VIRILE AGITUR

2008
HALF YEARLY
Examination

Number: _____

Teacher: _____

Year 12 Extension 2 Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen only
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in every question

Total Marks (120)

- Attempt Questions 1 – 8
- All questions are of equal value
- Use a SEPARATE Writing Booklet for each question

Teachers

Mr Vuletich
Mr Bradford

This paper MUST NOT be removed from the examination room

Number of Students in Course: 40

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Total marks (120)
Attempt questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = 2 + i$, and $w = 3 - 2i$, express in the form $x + iy$, where x and y are real:

(i) $\text{Re}(z)$ **1**

(ii) $z\,w$ **2**

(iii) iz **2**

(iv) $2z + \overline{w}$ **2**

(v) $\dfrac{1}{z - w}$ **2**

(b) Sketch the locus of z satisfying: $\arg(z-2) = \dfrac{\pi}{6}$ **2**

(c) Factorise $z^4 - 1$ over the field of

(i) Real numbers **2**

(ii) Complex numbers **2**

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The polynomial $P(x) = x^3 + 5x^2 + 17x + 22$ has only one real zero. Find the zero and hence factorise the polynomial over the complex field. **2**

(b) (i) Prove that if a polynomial $P(x)$ has a root of multiplicity, m, then $P'(x)$ has a root of multiplicity $(m - 1)$ **2**

(ii) Given that $ax^3 + bx^2 + d = 0$ has a double root, show that $27a^2d + 2b^3 = 0$. **2**

(c) Let α, β and γ be the roots of the equation $x^3 + 3x + 4 = 0$. Find the value of:

(i) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ **2**

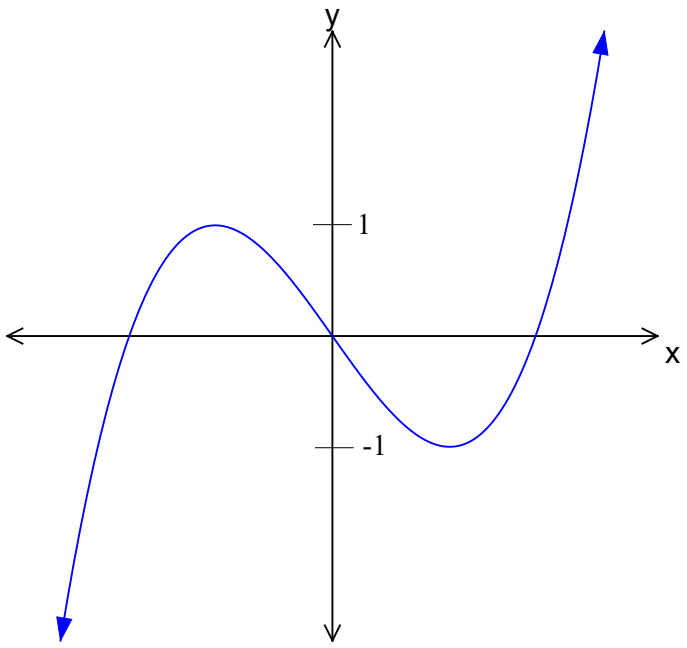
(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(d) Let $P(x) = x^3 - 6x^2 + 9x + \alpha$ where α is a constant. By sketching the graph of $y = P(x)$ for $\alpha = 0$ find the set of values of α for which the equation $P(x) = 0$ has only one real root. **2**

(e) Solve $z^6 + 1 = 0$. **3**

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram shows a graph of the function $y = f(x)$



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ (ii) $y = \dfrac{1}{f(x)}$

(iii) $y = f(|x|)$ (iv) $y = [f(x)]^2$

(v) $y^2 = f(x)$ (vi) $y = \log[f(x)]$

Label each graph with its equation. **11**

(b) Draw, without using calculus, a neat sketch of the graph $y = \dfrac{x^2 - 4}{x^2 - 9}$.

Show clearly its vertical and horizontal asymptotes and its intercepts with the coordinate axes. **4**

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) For the complex numbers $z = -1+\sqrt{3}\,i$ and $w = 1 - i$

(i) Write z and w in modulus-argument form. **4**

(ii) Find the exact value of z^{10}. Leave your answer in the form $x + iy$ where x and y are real numbers. **2**

(iii) By expressing zw in the form $x + iy$, find the exact value of $\cos\left(\frac{5\pi}{12}\right)$. **3**

(b) Sketch $\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$ on an Argand diagram. **1**

(c) Sketch the region of the Argand diagram whose points z satisfy both the inequalities: **2**

$$|z-2| \leq 4 \quad \text{and} \quad |\arg(z-1)| \leq \frac{\pi}{4}$$

(d) Solve completely $z^2 = 3 + 4i$, obtaining each root in the form $a + ib$. **3**

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The cubic equation $2x^3 - 4x^2 - 3x - 1 = 0$ has the roots α, β, and γ. Find the cubic equation whose roots are:

(i) $(\alpha + 2)$, $(\beta + 2)$, and $(\gamma + 2)$ **2**

(ii) α^2, β^2, and γ^2 **2**

(iii) $\alpha + \beta$, $\alpha + \gamma$, *and* $\beta + \gamma$ **3**

(b) Solve the equation $x^3 - 7x^2 + 14x - 8 = 0$, given that the roots of the equation are in geometric progression. **3**

(c) Given that $a + bi$ and $a - 2bi$ are zeros of the polynomial

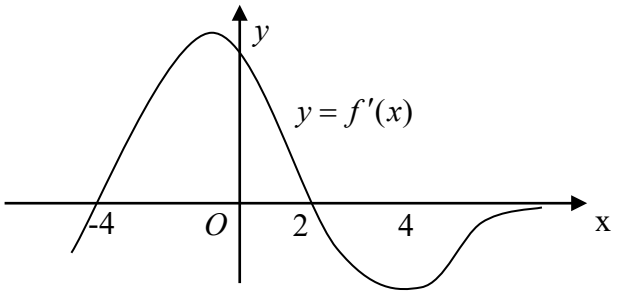
$$P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$$

(i) Find the values of a and b, given that a and b are real positive numbers. **3**

(ii) Hence, express $P(x)$ as a product of two quadratic factors with real coefficients. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The graph of $y = f'(x)$ is shown. **3**



Given that $f(0) = 0$ and for $x > 4$, $f(x) > 0$, sketch $y = f(x)$.

(b) For the domain $-2\pi \leq x \leq 2\pi$, first sketch $f(x) = e^{-x}$ and then

$g(x) = \sin x$ and hence the graph of $y = f(x) \,.\, g(x) = e^{-x} \,.\, \sin x$ **3**

(c) Consider the function $x^2 + xy + y^2 = 5$.

Find the x-coordinates of any stationary points and determine their nature. **4**

(d) Two point $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

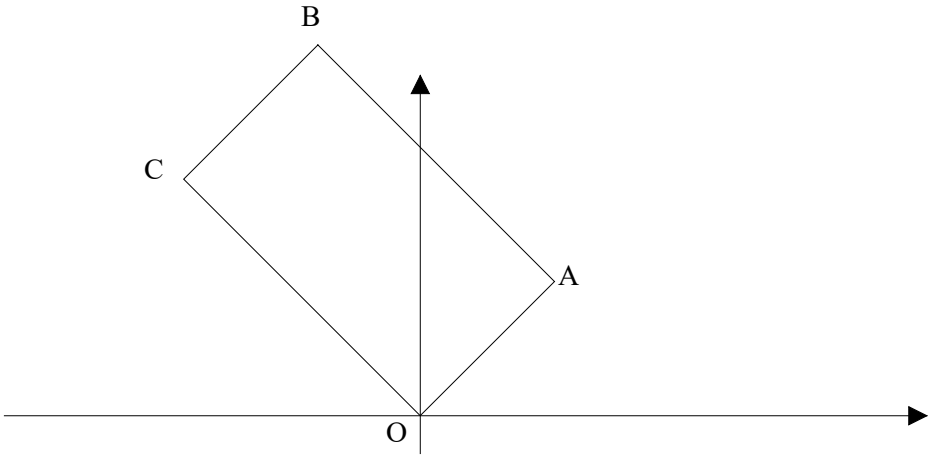
(i) You are given that the tangents at P and Q intersect at an angle of 45°. You may assume that the gradient of the tangents at P and Q are p and q respectively. Show that **2**

$$p - q = 1 + pq$$

(ii) The coordinates of the point of intersection, T, of the tangents to the parabola at P and Q are $\left(a(p+q),\ apq\right)$. Find the locus of T when the tangents at p and q intersect as given in part (i). **3**

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



The Argand diagram OABC is a rectangle, where $OC = 2OA$.

Let the vertex A corresponds to the complex number w.

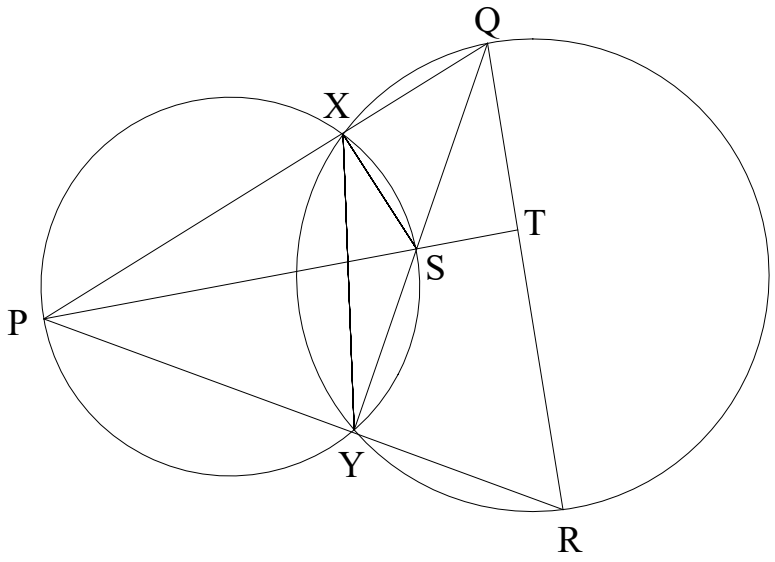
(i) What complex number, in terms of w, corresponds to the vertex C? **2**

(ii) What complex number, in terms of w, corresponds to the point of intersection D of the diagonals OB and AC? **2**

(b) Find the locus of w, if $w = \dfrac{z-1}{z}$, given that $|z| = 2$. Hence, sketch the locus of w. **4**

Question 7 continued on next page...

(c)



In the diagram, the circles *XPYS* and *XYRQ* intersect at the points *X* and *Y*, and *PXQ*, *PYR, QSY, PST,* and *QTR* are straight lines.

(i) Explain why $\angle STQ = \angle YRQ + \angle YPS$ **1**

(ii) Show that $\angle YRQ + \angle YPS + \angle SXQ = \pi$. **2**

(iii) Deduce that *STQX* is a cyclic quadrilateral. **1**

(iv) Let $\angle QPY = \alpha$ and $\angle PQY = \beta$. Show that $\angle STQ = \alpha + \beta$. **3**

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) When the polynomial $P(x)$ is divided by $(x - 4)$ the remainder is -5 and when $P(x)$ is divided by $(x + 1)$, the remainder is 5. Find the remainder when $P(x)$ is divided by $(x - 4)(x + 1)$ **3**

(b) Given that $2x+3 > 2\sqrt{(x+1)(x+2)}$, use mathematical induction to prove that **4**

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1}-1\right)$$

(c) If $z = \cos\theta + i\sin\theta$,

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and hence show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **3**

(ii) Using part (i) or otherwise, solve $8x^3 - 6x - 1 = 0$. **3**

(iii) Hence deduce that $\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$ **2**

End of Examination