VIRILE AGITUR

2010
HALF YEARLY
Examination

Number: _____

Teacher: _____

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value
- Use a SEPARATE Writing Booklet for each question

Teachers:

Mr Bradford (Examiner)
Mr Vuletich
Mr Harnwell

This paper MUST NOT be removed from the examination room.

Number of Students in Course: 29

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The complex number z is given by $z = \dfrac{-1+i}{i}$.

(i) Express z in the form $x + iy$, where x and y are real numbers. **1**

(ii) Find the modulus and argument of z. **2**

(b) On separate Argand diagrams, sketch the locus of z satisfying:

(i) $\arg(z) = \dfrac{3\pi}{4}$ **1**

(ii) $|z-1-i| = \sqrt{2}$ **2**

(c) In ΔABC, what is the value of $(\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)$? **2**

(d) Let $z = x + iy$ and $z_1 = 3 + 4i$ represent two complex numbers such that $|z + z_1| = |z - 2\overline{z_1}|$.

(i) Show that the locus of the point representing z is the line $18x - 8y = 75$. **2**

(ii) Find the point on the locus such that $z = \overline{z}$. **1**

(e) For t a real variable, the complex number z moves on the Argand plane according to the formula $z = \dfrac{1}{1+it}$ as t varies from $-\infty$ to $+\infty$.

(i) Find the value of t for which $z = \dfrac{1}{2} + \dfrac{1}{2}i$. **1**

(ii) The locus of z is, in fact, a circle. Find the cartesian equation of the locus, writing down its centre and radius. **3**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the polynomial function $F(x) = x^3 - 12x^2 + 36x$.

(i) Find its x and y intercepts. **2**

(ii) Hence, draw a neat sketch of $y = F(x)$. **2**

(b) $P(x)$ is a polynomial of degree four where $P(-2) = P(-1) = P(1) = P(2) = 0$ and the constant term is 2. Find the equation for $P(x)$. **2**

(c) The equation $x^3 - x^2 - 7x + 15 = 0$ has $2 + i$ as a solution. Find the remaining solutions. **2**

(d) When $x^3 + 4x^2 + ax + b$ is divided by $x^2 - 1$ the remainder is $2x + 3$. Find a and b. **3**

(e) Given $2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$ has a triple root, find all its roots. **3**

(f) A student proposed a function with the following two properties:

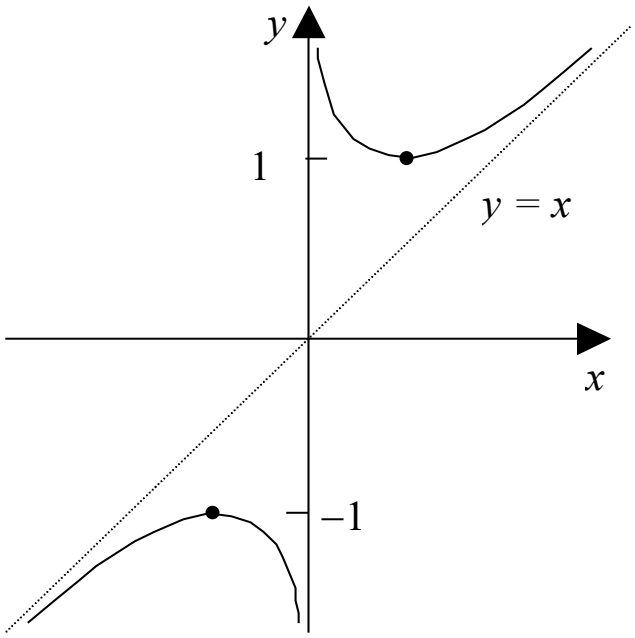
(i) $f(-x) = f(x)$ for all real x and

(ii) $f(-x) = -f(x)$ for all real x.

Can such a function exist? If so, give its equation; otherwise, explain why such a function is impossible. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The graph of a certain *odd* function $y = f(x)$ is shown above.

Provide separate half-page sketches of the graphs of the following, including asymptotes:

(i) $y = -f(x)$ **2**

(ii) $y = f(-x)$ **2**

(iii) $y = |f(x)|$ **2**

(iv) $y = \dfrac{1}{|f(x)|}$ **2**

(v) $y^2 = f(x)$ **2**

(vi) $y = \int f(x)\,dx$ **2**

(b) (i) Sketch the graph of $y = x^2 - |x| - x$. **2**

(ii) Hence, or otherwise, solve the inequality $x^2 - |x| - x < 0$. **1**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If n is a positive integer and $(1+i)^n = a+ib$ where a and b are real numbers, show that $a^2+b^2 = 2^n$. **2**

(b) The equation of a cubic curve is $9y = ax^3+bx^2+cx+d$ where a, b, c and d are constants. The curve passes through the origin and the tangent there makes an angle of $45°$ with the positive x axis. Moreover, the x axis is tangent to the curve when $x = 3$. Find the values of a, b, c and d. **4**

(c) (i) Simplify $\sin(\pi+x)$. **1**

(ii) Sketch the graph of $y = \dfrac{\sin(x-2\pi)+\sin(\pi-x)}{\sin(x+2\pi)-\sin(\pi+x)}$ noting its domain and range. **3**

(iii) Sketch $\sin y = \sin x$ on domain $-\pi \le x \le \pi$ and range $-\pi \le x \le \pi$. **4**

(d) Find the equation of the curve that passes through the points represented by $1+i,\ 2+8i,\ 3+27i,\ 4+64i, \ldots$ **1**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The cubic equation $x^3 + x + 1 = 0$ has roots α, β and γ.

Find a cubic equation in the form $ax^3 + bx^2 + cx + d = 0$ where a, b, c and d are *integers* and whose roots are:

(i) 2α, 2β and 2γ **2**

(ii) α^2, β^2 and γ^2 **3**

(b) Given α, β and γ are the roots of the equation $x^3 - 8x + 8 = 0$ where α is the only integral root, evaluate:

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) $\alpha\beta\gamma + \alpha(\beta + \gamma)$ **3**

(c) When studying the topic *Working with Numbers*, a Year 7 student wrote the following sums: $14 = 3 + 3 + 8$, $15 = 3 + 3 + 3 + 3 + 3$, $16 = 8 + 8$. On the basis of these three results, the student conjectured that, allowing for repetitions, every counting number above 13 can be written as a sum using only the numbers three and/or eight.

Formally, let $P(n)$ be the proposition stated below:

$P(n)$: *n can be written as a sum of 3's and/or 8's for all counting numbers greater than 13.*

Use mathematical induction to prove that the student's conjecture is, in fact, correct. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) When solving the inequality $0 \le -\sqrt{1-x} \le 2$ for real x, a student provided the following solution:-

$$\begin{aligned} &0 \le -\sqrt{1-x} \le 2 \\ &\Rightarrow 0 \le 1-x \le 4 \text{ upon squaring} \\ &\Rightarrow -1 \le -x \le 3 \\ &\Rightarrow -3 \le x \le 1. \end{aligned}$$

(i) Explain why the solution is wrong and write down the correct solution. **2**

(ii) Hence, or otherwise, find the range of values for k, for which $\sin\theta$ is a root of the equation $x^2 + 2x + k = 0$. **3**

(b) (i) Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ be a polynomial function with integral coefficients where $0 \le a_i < 10$, $i = 0, 1, 2, 3$. If $f(10) = 2709$, find $\sum_{i=0}^{3} a_i$. **3**

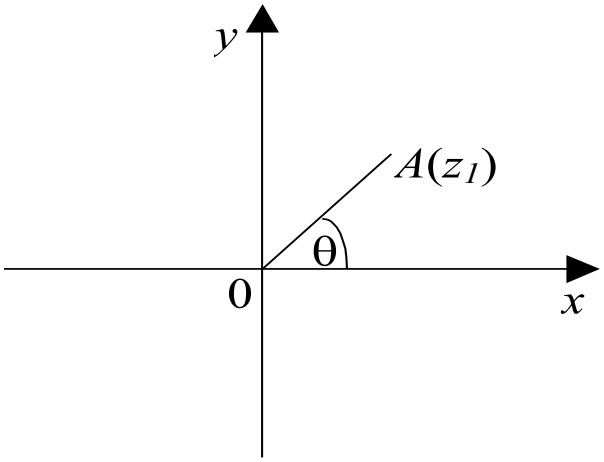
(ii) More generally, let $f(x) = a_0x^n + a_1x^{n-1} + \text{L} + a_n$ be any polynomial with integral coefficients where $0 \le a_i < 10$, $i = 0, 1, 2, \ldots, n$.

(a) Find the remainder when $f(x)$ is divided by $(x-1)$. **1**

(b) Hence, or otherwise, prove that if the sum of the digits of a positive integer is divisible by 9, then the positive integer is divisible by 9. **3**

(c) Evaluate $\lim\limits_{h\to 0} \dfrac{1}{h}\left[\dfrac{2x+3+2h}{x+4+h} - \dfrac{2x+3}{x+4}\right]$ **3**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**



On an Argand plane, A, B and C represent three non-zero complex numbers z_1, z_2 and z_3 respectively. The position of A is shown in the diagram above and $\arg(z_1) = \theta$.
Furthermore, $z_2 = \omega z_1$ and $z_3 = \omega z_2$ where $\omega^3 = 1$ and $\arg(\omega) = \frac{2\pi}{3}$.

(a) Explain how z_2 and z_3 can be obtained geometrically from z_1 and hence, copy and complete the Argand diagram to include the points B and C. **3**

(b) If $|z_1| = 3$, find $|z_2|$ and $|z_3|$. Deduce that A, B and C lie on a circle. **2**

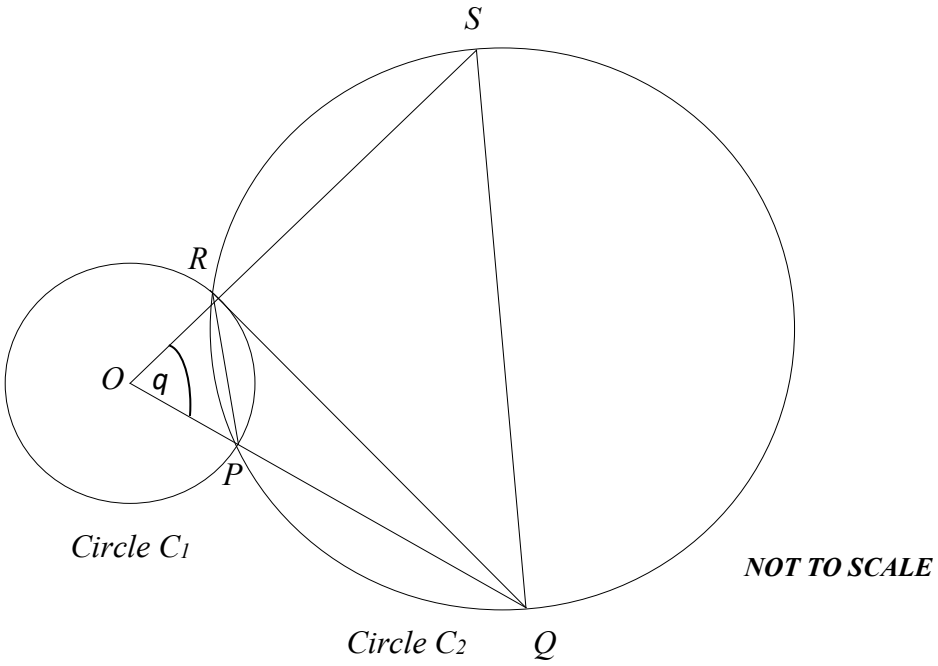
(c) Explain, using your knowledge of plane or circle geometry, why A, B and C are also the vertices of an equilateral triangle. **1**

(d) Calculate $(z_1 - z_2)^2 + (z_1 - z_3)^2 + (z_2 - z_3)^2$. **3**

(e) Calculate $|z_1 - z_2|^2 + |z_1 - z_3|^2 + |z_2 - z_3|^2$. **2**

(f) Show that $\arg\left(\frac{z_1 - z_3}{z_1 - z_2}\right) = \frac{\pi}{3}$ **4**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**



In the diagram above PR is the common chord of the two circles. O is the centre of the smaller circle C_1 and OR is produced to meet the larger circle C_2 at S. OP is also produced to Q so that $PQ = n \times OP$ for some positive number n. QR is tangent to circle C_1 and $\angle POR = \theta$ is acute.

(a) Copy the diagram into your writing booklet.

(b) Explain why QS is a diameter of circle C_2. **2**

(c) Prove that $PR \parallel QS$. **3**

(d) Prove that $QS = (n+1)PR$. **2**

(e) Show that $\cos\theta = \dfrac{1}{n+1}$. **1**

(f) Let A_S and A_L represent the areas of circles C_1 and C_2 respectively.

Using the result in part (e), show that $A_L : A_S = \dfrac{1}{2}\left(\sec^2\theta - \sec\theta\right)$. **4**

(g) If $A_L = 3A_S$, prove that $PQ = 2OP$. **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$