VIRILE AGITUR

2009
HALF YEARLY
Examination

Number: _____

Teacher: _____

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value
- Use a SEPARATE Writing Booklet for each question

Teachers:

Mr Bradford (Examiner)
Mr Maxwell
Mr Dinnell
Mrs Wassef

This paper MUST NOT be removed from the examination room.

Number of Students in Course: 46

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Two complex numbers are given by $z = 3 - 4i$ and $w = 2 - 2i$.

(i) Find the value of the product $\bar{z}\,w$ expressing your answer in the form $x + iy$. **1**

(ii) Find the two square roots of z expressing your answers in the form $x + iy$. **3**

(iii) Express w in modulus-argument form and hence show that w^4 is purely real. **2**

(b) Sketch the locus of z satisfying each of the following:

(i) $\arg(z-1) = \dfrac{\pi}{4}$ **1**

(ii) $z + \bar{z} \le 2$ **1**

(c) Given $z = 1 - i$ find real numbers a and b such that $az = 1 - \dfrac{b}{z}$. **3**

(d) The locus of a point P which moves in the complex plane is represented by the equation $|z - (3 + 4i)| = 5$.

(i) Sketch the locus of the point P, noting the x and y intercepts. **2**

(ii) Using your sketch, find the maximum value of the modulus of z. **1**

(iii) Find the exact value of $\arg z$ when P is in the position of maximum modulus. **1**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given that $z = 1 - i$ is one root of $z^2 - 3z + 3 - i = 0$, find the other root. **1**

(b) Find the three roots of the equation $x^3 + 3x - 4 = 0$. **3**

(c) Consider the polynomial $P(x) = x^4 - 6x^3 + 12x^2 - 10x + 3$.

(i) Solve the equation $x^4 - 6x^3 + 12x^2 - 10x + 3 = 0$, given that it has a three-fold root. **3**

(ii) Hence, or otherwise, sketch the function $y = P(x)$ noting its intercepts with the axes. **2**

(d) You are given the following system of equations in α, β and γ :

$$\alpha + \beta + \gamma = -3$$
$$\alpha^2 + \beta^2 + \gamma^2 = 29$$
$$\alpha\beta\gamma = -6$$

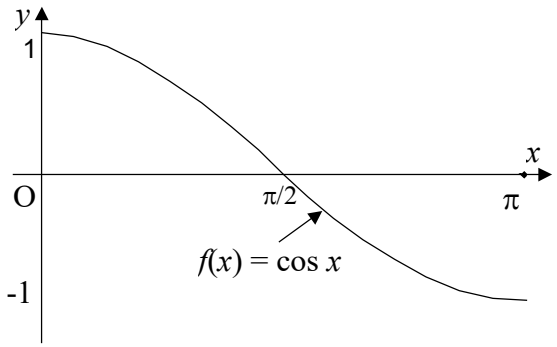
(i) Show that $\alpha\beta + \alpha\gamma + \beta\gamma = -10$. **2**

(ii) Form the cubic equation with roots α, β and γ. **1**

(iii) Hence, or otherwise, find the values of α, β and γ where $\alpha < \beta < \gamma$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The graph of $f(x) = \cos x$ is shown above for the interval $0 \le x \le \pi$.

Provide separate half-page sketches of the graphs of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = f(x - \pi)$ **2**

(iii) $y = \dfrac{1}{|f(x)|}$ **2**

(iv) $y = [f(x)]^2$ **2**

(v) $y^2 = f(x)$ **2**

(vi) $y = e^{f(x)}$ **2**

(b) Describe what graph is represented by the equation $x^2 + y^2 = 25(x - y)^2$. **3**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) $ABCD$ is a square. The points A, B, C and D represent the complex numbers $z_1 = -1 + 4i$, $z_2 = -3 + 0i$, z_3 and z_4 respectively.

(i) Draw the above information on an Argand diagram. **1**

(ii) Find the complex numbers representing the remaining vertices. **4**

(iii) Calculate the area of the square. **1**

(b) Find the equation of the curve that passes through the points represented by $1+i, 8+2i, 27+3i, 64+4i, \ldots$ **1**

(c) If $\text{Re}\left(\dfrac{z+1}{z+i}\right) = 1$, find the equation of the locus of the point representing z. **3**

(d) Solve completely $z^3 = -8i$, obtaining each root in the form $a + ib$. **3**

(e) Describe the locus of the point Q representing the complex number z satisfying the equation $|z| + |z-2| = 2$. **2**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The cubic equation $x^3 + x^2 - 3x - 1 = 0$ has roots α, β and γ.

Find the cubic equation whose roots are:

(i) $(\alpha + 1)$, $(\beta + 1)$ and $(\gamma + 1)$ **2**
(*You may assume* $(y-1)^3 = y^3 - 3y^2 + 3y - 1$.)

(ii) α^2, β^2 and γ^2 **2**

(iii) $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ **2**

(b) Consider the polynomial $P(x) = x^3 + ax - 2$ where a is real.

(i) Find the range of values of a if the graph of $y = P(x)$ has *no* turning points. **2**

(ii) Show that the equation of the tangent line through the point of inflexion is given by $y = ax - 2$. **1**

(iii) If the polynomial equation $P(x) = 0$ has a double root at $x = \alpha$, show that $\alpha = -\sqrt{\frac{-a}{3}}$ *only* and hence find the numerical value of a and the double root. **4**

(iv) Suppose instead of a double root, $P(x) = 0$ has three distinct real roots. **2**

Use the above results to find the range of values of a for which this occurs.

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Given that ω is a complex cube root of unity, prove that the other complex root is ω^2. **1**

(ii) Prove that $1+\omega+\omega^2=0$. **1**

(iii) Hence, evaluate $(1-\omega+\omega^2)(1+\omega-\omega^2)$. **1**

(b) (i) By realising the denominator, show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$. **2**

(ii) Hence simplify $\left(\dfrac{1+\sin\dfrac{\pi}{5}+i\cos\dfrac{\pi}{5}}{1+\sin\dfrac{\pi}{5}-i\cos\dfrac{\pi}{5}}\right)^5$, leaving your answer in the form $x + iy$. **2**

(c) (i) By using the binomial expansion of $(\cos\theta+i\sin\theta)^5$ show that: **3**

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(*You may assume* $(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.)

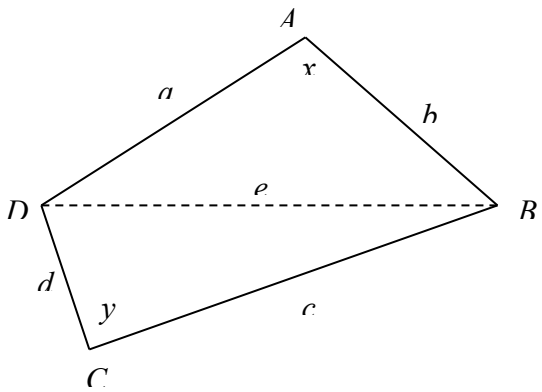
(ii) Write down all solutions of $\sin 5\theta = 0$ for $0<\theta<\pi$. **2**
Leave your answers in terms of π.

(iii) Hence write down the four roots of $16x^4-20x^2+5=0$. **1**

(iv) Deduce that $\sin^2\dfrac{\pi}{5}+\sin^2\dfrac{2\pi}{5}=\dfrac{5}{4}$. **2**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



In the above diagram $ABCD$ is a quadrilateral where $\angle DAB = x$ *radians* and $\angle BCD = y$ *radians*. The lengths a, b, c and d are fixed, but the interior angles can vary.

(i) By using the cosine rule twice show that $\dfrac{dy}{dx} = \dfrac{ab\sin x}{cd\sin y}$. **2**

(ii) Hence show that the area of the quadrilateral is a maximum when it is a cyclic quadrilateral. **4**

(b) A student believed that $\displaystyle\sum_{r=1}^{n} r = \frac{1}{2}\left(n+\frac{1}{2}\right)^2$ and set about proving the formula using the the Principle of Mathematical Induction.

(i) Assuming the formula is true for some arbitrary counting number $n = k$, show that it is also true for the next counting number $n = k+1$. **2**

(ii) The student incorrectly concluded that the formula was true for all counting numbers. From your understanding of mathematical induction, explain why the formula is incorrect. **1**

(iii) Write down the correct formula for $\displaystyle\sum_{r=1}^{n} r$. **1**

Question 7 continues on next page...

Question 7 (continued)

(c) Suppose that α is a real number with $0 < \alpha < \pi$.

Let $P_n = \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\text{L}\ \cos\left(\frac{\alpha}{2^n}\right)$.

(i) Show that $P_n \sin\left(\frac{\alpha}{2^n}\right) = \frac{1}{2} P_{n-1} \sin\left(\frac{\alpha}{2^{n-1}}\right)$. **2**

(ii) By repeated use of the double angle formula $\sin 2A = 2\sin A \cos A$, **1**

deduce that $P_n = \dfrac{\sin\alpha}{2^n \sin\left(\frac{\alpha}{2^n}\right)}$.

(iii) Given that $\sin x < x$ for $x > 0$, show that: **2**

$$\frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\text{L}\ \cos\left(\frac{\alpha}{2^n}\right)} < \alpha .$$

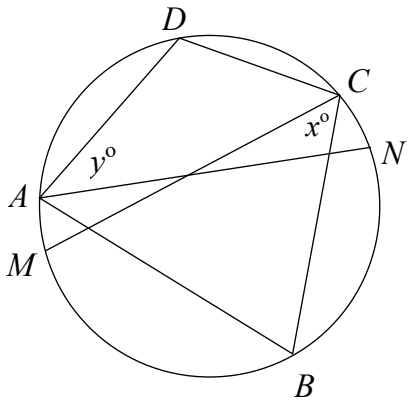
End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use mathematical induction to prove that $2^n < n!$ for $n \geq 4$. **3**

(*Note:* $n! = 1 \times 2 \times 3 \times 4 \times \cdots \times n$. *For example* $3! = 3 \times 2 \times 1$.)

(b)



In the diagram *ABCD* is a cyclic quadrilateral.

M and *N* are points on the circle through *A*, *B*, *C* and *D* such that *CM* bisects $\angle BCD$ and *AN* bisects $\angle DAB$. Let $\angle BCM = x^\circ$ *and* $\angle DAN = y^\circ$.

(i) Copy the diagram into your writing booklet.

(ii) Prove that *MN* is a diameter of the circle. **4**

(c) A continuous function $g(x)$ exists satisfying the following two properties:

α. $g'(x) = g(x)$ for all real numbers *x*

β. $g(0) = 1$

(i) Consider the function *y* given by $y = g(x) \cdot g(-x)$. By using the product rule for differentiation prove that *y* = 1, that is, $g(x) \cdot g(-x) = 1$. **3**

(*You may assume the chain rule result* $[g(-x)]' = g'(-x) \cdot (-1)$.)

(ii) Explain why $g(x) \neq 0$ for all real numbers *x*. **1**

(iii) Explain why $g(x) > 0$ for all real numbers *x*. **1**

(iv) Hence, show $g(x)$ is a strictly increasing function. **1**

(v) Draw a sketch of a suitable function *g* satisfying the above properties. **2**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \ , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$