St George Girls High School

Year 12

Common Test 3

May 2007

Mathematics Extension 2

Time Allowed: 75 Minutes

Instructions to Candidates

1. All 4 questions may be attempted.
2. All necessary working must be shown.
3. Start each question on a new page.

Question 1 – (15 marks) – Start a New Page **Marks**

a) Consider the hyperbola $H: \frac{x^2}{16} - \frac{y^2}{9} = 1$ 7

(i) Find the x-intercepts

(ii) Find the eccentricity e

(iii) Find the foci

(iv) Find the directrices

(v) Find the asymptotes

(v) Sketch H clearly showing all of the above.

b) The equation $x^3 - 3x - 1 = 0$ has roots α, β, γ 8

(i) Evaluate $\alpha^3 + \beta^3 + \gamma^3$

(ii) Find the equation with roots of $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$

(iii) Find the equation with roots of

$$\frac{1}{\alpha^2+1}, \frac{1}{\beta^2+1}, \frac{1}{\gamma^2+1}$$

(iv) Evaluate $\frac{1}{\alpha^2+1} + \frac{1}{\beta^2+1} + \frac{1}{\gamma^2+1}$

Question 2 – (15 marks) – Start a new page **Marks**

a) The point $P(a\cos\theta, b\sin\theta)$ is a variable point on the ellipse $H: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 11

(i) Find the gradient of the tangent to H at P.

(ii) Find the equation of the line L through the origin parallel to the tangent at P.

(iii) Find the points A and B where the line L cuts H.

(iv) Find the distance AB in simplest form.

(v) Find the perpendicular distance from P to L in simplest form.

(vi) Hence show that the area of the triangle ABP is independent of P.

b) (i) Find the coordinates and the nature of the stationary points on the curve $y=x^3+6x^2+9x+k$ where k is real. 2

(ii) Hence determine the set of values of k for which the equation $x^3+6x^2+9x+k=0$ has three real, distinct roots. 2

Question 3 – (15 marks) – Start a new page **Marks**

a) The points $T_1\left(t, \frac{1}{t}\right)$ and $T_2\left(3t, \frac{1}{3t}\right)$ are two variable points on the hyperbola $H: xy = 1$. 8

(i) Show that the locus of the mid-point, M, of T_1T_2 as t varies is $3xy = 4$

(ii) Show that the chord T_1T_2 is tangential to the locus in (i) above.

(iii) Show that the equation of the normal to H at $T_1\left(t, \frac{1}{t}\right)$ may be written in the form $t^4 - t^3x + ty - 1 = 0$

(iv) Show that there are exactly two points on H with normals which pass through $(0, h)$ where h is real.

b) The roots of $x^3 - 4x^2 + x + 6 = 0$ are α, β and γ where $\gamma = \alpha + \beta$. 3
Find α, β, γ

c) (i) Prove that if $x = \alpha$ is a root of multiplicity m of $P(x) = 0$ then $x = \alpha$ is a root of multiplicity $(m-1)$ of $P'(x) = 0$ 2

(ii) If $x = \alpha$ is a double root of $x^3 + px + q = 0$ show that $4p^3 + 27q^2 = 0$ 2

Question 4 – (15 marks) – Start a new page **Marks**

a) (i) If $y=\dfrac{1+x}{1-x}$ make x the subject of the formula. 4

(ii) If α, β, γ are the roots of $x^3-x-1=0$, find the polynomial equation with roots $\dfrac{1+\alpha}{1-\alpha}, \dfrac{1+\beta}{1-\beta}, \dfrac{1+\gamma}{1-\gamma}$

b) It is given that 6

$2\cos A\sin B=\sin(A+B)-\sin(A-B)$ and $\cos 3A=4\cos^3 A-3\cos A$

Let $S=1+2\cos\theta+2\cos 2\theta+2\cos 3\theta$

(i) Prove that $S\sin\dfrac{\theta}{2}=\sin\dfrac{7\theta}{2}$

(ii) Hence show that if $\theta=\dfrac{2\pi}{7}$ then $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta=0$

(iii) By expressing S in terms of $\cos\theta$, prove that $\cos\dfrac{2\pi}{7}$ is a solution of $8x^3+4x^2-4x-1=0$

c) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $E:\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. The normal at P cuts the x-axis at G, and N is the foot of the perpendicular from P to the x-axis. 5

(i) Show that the equation of the normal at P is $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$

(ii) Find the coordinates of G.

(iii) Prove that $OG:ON=e^2:1$

End of Paper