NAME MASTER

SYDNEY GRAMMAR SCHOOL



2017 Half-Yearly Examination

FORM IV

MATHEMATICS

Monday 22nd May 2017

General Instructions

- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 108 Marks

- All questions may be attempted.
- All necessary working should be shown.
- Start each question on a new page.

Collection

- Write your name, class and Master on each page of your answers.
- Staple your answers in a single bundle.
- Write your name, class and Master on this question paper and hand it in with your answers.

4A: DWH	4B: LYL	4C: CMDB	4D: NJL
4E: GMC	4F: BR	4G: LL	4H: RCF
4I: DNW	4J: LRP		

Checklist

- Writing paper required.
- Candidature — 187 boys

Examiner

DNW

QUESTION ONE (12 marks) Start a new page. **Marks**

(a) Expand $3(2x - 7)$. **1**

(b) Simplify $3a + 5b - 7a + b$. **1**

(c) Solve $3x - 1 = 5$. **1**

(d) Solve $x^2 = 25$. **1**

(e) A cube has edge length 3 cm. What is its volume? **1**

(f) What is the value of $-x^2$ when $x = 7$? **1**

(g) Simplify:

(i) $2\sqrt{3} \times 4\sqrt{3}$ **1**

(ii) $\sqrt{18}$ **1**

(h) Solve $(x - 6)(x + 1) = 0$. **1**

(i) What is the equation of the horizontal line through $(-2, 1)$? **1**

(j) What is the y-intercept of the line with equation $3x + 5y = 30$? **1**

(k) A certain parabola has x-intercepts at $x = -2$ and $x = 4$. What is the equation of its axis of symmetry? **1**

Examination continues next page ...

QUESTION TWO (12 marks) Start a new page. **Marks**

(a) Factorise $x^2 + 10x + 25$. **1**

(b) (i) Solve $5 - 2x > 7$. **1**

(ii) Graph your solution to part (i) on a number line. **1**

(c) Find the distance PQ for the points $P(1, -1)$ and $Q(4, 4)$. **2**

(d) Expand and simplify $(x + 3)(2x - 5)$. **2**

(e) Factorise $px + py - 2y - 2x$ by grouping in pairs. **2**

(f) Consider the formula $y = ut - at^2$.

(i) Make u the subject of this formula. **1**

(ii) Find the value of u when $y = 3{\cdot}2$, $a = 4{\cdot}9$ and $t = 2$. **1**

(g) A particular rectangular prism has surface area $11\,\text{cm}^2$. Each of its edge lengths are doubled to create a new rectangular prism. What is the surface area of the new prism? **1**

QUESTION THREE (12 marks) Start a new page. **Marks**

(a) Simplify:

(i) $\sqrt{12} - \sqrt{48}$ **2**

(ii) $\sqrt{10} \times \sqrt{15}$ **2**

(iii) $\dfrac{\sqrt{21}}{\sqrt{3}}$ **1**

(b) Solve $x^2 + 3x - 28 = 0$ by any appropriate method. **2**

(c) Solve $\dfrac{x + 2}{4} + x = 2$. **2**

(d) A particular parabola has vertex $V(2, 1)$ and y-intercept $W(0, 5)$.

(i) Sketch this parabola, showing the points V and W. **1**

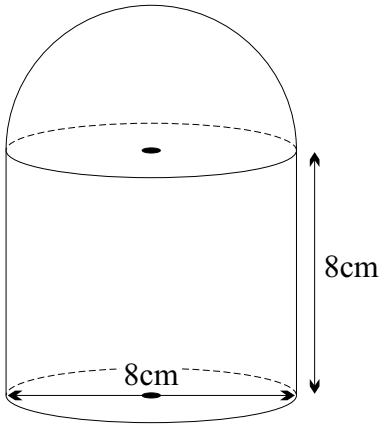
(ii) Use symmetry to find another point on the parabola. **1**

(iii) Suppose the equation of this parabola is $y = ax^2 + bx + c$. Without finding the values of a, b or c, what can be said about the value of $b^2 - 4ac$? **1**

Examination continues overleaf ...

QUESTION FOUR (12 marks) Start a new page. **Marks**

(a) **3**



The diagram above shows an object made up of a cylinder and a hemisphere. Find the total volume in cubic centimetres, correct to two decimal places.

(b) In each case, express the given fraction with a rational denominator.

(i) $\dfrac{5}{\sqrt{6}}$ **1**

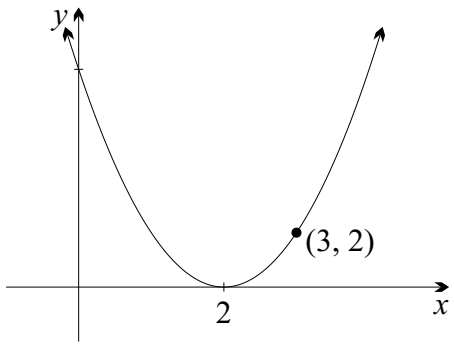
(ii) $\dfrac{3}{4-\sqrt{2}}$ **2**

(c) (i) Find the gradient of the line through $A(1,1)$ and $B(-1,4)$. **1**

(ii) The line ℓ is parallel with AB. What is the gradient of ℓ? **1**

(d) Solve $3x^2 + 11x - 4 = 0$ by factoring. **2**

(e) **2**



Determine the equation of the parabola graphed above.

Examination continues next page ...

QUESTION FIVE (12 marks) Start a new page. **Marks**

(a) Expand and simplify $\left(3\sqrt{2}-4\right)\left(\sqrt{2}+1\right)$. **2**

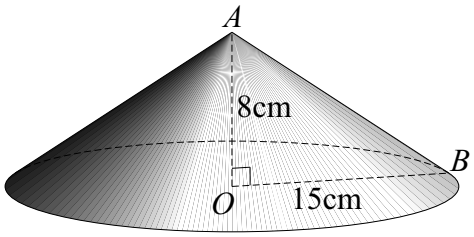
(b) (i) Solve $x^2+5x+2=0$, using the quadratic formula. **2**

(ii) Solve $x^2+4x-1=0$, by completing the square. **2**

(c) Solve simultaneously: **3**

$$\begin{aligned} 3x+\ \ y &= 4 \\ x-5y &= 4 \end{aligned}$$

(d)



The diagram above shows a solid cone with base radius $OB = 15\,\text{cm}$ and perpendicular height $OA = 8\,\text{cm}$.

(i) Find the length of AB. **1**

(ii) Hence determine the total surface area of this cone. Give your answer in exact form. **2**

QUESTION SIX (12 marks) Start a new page. **Marks**

(a) Consider the parabola with equation $y=-x^2+6x-5$.

(i) State the concavity. **1**

(ii) Write down the y-intercept. **1**

(iii) Find the two x-intercepts. **1**

(iv) Find the coordinates of the vertex. **1**

(v) Sketch the parabola, showing all this information. **2**

(b) Simplify $\dfrac{x^2+x-6}{x^2-9}$. **2**

(c) The interval PQ has endpoints at $P(-2,-1)$ and $Q(4,-3)$.

(i) Find the coordinates of M, the mid-point of PQ. **1**

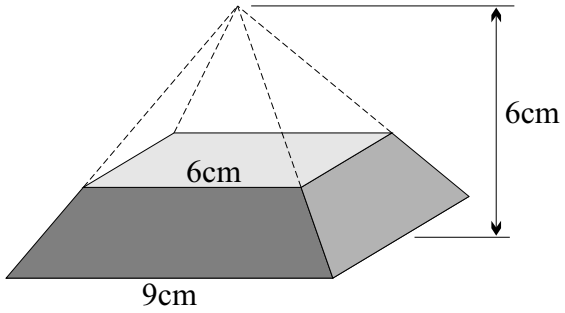
(ii) Find the gradient of PQ. **1**

(iii) Hence find the equation of the perpendicular bisector of PQ. Give your answer in general form. **2**

Examination continues overleaf ...

QUESTION SEVEN (12 marks) Start a new page. **Marks**

(a)



The diagram above shows a square pyramid with a portion removed from the top. The remaining solid is called a frustum.

(i) The portion removed and the original pyramid are similar figures. What is the ratio of their volumes? **2**

(ii) Hence or otherwise determine the volume of the frustum. **2**

(b) Simplify $\dfrac{2x^2+7x-4}{x^2-3x} \div \dfrac{2x-1}{x}$. **3**

(c) Show that the points $A(3,4)$, $B(1,1)$ and $C(-\frac{1}{6},-\frac{3}{4})$ are collinear. **2**

(d) Solve the equation $2x-1=\dfrac{5}{4-x}$. **3**

Examination continues next page ...

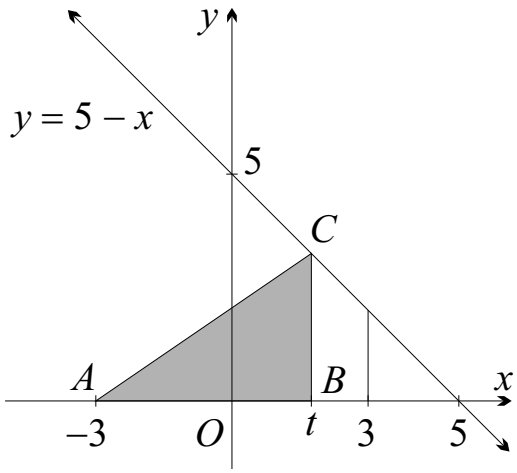
QUESTION EIGHT (12 marks) Start a new page. **Marks**

(a) Find and simplify the gradient of the line through $P\left(2t, \frac{2}{t}\right)$ and $Q\left(\frac{2}{t}, 2t\right)$. **2**

(b) Express $\frac{2}{(x-1)^2} - \frac{1}{x^2-1}$ as a single fraction in simplest form. **3**

(c) Solve the equation $\sqrt{x-1} = x - 7$. **3**

(d)



The graph above shows $\triangle ABC$ with one vertex at $A(-3, 0)$. The second vertex $B(t, 0)$ lies between the origin and $x = 3$, so that $0 \leq t \leq 3$. The third vertex C lies on the line $y = 5 - x$ and is directly above B.

Let R be the area of $\triangle ABC$.

(i) Show that $R = \frac{1}{2}(15 + 2t - t^2)$. **1**

(ii) Find the maximum area of $\triangle ABC$. **2**

(iii) Find the minimum area of $\triangle ABC$. **1**

Examination continues overleaf ...

QUESTION NINE (12 marks) Start a new page. **Marks**

(a) A chemist is asked to make a solution containing the secret compound called mysterium. The chemist mixes a strong solution with a concentration of 30% mysterium, and a weak solution with a concentration of only 6%. The volume of the resulting mixture is 1200 mL, at a concentration of 15% mysterium. **3**

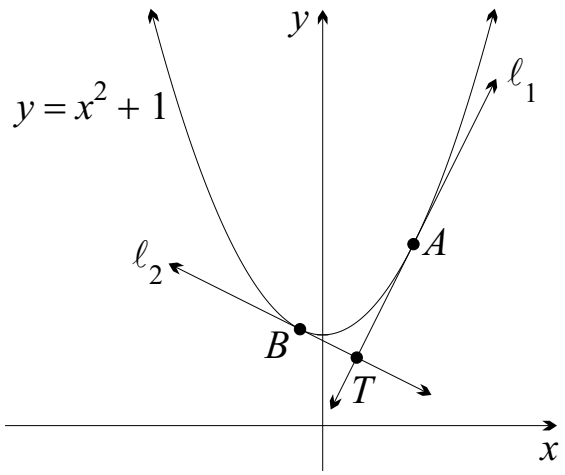
Use the given information to form a pair of equations and solve them simultaneously. Hence state the volumes of the strong and weak solutions used to create the mixture.

(b) Suppose that the quadratic equation $ax^2 + bx + c = 0$ has only one solution.

(i) What is the value of $b^2 - 4ac$? **1**

(ii) What is the value of x? **1**

(c)



Consider the parabola with equation $y = x^2 + 1$, shown above.

(i) The line ℓ_1 with equation $y = mx + b$ is tangent to the parabola at A. That is, ℓ_1 touches the parabola only at one point, which is at A. **2**

Find the coordinates of the point A in terms of m.

(ii) Hence show that the equation of the tangent ℓ_1 is: **1**

$$y = mx + 1 - \tfrac{1}{4}m^2 \,.$$

(iii) The line ℓ_2 is perpendicular to ℓ_1 and is tangent to the parabola at B. Write down the coordinates of B and the equation of ℓ_2. [You may assume that $m \neq 0$.] **2**

(iv) The lines ℓ_1 and ℓ_2 intersect at the point T.

(α) Find the coordinates of T. **1**

(β) Where precisely is T in relation to A and B? **1**

END OF EXAMINATION

QUESTION ONE (12 marks)

(a) $3(2x-7) = 6x - 21$ √

(b) $3a + 5b - 7a + b = 6b - 4a$ √

(c) $3x - 1 = 5$
so $3x = 6$
thus $x = 2$ [Working need not be shown.] √

(d) $x^2 = 25$
so $x = \pm 5$ √

(e) $V_{\text{cube}} = 3^3$
$= 27\,\text{cm}^3$ [Do not penalise lack of units.] √

(f) $-x^2 = -(7^2)$
$= -49$ √

(g) (i) $2\sqrt{3} \times 4\sqrt{3} = 24$ √

(ii) $\sqrt{18} = 3\sqrt{2}$ √

(h) $(x-6)(x+1) = 0$
so $x = 6$ or -1 √

(i) $y = 1$ √

(j) At $x = 0$: $5y = 30$
so $y = 6$. √
Thus $(0, 6)$ is the y-intercept. [No penalty for omitting this.]

(k) Axis: $x = \dfrac{-2+4}{2}$ (The average of the intercepts.)
so $x = 1$ [They must have an equation.] √

Total for Question 1: 12 Marks

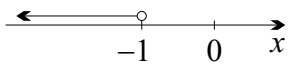
QUESTION TWO (12 marks)

(a) $x^2 + 10x + 25 = (x+5)^2$ √

(b) (i) $5 - 2x > 7$

so $-2x > 2$

thus $x < -1$ √

(ii) 

√

(c) $PQ^2 = (4-1)^2 + (4+1)^2$ √

$= 34$

hence $PQ = \sqrt{34}$ √

(d) $(x+3)(2x-5) = 2x^2 - 5x + 6x - 15$ √

$= 2x^2 + x - 15$ √

(e) $px + py - 2y - 2x = p(x+y) - 2(x+y)$ √

$= (p-2)(x+y)$ √

(f) (i) $y = ut - at^2$

$ut = y + at^2$

so $u = \dfrac{y + at^2}{t}$ [or equivalent] √

(ii) $u = 11{\cdot}4$ √

[Answer must be consistent with either the question or part (i) answer.]

(g) Area ratio $= 1 : 2^2 = 1 : 4$

so new area $= 4 \times 11 = 44\,\text{cm}^2$ √

Total for Question 2: 12 Marks

QUESTION THREE (12 marks)

(a) (i) $\sqrt{12} - \sqrt{48} = 2\sqrt{3} - 4\sqrt{3}$ √

$= -2\sqrt{3}$ √

(ii) $\sqrt{10} \times \sqrt{15} = \sqrt{150}$ √

$= 5\sqrt{6}$ √

(iii) $\dfrac{\sqrt{21}}{\sqrt{3}} = \sqrt{7}$ √

(b) $x^2 + 3x - 28 = 0$

$(x+7)(x-4) = 0$ √

so $x = -7$ or 4 √

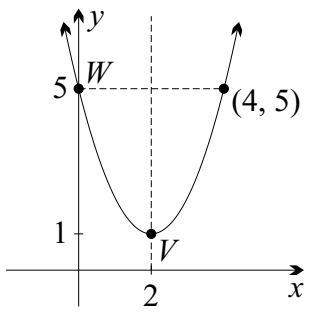
(c) $\dfrac{x+2}{4} + x = 2$

$x + 2 + 4x = 8$ √

$5x = 6$

$x = \frac{6}{5}$ √

(d) (i) √



(ii) $(4, 5)$ [Or an answer consistent with part (i).] √

(iii) $b^2 - 4ac < 0$ (There are no x-intercepts.) √

Total for Question 3: 12 Marks

QUESTION FOUR (12 marks)

(a) $r = 4$ ☑

Volume = cylinder + hemisphere

$= \pi \times 4^2 \times 8 + \frac{2}{3} \times \pi \times 4^3 \qquad [= \frac{512\pi}{3}]$ ☑

so Volume $\div 536{\cdot}17\,\text{cm}^3$, to 2 decimal places. ☑

[Penalise a lack of approximation or an incorrect approximation.]

(b) (i) $\dfrac{5}{\sqrt{6}} = \dfrac{5\sqrt{6}}{6}$ ☑

(ii) $\dfrac{3}{4-\sqrt{2}} = \dfrac{3}{4-\sqrt{2}} \times \dfrac{4+\sqrt{2}}{4+\sqrt{2}}$ ☑

$= \dfrac{12+3\sqrt{2}}{14}$ [No need to expand numerator.] ☑

(c) (i) Gradient $AB = \dfrac{4-1}{-1-1}$

$= -\frac{3}{2}$ ☑

(ii) Gradient $\ell = -\frac{3}{2}$ ☑

(d) $3x^2 + 11x - 4 = 0$

$(3x-1)(x+4) = 0$ ☑

$x = -4$ or $\frac{1}{3}$ ☑

(e) The general equation is $y = a(x-2)^2$ ☑

At $(3, 2)$: $2 = a \times 1^2$

so $a = 2$ ☑

That is, $y = 2(x-2)^2$ [OK to omit this line, but grumble.]

Total for Question 4: 12 Marks

QUESTION FIVE (12 marks)

(a) $(3\sqrt{2}-4)(\sqrt{2}+1) = 6+3\sqrt{2}-4\sqrt{2}-4$ √
$= 2-\sqrt{2}$ √

(b) (i) $x^2+5x+2=0$
so $b^2-4ac=17$ √
thus $x = \dfrac{-5+\sqrt{17}}{2}$ or $\dfrac{-5-\sqrt{17}}{2}$ √

(ii) $x^2+4x-1=0$
so $(x+2)^2=5$ √
thus $x=-2+\sqrt{5}$ or $-2-\sqrt{5}$ √

(c) $3x+\ y=4$ [1]
$x-\ 5y=4$ [2]
Now $3\times$ [2] gives [this, or some other appropriate first step]
$3x-15y=12$ [3] √
Next [1] − [3] yields
$16y=-8$
so $y=-\frac{1}{2}$ √
and from [2]
$x=4+5\times\frac{-1}{2}$
$=\frac{3}{2}$ √

(d) (i) $AB=17\,\text{cm}$ (Pythagorean triad) √

(ii) Thus the surface area is
$A=\pi r^2+\pi r\ell$ (using the textbook notation)
$=15^2\pi+15\times17\times\pi$ √
$=480\pi\,\text{cm}^2$ √

Total for Question 5: 12 Marks

QUESTION SIX (12 marks)

(a) $y = -x^2 + 6x - 5$

(i) The parabola is concave down ($a = -1$). √

(ii) y-intercept $= -5$ √

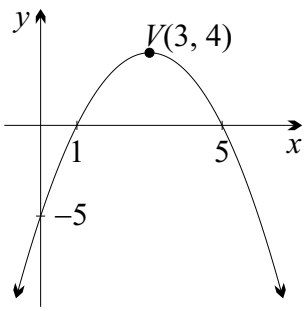
(iii) x-intercepts:

$$x^2 - 6x + 5 = 0$$
$$(x-5)(x-1) = 0$$
so $x = 1$ or 5 √

(iv) Axis of symmetry $x = 3$ (average of x-intercepts)

At the vertex $y = -9 + 18 - 5$
$= 4$ √

So the vertex is $V(3, 4)$. [OK to omit this, but grumble.]

(v) 

[All or nothing.] √√

(b) $\dfrac{x^2 + x - 6}{x^2 - 9} = \dfrac{(x+3)(x-2)}{(x+3)(x-3)}$ [One mark for one correct factoring] √

$= \dfrac{x-2}{x-3}$ [2nd mark only if ALL correct] √

(c) (i) $M = (1, -2)$ (average of the coordinates) √

(ii) gradient $PQ = -\frac{1}{3}$ √

(iii) Required gradient $= 3$ √

So the equation of the perpendicular bisector is:

$$y + 2 = 3(x - 1)$$
thus $y - 3x + 5 = 0$ √

Total for Question 6: 12 Marks

QUESTION SEVEN (12 marks)

(a) (i) Ratio of lengths $= 2 : 3$ √

Hence the ratio of volumes $= 8 : 27$ √

(ii) Fraction of the volume left $= \frac{19}{27}$ √

Thus $V = \frac{19}{27} \times \left(\frac{1}{3} \times 9^2 \times 6\right)$

$= 114\,\text{cm}^3$ √

[This or any other correct working.]

(b)
$$\frac{2x^2+7x-4}{x^2-3x} \div \frac{2x-1}{x}$$

$$= \frac{2x^2+7x-4}{x^2-3x} \times \frac{x}{2x-1}$$ [1st mark for reciprocal] √

$$= \frac{(2x-1)(x+4)}{x(x-3)} \times \frac{x}{2x-1}$$ [2nd mark for one correct factoring] √

$$= \frac{x+4}{x-3}$$ [Last mark only if ALL correct] √

(c) Gradient $AB = \frac{3}{2}$. √

Gradient $BC = \dfrac{1+\frac{3}{4}}{1+\frac{1}{6}}$

$= \frac{3}{2}$

Since the gradients are equal and B is common, the points are collinear. √

[A reason must be given to award the 2nd mark.]

(d)
$$2x - 1 = \frac{5}{4-x}$$

so $(2x-1)(4-x) = 5$ √

$-2x^2 + 9x - 4 = 5$

thus $2x^2 - 9x + 9 = 0$ √

or $(2x-3)(x-3) = 0$

hence $x = \frac{3}{2}$ or 3 √

Total for Question 7: 12 Marks

QUESTION EIGHT (12 marks)

(a) Gradient $PQ = \dfrac{2t - \frac{2}{t}}{\frac{2}{t} - 2t}$ (multiply top and bottom by t) √

$= \dfrac{2t^2 - 2}{2 - 2t^2}$

$= -1\,.$ √

(b) $\dfrac{2}{(x-1)^2} - \dfrac{1}{x^2-1} = \dfrac{2}{(x-1)^2} - \dfrac{1}{(x-1)(x+1)}$ √

$= \dfrac{2(x+1) - (x-1)}{(x-1)^2(x+1)}$ [or equivalent] √

$= \dfrac{x+3}{(x-1)^2(x+1)}$ √

(c) $\sqrt{x-1} = x - 7$

so $x - 1 = x^2 - 14x + 49$ √

$0 = x^2 - 15x + 50$

$= (x-5)(x-1)$

thus $x = 10$ or 5 √

But, when $x = 5$

LHS $= \sqrt{5-1} = 2$

and RHS $= 5 - 7 = -2$

thus $x = 5$ cannot be a solution.

Hence the only solution is $x = 10$. √

(d) (i) $C = (t, 5-t)$

thus $R = \frac{1}{2} \times AB \times BC$

$= \frac{1}{2}(t+3)(5-t)$ √

$= \frac{1}{2}(15 + 2t - t^2)$

(ii) The maximum area is at the vertex of the parabola $R = \frac{1}{2}(15 + 2t - t^2)$

which occurs at $t = \frac{1}{2}(5 + -3)$

$= 1$ √

where $R_{\max} = 8$ square units √

(iii) The minimum must be at either $t = 0$ or $t = 3$.

At $t = 0$: $R = 7\frac{1}{2}$ square units

At $t = 3$: $R = 6$ square units

Hence $R_{\min} = 6$ square units. √

Total for Question 8: 12 Marks

QUESTION NINE (12 marks)

(a) Let the volume of strong solution be S
and the volume of weak solution be W.

Then $\frac{6}{100}W + \frac{30}{100}S = \frac{15}{100}1200$ (adding the amounts of mysterium)

so $6W + 30S = 18000$

or $W + 5S = 3000$ [1] √

and $W + S = 1200$ [2] (adding the volumes of liquid.) √

[One mark each for any appropriate pair of simultaneous equations.]

Subtracting these equations gives ([1] − [2])

$4S = 1800$

so $S = 450\,\text{mL}$

and $W = 750\,\text{mL}$ (from equation [2].) √

Thus 450 mL of strong solution and 750 mL of weak solution are used.

(b) (i) $b^2 - 4ac = 0$ √

(ii) $x = \dfrac{-b}{2a}$ √

(c) (i) The line and parabola intersect when

$mx + b = x^2 + 1$ √

or $x^2 - mx + (1 - b) = 0\,.$

Since there is only one solution, by part (b), it must be

$x = \frac{m}{2}$

$y = \frac{m^2}{4} + 1$ √

(ii) By the point gradient formula

$y - \frac{m^2}{4} - 1 = m(x - \frac{m}{2})$

$= mx - \frac{m^2}{2}$ √

thus $y = mx + 1 - \frac{m^2}{4}\,.$

(iii) At B the gradient is $\frac{-1}{m}$, so by parts (i) and (ii), replacing m with $\frac{-1}{m}$:

$B = (\frac{-1}{2m}, 1 + \frac{1}{4m^2})$ √

and ℓ_2 has equation

$y = -\frac{1}{m}x + 1 - \frac{1}{4m^2}$ √

(iv) (α) At T $\quad y = \quad mx + 1 - \frac{1}{4}m^2$ (from part (ii))

and $\quad m^2y = -mx - \frac{1}{4} + \quad m^2$ (from part (iii))

so adding these gives

$$(1+m^2)y = \tfrac{3}{4} + \tfrac{3}{4}m^2$$
$$= \tfrac{3}{4}(1+m^2)$$

so $\quad y = \frac{3}{4}\,.$

Hence, from the equation in part (ii)

$$\tfrac{3}{4} = mx + 1 - \tfrac{m^2}{4}$$
$$mx = \tfrac{m^2}{4} - \tfrac{1}{4}$$

thus $\quad x = \frac{m}{4} - \frac{1}{4m}$ $\boxed{\surd}$

(β) Clearly T is on the horizontal line $y = \frac{3}{4}$, from the y-coordinate.

Rearranging the x-coordinate:

$x = \frac{1}{2}(\frac{m}{2} - \frac{1}{2m})\,.$

That is, the x-coordinate of T is the average of A and B $\boxed{\surd}$

[Both results are needed for the mark.]

Total for Question 9: 12 Marks

DNW