_0 = a_1 = 2$. **3**

Use mathematical induction to prove that $a_n = \left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n$ for all $n \geq 0$.

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Sketch the common region given by $y \le \sqrt{9-x^2}$ and $y \ge 2$ noting intercepts and points of intersection. **2**

(ii) *Without* using calculus, determine the shaded area found in (i) above, correct to three decimal places. **3**

(b) Let $a^2 + b^2 = 7ab$ with $a, b > 0$.

(i) Prove that $\ln\left(\frac{a+b}{3}\right) = \frac{1}{2}(\ln a + \ln b)$. **2**

(ii) Let two functions f and g be defined by $f(x) = \ln x$ and $g(x) = 2\ln\left(\frac{x+1}{3}\right)$. **3**

Use one application of Newton's method to find the x coordinate of the point of intersection of their graphs near $x = 7$. Give your answer to three decimal places.

(iii) Using the result of part (i), find the **exact** value of this x coordinate. **3**

(iv) More generally, for the function $h(x) = \ln\left(\frac{x+b}{3}\right) - \frac{1}{2}(\ln x + \ln b)$ with $b \in \mathrm{N}$, **2**

show that the corresponding equation $h(x) = 0$ can **never** have rational roots.

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Solve $2^x = 3x - 1$. **1**

(ii) Hence, or otherwise, find the *largest* interval $a \le x \le b$ such that $2^x \le 3x - 1$. **1**

(b) Let $P(x) = (n-1)x^n - nx^{n-1} + 1$ where n is an odd integer, $n \ge 3$.

(i) Show that $P(x)$ has exactly two stationary points and find their coordinates. **2**

(ii) Show that $P(x)$ has a double zero at $x = 1$. **1**

(iii) Sketch the graph of $y = P(x)$ and deduce that $P(x)$ has exactly one real zero other than 1. **2**

(iv) Let α be the real zero of $P(x)$ other than 1. **4**

Using part (a), or otherwise, show that $-1 < \alpha \le -\frac{1}{2}$.

(v) Explain why the real zeros of the polynomial $4x^5 - 5x^4 + 1$ each have a modulus satisfying the inequality $0 \le \text{mod}(zero) \le 1$, where $\text{mod}(zero)$ means 'modulus of the zero'. **1**

(vi) Deduce that the complex zeros of $4x^5 - 5x^4 + 1$ each have modulus satisfying: **3**

$$\frac{1}{2} < \text{mod}(zero) \le \frac{1}{\sqrt{2}}.$$

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find $\frac{d}{dx}\left(\tan x^\circ\right)$. **2**

(b) A farmer wishes to travel from a point A on the edge of a circular paddock of radius 1 km to a point B directly opposite. His tractor can travel at a speed of π km per hour around the perimeter track or at 2km per hour across the paddock. Suppose he drives part of the way around the circular track to a point C and then in a straight line across the paddock to a point B.

(i) Sketch the paddock, clearly labelling the centre O and the points A, B and C. **1**

(ii) Let $2x$ be the angle in radians subtended at the centre O of the paddock by the arc AC. Show that the total time taken from A to B along the arc AC and then along the straight line CB is given by the function $T(x)$ where: **3**

$$T(x) = \frac{2x}{\pi} + \cos x$$

(iii) What is the domain of $T(x)$? **1**

(iv) Sketch the graph of $y = T(x)$ on this domain, noting the coordinates of any stationary points. *Round all values to two decimal places where necessary*. **3**

(v) Deduce the maximum time for the trip from A to B via C. **1**
Give your answer to the nearest minute.

(c) The points A, B, C, D and E represent the five complex numbers a, b, c, d and e respectively. These complex numbers are connected by the equations:

$$b = ia$$
$$c - 1 = i(b - 1)$$
$$d = ic$$
$$e - 1 = i(d - 1)$$

where $i^2 = -1$.

(i) Prove that $a = e$. **2**

(ii) Furthermore, if $c = d$, prove that $\bar{a} = b$. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$