Solutions to Exercise 1A Development questions

10a $(x + yi)(2 - 3i) = -13i$

$2x - 3xi + 2yi - 3yi^2 = -13i$

$2x - 3xi + 2yi - 3y(-1) = -13i$

$2x - 3xi + 2yi + 3y = -13i$

$(2x + 3y) + (-3x + 2y)i = -13i$

Equating real and imaginary parts,

$2x + 3y = 0$ (1)

$-3x + 2y = -13$ (2)

$3 \times (1) + 2 \times (2)$ gives:

$13y = -26$

$y = -2$

(1) becomes:

$2x - 6 = 0$

$x = 3$

Thus, $x = 3$ and $y = -2$.

10b $(1 + 4i)(x + yi) = 6 + 7i$

$x + yi + 4xi + 4yi^2 = 6 + 7i$

$x + yi + 4xi - 4y = 6 + 7i$

$(x - 4y) + (4x + y)i = 6 + 7i$

Equating real and imaginary parts,

$x - 4y = 6$ (1)

$4x + y = 7$ (2)

$4 \times (2) + (1)$ gives:

$17x = 34$

$x = 2$

(2) becomes:

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$8 + y = 7$

$y = -1$

Thus, $x = 2$ and $y = -1$.

10c $(1 + i)x + (2 - 3i)y = 10$

$(x + 2y) + (x - 3y)i = 10$

Equating real and imaginary parts,

$x + 2y = 10$ (1)

$x - 3y = 0$ (2)

$(1) - (2)$ gives:

$5y = 10$

$y = 2$

(2) becomes:

$x - 6 = 0$

$x = 6$

Thus, $x = 6$ and $y = 2$.

10d $x(1 + 2i) + y(2 - i) = 4 + 5i$

$(x + 2y) + (2x - y)i = 4 + 5i$

Equating real and imaginary parts,

$x + 2y = 4$ (1)

$2x - y = 5$ (2)

$(1) + 2 \times (2)$ gives:

$5x = 14$

$x = \frac{14}{5}$

(2) becomes:

$\frac{28}{5} - y = 5$

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$y = \frac{3}{5}$

Thus $x = \frac{14}{5}$ and $y = \frac{3}{5}$.

10e

$$\frac{x}{2+i} + \frac{y}{2+3i} = 4 + i$$

$$\frac{x(2+3i) + y(2+i)}{(2+i)(2+3i)} = 4 + i$$

$$\frac{2x + 3xi + 2y + yi}{4 + 6i + 2i - 3} = 4 + i$$

$$\frac{2x + 3xi + 2y + yi}{1 + 8i} = 4 + i$$

$$2x + 3xi + 2y + yi = (4 + i)(1 + 8i)$$

$$2x + 3xi + 2y + yi = 4 + 32i + i - 8$$

$$2x + 2y + 3xi + yi = -4 + 33i$$

Equating real and imaginary parts,

$2x + 2y = -4$ or

$x + y = -2$ (1)

$3x + y = 33$ (2)

$(2) - (1)$ gives:

$2x = 35$

$x = \frac{35}{2}$

(1) becomes:

$\frac{35}{2} + y = -2$

$y = -\frac{39}{2}$

Thus, $x = \frac{35}{2}$ and $y = -\frac{39}{2}$.

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Chapter 1 worked solutions – Complex numbers I

11a

$$\frac{1}{1+i}+\frac{2}{1+2i}$$

$$=\frac{1(1-i)}{(1+i)(1-i)}+\frac{2(1-2i)}{(1+2i)(1-2i)}$$

$$=\frac{1-i}{1^2-i^2}+\frac{2-4i}{1-4i^2}$$

$$=\frac{1-i}{1+1}+\frac{2-4i}{1+4}$$

$$=\frac{1}{2}-\frac{i}{2}+\frac{2}{5}-\frac{4}{5}i$$

$$=\left(\frac{1}{2}+\frac{2}{5}\right)+\left(-\frac{1}{2}-\frac{4}{5}\right)i$$

$$=\frac{5+4}{10}+\frac{-5-8}{10}i$$

$$=\frac{9}{10}-\frac{13}{10}i$$

11b

$$\frac{1+i\sqrt{3}}{2}+\frac{2}{1+i\sqrt{3}}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2}{1+i\sqrt{3}}\times\frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2-2\sqrt{3}i}{1+3}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2-2\sqrt{3}i}{4}$$

$$=\frac{1+i\sqrt{3}+1-i\sqrt{3}}{2}$$

$$=\frac{2}{2}$$

$$=1$$

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11c

$$\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$$

$$=\frac{(3+2i)(2+5i)+(3-2i)(2-5i)}{(2+5i)(2-5i)}$$

$$=\frac{6+15i+4i-10+6-15i-4i-10}{4+25}$$

$$=\frac{12-20+19i-19i}{29}$$

$$=-\frac{8}{29}$$

11d

$$\frac{-8+5i}{-2-4i}-\frac{3+8i}{1+2i}$$

$$=\frac{-8+5i}{-2(1+2i)}-\frac{3+8i}{1+2i}$$

$$=\frac{-8+5i}{-2(1+2i)}-\frac{-2(3+8i)}{-2(1+2i)}$$

$$=\frac{-8+5i+2(3+8i)}{-2(1+2i)}$$

$$=\frac{-8+5i+6+16i}{-2(1+2i)}$$

$$=\frac{-2+21i}{-2(1+2i)}$$

$$=\frac{-2+21i}{-2(1+2i)}\times\frac{1-2i}{1-2i}$$

$$=\frac{-2+4i+21i-42i^2}{-2(1-4i^2)}$$

$$=\frac{-2+25i+42}{-2(5)}$$

$$=\frac{40+25i}{-10}$$

$$=-4-\frac{5}{2}i$$

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Chapter 1 worked solutions – Complex numbers I

12a $\overline{z+w}$

$= \overline{x+\imath y+a+\imath b}$

$= \overline{(a+x)+(b+y)\imath}$

$= (a+x)-(b+y)i$

$= (x-iy)+(a-bi)$

$= \bar{z}+\bar{w}$

12b $\overline{z-w}$

$= \overline{x+\imath y-a-\imath b}$

$= \overline{(-a+x)+(-b+y)\imath}$

$= (-a+x)-(-b+y)i$

$= (x-yi)-(a-bi)$

$= \bar{z}-\bar{w}$

12c $\overline{zw}$

$= \overline{(x+\imath y)(a+\imath b)}$

$= \overline{xa+bx\imath+ay\imath+by\imath^2}$

$= \overline{xa+bx\imath+ay\imath+by\imath^2}$

$= \overline{xa+bx\imath+ay\imath-by}$

$= \overline{(xa-by)+(bx+ay)\imath}$

$= (xa-by)-(bx+ay)i$

$= xa-bxi-ayi-by$

$= xa-bxi-ayi+byi^2$

$= (x-iy)(a-bi)$

$= \bar{z}\bar{w}$

12d Consider the case where $a=x$ and $b=y$, we then have that $z=w$. The result in part c then gives that $\overline{zz}=\bar{z}\bar{z}$ and hence $\overline{z^2}=(\bar{z})^2$.

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12e

$$\overline{\left(\frac{1}{z}\right)}$$

$$= \overline{\left(\frac{1}{z} \times \frac{\bar{z}}{\bar{z}}\right)}$$

$$= \overline{\left(\frac{\bar{z}}{z\bar{z}}\right)}$$

$$= \overline{\left(\frac{\bar{z}}{|z|^2}\right)}$$

$$= \frac{1}{|z|^2}\bar{\bar{z}}$$

$$= \frac{1}{|z|^2}\overline{\overline{(x + \imath y)}}$$

$$= \frac{1}{|z|^2}\overline{x - \imath y}$$

$$= \frac{1}{|z|^2}(x + iy)$$

$$= \frac{z}{|z|^2}$$

$$= \frac{z}{z\bar{z}}$$

$$= \frac{1}{\bar{z}}$$

12f

$$\overline{\left(\frac{z}{w}\right)}$$

$$= \overline{\left(\frac{x + \imath y}{a + \imath b}\right)}$$

$$= \overline{\left(\frac{(x + \imath y)(a - \imath b)}{(a + \imath b)(a - \imath b)}\right)}$$

$$= \overline{\left(\frac{ax - bx\imath + ay\imath - by\imath^2}{a^2 + b^2}\right)}$$

$$= \overline{\left(\frac{(ax + by) + (ay - bx)\imath}{a^2 + b^2}\right)}$$

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$$= \overline{\left(\frac{(ax+by)}{a^2+b^2} + \frac{(ay-bx)}{a^2+b^2}\iota\right)}$$

$$= \frac{(ax+by)}{a^2+b^2} - \frac{(ay-bx)}{a^2+b^2}i$$

$$= \frac{ax+by-ayi+bxi}{a^2+b^2}$$

$$= \frac{ax-ayi+bxi-byi^2}{a^2+b^2}$$

$$= \frac{(x-iy)(a+ib)}{(a+ib)(a-ib)}$$

$$= \frac{x-iy}{a-ib}$$

$$= \frac{\bar{z}}{\bar{w}}$$

13a $z+\bar{z}$

$= a+ib+\overline{a+\iota b}$

$= a+ib+a-ib$

$= 2a$

Since a is real, $2a$ and hence $z+\bar{z}$ must also be real.

13b $z-\bar{z}$

$= a+ib-\left(\overline{a+\iota b}\right)$

$= a+ib-(a-ib)$

$= a+ib-a+ib$

$= 2bi$

Since b is real, $2bi$ and hence $z-\bar{z}$ must be imaginary.

13c $z^2+(\bar{z})^2$

$= (a+ib)^2+\left(\overline{a+\iota b}\right)^2$

$= (a+ib)^2+(a-ib)^2$

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$= a^2 + 2abi + b^2i^2 + a^2 - 2abi + b^2i^2$

$= 2a^2 + 2b^2i^2$

$= 2a^2 - 2b^2$

Since a and b are real, $2a^2 - 2b^2$ and hence $z^2 + (\bar{z})^2$ must be real.

13d $z\bar{z}$

$= (a + ib)\overline{a + ib}$

$= (a + ib)(a - ib)$

$= a^2 - i^2b^2$

$= a^2 + b^2$

Since a and b are real, $a^2 \geq 0$ and $b^2 \geq 0$ so $a^2 + b^2 \geq 0$. This in turn means that $z\bar{z} \geq 0$. Hence $z\bar{z}$ is real and positive.

14

$$\frac{z}{z - i}$$

$$= \frac{a + bi}{a + bi - i}$$

$$= \frac{a + bi}{a + (b - 1)i}$$

$$= \frac{(a + bi)(a - (b - 1)i)}{(a + (b - 1)i)(a - (b - 1)i)}$$

$$= \frac{(a + bi)(a - (b - 1)i)}{a^2 - (b - 1)^2i^2}$$

$$= \frac{a^2 - a(b - 1)i + abi - b(b - 1)i^2}{a^2 + (b - 1)^2}$$

$$= \frac{a^2 - abi + ai + abi + b(b - 1)}{a^2 + (b - 1)^2}$$

$$= \frac{a^2 - ai + b^2 - b}{a^2 + (b - 1)^2}$$

$$= \frac{a^2 + b^2 - b}{a^2 + (b - 1)^2} - \frac{a}{a^2 + (b - 1)^2}i$$

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If $\frac{z}{z-i}$ is real, then the imaginary component must be equal to zero, this means that

$$\frac{a}{a^2+(b-1)^2}=0$$

$$a=0$$

Hence, $z=0+bi=bi$. Thus z is either imaginary or zero.

15 z^2

$=(a+bi)^2$

$=a^2+2abi+b^2i^2$

$=a^2+2abi-b^2$

$\bar{z}^2$

$=(a-bi)^2$

$=a^2-2abi+b^2i^2$

$=a^2-2abi-b^2$

If $z^2=\bar{z}^2$ then

$a^2+2abi-b^2=a^2-2abi-b^2$

$4abi=0$

Hence $a=0$ or $b=0$. This means that either z is purely real or purely imaginary.

16a z^{-1}

$=\frac{1}{z}$

$=\frac{1}{x+iy}$

$=\frac{(x-iy)}{(x-iy)(x+iy)}$

$=\frac{x-iy}{x^2-i^2y^2}$

$=\frac{x-iy}{x^2+y^2}$

Uncorrected proofs

16b z^{-2}

$= (z^{-1})^2$

$= \left(\frac{x - iy}{x^2 + y^2}\right)^2$

$= \frac{x^2 - 2ixy + i^2y^2}{(x^2 + y^2)^2}$

$= \frac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}$

16c

$\frac{z - 1}{z + 1}$

$= \frac{x - 1 + iy}{x + 1 + iy}$

$= \frac{(x - 1 + iy)(x + 1 - iy)}{(x + 1)^2 + y^2}$

$= \frac{x^2 - x + ixy + x - 1 + iy - ixy + iy + y^2}{(x + 1)^2 + y^2}$

$= \frac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2}$

Uncorrected proofs

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Solutions to Exercise 1B Development questions

5a Let $z = a + bi$ be a square root of $-3 - 4i$

$$z^2 = (a + bi)^2$$
$$= a^2 + 2abi + b^2i^2$$
$$= a^2 - b^2 + 2abi$$

Thus $a^2 - b^2 + 2abi = -3 - 4i$.

Equating real parts,

$a^2 - b^2 = -3$ (1)

Equating imaginary parts,

$2ab = -4$

$b = -\frac{2}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(-\frac{2}{a}\right)^2 = -3$

$a^2 - \frac{4}{a^2} = -3$

$a^4 - 4 = -3a^2$

$a^4 + 3a^2 - 4 = 0$

$(a^2 + 4)(a^2 - 1) = 0$

Hence $a = \pm 1$ (as a is real)

When $a = 1, b = -2$ and $z = 1 - 2i$.

When $a = -1, b = 2$ and $z = -1 + 2i$.

5b $z^2 - 3z + (3 + i) = 0$

$\left(z - \frac{3}{2}\right)^2 - \frac{9}{4} + 3 + i = 0$

$\left(z - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{12 + 4i}{4} = 0$

$\left(z - \frac{3}{2}\right)^2 + \frac{3 + 4i}{4} = 0$

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$$\left(z-\frac{3}{2}\right)^2=-\frac{3+4i}{4}$$

$$\left(z-\frac{3}{2}\right)^2=\frac{-3-4i}{4}$$

$$z-\frac{3}{2}=\pm\frac{\sqrt{-3-4i}}{2}$$

$$z=\frac{3}{2}\pm\frac{\sqrt{-3-4i}}{2}$$

Now, using the square roots found in part (a),

$$z=\frac{3}{2}\pm\frac{(1-2i)}{2}$$

$$z=\frac{3}{2}\pm\left(\frac{1}{2}-i\right)$$

$$z=2-i,1+i$$

6a Let $z=a+bi$ be a square root of $-8+6i$.

$$\begin{aligned}z^2&=(a+bi)^2\\&=a^2+2abi+b^2i^2\\&=a^2-b^2+2abi\end{aligned}$$

Thus $a^2-b^2+2abi=-8+6i$.

Equating real parts,

$a^2-b^2=-8$ (1)

Equating imaginary parts,

$2ab=6$

$$b=\frac{3}{a}\qquad(2)$$

Substituting (2) into (1):

$$a^2-\left(\frac{3}{a}\right)^2=-8$$

$$a^2-\frac{9}{a^2}=-8$$

$$a^4-9=-8a^2$$

$$a^4+8a^2-9=0$$

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$(a^2 + 9)(a^2 - 1) = 0$

Hence $a = \pm 1$ (as a is real)

When $a = 1$, $b = 3$ and $z = 1 + 3i$.

When $a = -1$, $b = -3$ and $z = -1 - 3i$.

6b $\Delta = \left(-(7 - i)\right)^2 - 4(1)(14 - 5i)$

$= 49 - 14i + i^2 - 56 + 20i$

$= 49 - 14i - 1 - 56 + 20i$

$= -8 + 6i$

$\lambda^2 = -8 + 6i$

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2 i^2$
$= a^2 - b^2 + 2abi$

Equating real parts,

$a^2 - b^2 = -8$ (1)

Equating imaginary parts,

$6 = 2ab$

$b = \frac{3}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(\frac{3}{a}\right)^2 = -8$

$a^2 - \frac{9}{a^2} = -8$

$a^4 - 9 = -8a^2$

$a^4 + 8a^2 - 9 = 0$

$(a^2 + 9)(a^2 - 1) = 0$

Since a is real, $a^2 > 0$ and $a^2 = 1$

$a = \pm 1$

When $a = 1$, $b = 3$ and when $a = -1$, $b = -3$.

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$\lambda = \pm(1 + 3i)$

$z = \dfrac{-b - \lambda}{2a}$ or $\dfrac{-b + \lambda}{2a}$

$z = \dfrac{-\big(-(7 - i)\big) \pm (1 + 3i)}{2(1)}$

$z = 4 + i$ or $3 - 2i$

7a Using the method in Box 12 to solve $z^2 - z + (1 + i) = 0$:

1. $\Delta = b^2 - 4ac$
   $= (-1)^2 - 4(1)(1 + i)$
   $= 1 - 4(1 + i)$
   $= 1 - 4 - 4i$
   $= -3 - 4i$
2. $\lambda^2 = -3 - 4i$ so, from **5**, $\lambda = \pm(1 - 2i)$
3. Thus $z = \frac{-(-1)-(1-2i)}{2(1)}$ or $\frac{-(-1)+(1-2i)}{2(1)}$

   $z = \dfrac{2i}{2(1)}$ or $\dfrac{2 - 2i}{2(1)}$

   $z = i$ or $1 - i$

7b Using the method in Box 12 to solve $z^2 + 3z + (4 + 6i) = 0$:

1. $\Delta = b^2 - 4ac$
   $= (3)^2 - 4(1)(4 + 6i)$
   $= 9 - 16 - 24i$
   $= -7 - 24i$
2. $\lambda^2 = -7 - 24i$

   Let $\lambda = a + bi$ be a square root of $-7 - 24i$

   $\lambda^2 = (a + bi)^2$
   $= a^2 + 2abi + b^2i^2$
   $= a^2 - b^2 + 2abi$

   Thus $a^2 - b^2 + 2abi = -7 - 24i$.

   Equating real parts,

   $a^2 - b^2 = -7$ (1)

Uncorrected proofs

Equating imaginary parts,

$2ab = -24$

$b = -\frac{12}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(-\frac{12}{a}\right)^2 = -7$

$a^2 - \frac{144}{a^2} = -7$

$a^4 - 144 = -7a^2$

$a^4 + 7a^2 - 144 = 0$

$(a^2 - 9)(a^2 + 16) = 0$

$a = \pm 3$ (as a is real)

From (2) when $a = 3$, $b = -4$ and when $a = -3$, $b = 4$.

This gives $\lambda = \pm(3 - 4i)$

3. Thus $z = \frac{-(3)+(3-4i)}{2(1)}$ or $\frac{-(3)-(3-4i)}{2(1)}$

$z = \frac{-4i}{2(1)}$ or $\frac{-6+4i}{2(1)}$

$z = -2i$ or $-3 + 2i$

7c Using the method in Box 12 to solve $z^2 - 6z + (9 - 2i) = 0$:

1. $\Delta = b^2 - 4ac$
$= (-6)^2 - 4(1)(9 - 2i)$
$= 36 - 36 + 8i$
$= 8i$

2. $\lambda^2 = 8i$

Let $\lambda = a + bi$ be a square root of $8i$.

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

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Thus $a^2 - b^2 + 2abi = 8i$.

Equating real parts,

$a^2 - b^2 = 0$ (1)

Equating imaginary parts,

$2ab = 8$

$b = \frac{4}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(\frac{4}{a}\right)^2 = 0$

$a^2 - \frac{16}{a^2} = 0$

$a^4 - 16 = 0$

$(a^2 - 4)(a^2 + 4) = 0$

$a = \pm 2$ (as a is real)

From (2) when $a = 2$, $b = 2$ and when $a = -2$, $b = -2$.

This gives $\lambda = \pm 2(1 + i)$.

3. Thus $z = \frac{-(6)+2(1+i)}{2(1)}$ or $\frac{-(6)-2(1+i)}{2(1)}$

$z = -\frac{-4+2i}{2(1)}$ or $-\frac{-8-2i}{2(1)}$

$z = 2 - i$ or $4 + i$

7d Using the method in Box 12 to solve $(1 + i)z^2 + z - 5 = 0$:

1. $\Delta = b^2 - 4ac$
$= 1 - 4(1 + i)(-5)$
$= 1 - (-20 - 20i)$
$= 21 + 20i$

2. $\lambda^2 = 21 + 20i$

Let $\lambda = a + bi$ be a square root of $21 + 20i$.

Uncorrected proofs

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

Thus $a^2 - b^2 + 2abi = 21 + 20i$.

Equating real parts,

$a^2 - b^2 = 21$ (1)

Equating imaginary parts,

$2ab = 20$

$b = \frac{10}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(\frac{10}{a}\right)^2 = 21$

$a^2 - \frac{100}{a^2} = 21$

$a^4 - 100 = 21a^2$
$a^4 - 21a^2 - 100 = 0$

$(a^2 - 25)(a^2 + 4) = 0$

$a^2 = 25$ or -4

Hence $a = \pm 5$

When $a = 5$, $b = \frac{10}{5} = 2$.

When $a = -5$, $b = \frac{10}{(-5)} = -2$.

3. Thus $z = \frac{-(1)+5+2i}{2(1+i)}$ or $\frac{-(1)-(5+2i)}{2(1+i)}$

$z = \frac{2+i}{1+i}$ or $\frac{-3-i}{1+i}$

$z = \frac{(2+i)(1-i)}{(1-i)(1+i)}$ or $\frac{(-3-i)(1-i)}{(1-i)(1+i)}$

$z = -2 + i$ or $\frac{1}{2}(3 - i)$

Uncorrected proofs

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7e Using the method in Box 12 to solve $z^2 + (2+i) - 13(1-i) = 0$:

1. $\Delta = b^2 - 4ac$
$= (2+i)^2 - 4(1)\big(-13(1-i)\big)$
$= 4 + 4i + i^2 + 52(1-i)$
$= 4 + 4i - 1 + 52 - 52i$
$= 55 - 48i$

2. $\lambda^2 = 55 - 48i$

Let $\lambda = a + bi$ be a square root of $55 - 48i$.

$\lambda^2 = (a+bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

Thus $a^2 - b^2 + 2abi = 55 - 48i$.

Equating real parts,

$a^2 - b^2 = 55$ (1)

Equating imaginary parts,

$2ab = -48$

$b = \dfrac{-24}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(-\dfrac{24}{a}\right)^2 = 55$

$a^2 - \dfrac{576}{a^2} = 55$

$a^4 - 576 = 55a^2$
$a^2 - 55a^2 - 576 = 0$

$a^2 = \dfrac{-(-55) \pm \sqrt{(-55)^2 - 4(1)(-576)}}{2}$

$a^2 = \dfrac{55 \pm \sqrt{5329}}{2}$

$a^2 = \dfrac{55 \pm 73}{2}$

$a^2 = 64$ or -9

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Hence $a = \pm 8$

When $a = 8, b = \frac{-24}{8} = -3.$

When $a = -8, b = \frac{-24}{-8} = 3.$

3. Thus $z = \frac{-(2+i)+(8-3i)}{2(1)}$ or $\frac{-(2+i)-(8-3i)}{2(1)}$

$z = \frac{6-4i}{2(1)}$ or $\frac{-10+2i}{2(1)}$

$z = 3 - 2i$ or $-5 + i$

7f Using the method in Box 12 to solve $iz^2 - 2(1 + i) + 10 = 0$:

1. $\Delta = b^2 - 4ac$
$= \left(-2(1 + i)\right)^2 - 4(i)(10)$
$= 4(1 + 2i + i^2) - 40i$
$= 4(1 + 2i - 1) - 40i$
$= 8i - 40i$
$= -32i$

2. $\lambda^2 = -32i$

Let $\lambda = a + bi$ be a square root of $-32i$

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

Thus $a^2 - b^2 + 2abi = -32i.$

Equating real parts,

$a^2 - b^2 = 0$ (1)

Equating imaginary parts,

$2ab = -32$

$b = \frac{-16}{a}$ (2)

Substituting (2) into (1):

$a^2 - \left(-\frac{16}{a}\right)^2 = 0$

Uncorrected proofs

$a^2 - \frac{16^2}{a^2} = 0$

$a^4 - 16^2 = 0$

$a^2 - 16)(a^2 + 16) = 0$

Hence $a = \pm 4$

When $a = 4, b = -\frac{16}{4} = -4.$

When $a = -4, b = \frac{-16}{-4} = 4.$

3. Thus $z = -\frac{-2(1+i)+(4-4i)}{2(i)}$ or $-\frac{-2(1+i)-(4-4i)}{2(i)}$

$z = -\frac{2-6i}{2i}$ or $\frac{-6+2i}{2i}$

$z = -\frac{2i-6i^2}{2i^2}$ or $-\frac{-6i+2i^2}{2i^2}$

$z = -\frac{2i+6}{-2}$ or $-\frac{-6i-2}{-2}$

$z = 3 + i$ or $-1 - 3i$

8a $z^2 + wz + (1 + i) = 0$

Since i is a root of the equation,

$i^2 + wi + (1 + i) = 0$

$-1 + wi + (1 + i) = 0$

$wi + i = 0$

$wi = -i$

$w = -1$

8b Since a and b are real, the conjugate $3 + 2i$ must also be a root of the equation.

The sum of the roots is $-\frac{a}{1}$, hence

$3 + 2i + 3 - 2i = -a$

$6 = -a$

$a = -6$

Uncorrected proofs

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The products of the roots is $\frac{b}{1} = b$, hence

$(3 + 2i)(3 - 2i) = b$

$9 - 4i^2 = b$

$b = 9 + 4$

$b = 13$

8c Since $1 - 2i$ is a root of the equation, we can find k by substituting it into the equation.

$(1 - 2i)^2 - (3 + i)(1 - 2i) + k = 0$

$1 - 4i + 4i^2 - (3 - 6i + i - 2i^2) + k = 0$

$1 - 4i - 4 - (3 - 5i + 2) + k = 0$

$-3 - 4i - (5 - 5i) + k = 0$

$-8 + i + k = 0$

$k = 8 - i$

Now, the sum of the roots is $-\left(-\frac{3+i}{1}\right) = 3 + i$, hence, if our second root is α

$1 - 2i + \alpha = 3 + i$

$\alpha = 2 + 3i$

9 $z\bar{z} = 5$

$z = \frac{5}{\bar{z}}$

$z^2 = \frac{5z}{\bar{z}}$

$\frac{z^2}{5} = \frac{z}{\bar{z}}$

$\frac{z}{\bar{z}} = \frac{1}{5}(3 + 4i)$

$\frac{z^2}{5} = \frac{1}{5}(3 + 4i)$

$z^2 = 3 + 4i$

Let $z = a + bi$.

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$z^2 = a^2 - b^2 + 2abi$

Hence $a^2 - b^2 + 2abi = 3 + 4i$

Equating real parts,

$a^2 - b^2 = 3 \qquad (1)$

Equating imaginary parts,

$2ab = 4$

$b = \frac{2}{a} \qquad (2)$

Substituting (2) in (1):

$a^2 - \left(\frac{2}{a}\right)^2 = 3$

$a^2 - \frac{4}{a^2} = 3$

$a^4 - 4 = 3a^2$

$a^4 - 3a^2 - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

As a is real, $a^2 \geq 0$ and hence $a = \pm 2$.

When $a = 2, b = 1$ and when $a = -2, b = -1$. Thus $\pm(2 + i)$.

10a $z^2 - 2z\cos\theta + 1 = 0$

$z^2 - 2z\cos\theta + \cos^2\theta - \cos^2\theta + 1 = 0$

$(z - \cos\theta)^2 - \cos^2\theta + 1 = 0$

$(z - \cos\theta)^2 + 1 - \cos^2\theta = 0$

$(z - \cos\theta)^2 + \sin^2\theta = 0$

$(z - \cos\theta)^2 = -\sin^2\theta$

$(z - \cos\theta) = \pm i\sin\theta$

$z = \cos\theta \pm i\sin\theta$

$z = \text{cis}(\pm\theta)$

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10b $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Substituting $z = \text{cis}\,\theta$ into the right-hand side of the equation gives

$$\frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left(\text{cis}\,\theta + \frac{1}{\text{cis}\,\theta}\right)$$

$$= \frac{1}{2}\left(\text{cis}\,\theta + \text{cis}(-\theta)\right) \qquad \text{(using De Moivres theorem)}$$

$$= \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$$

$$= \frac{1}{2}(2\cos\theta)$$

$$= \cos\theta$$

Substituting $z = \text{cis}(-\theta)$ into the right-hand side of the equation gives

$$\frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left(\text{cis}(-\theta) + \frac{1}{\text{cis}(-\theta)}\right)$$

$$= \frac{1}{2}(\text{cis}(-\theta) + \text{cis}\,\theta) \qquad \text{(using De Moivres theorem)}$$

$$= \frac{1}{2}(\text{cis}(-\theta) + \text{cis}\,\theta)$$

$$= \frac{1}{2}(\cos\theta - i\sin\theta + \cos\theta + i\sin\theta)$$

$$= \frac{1}{2}(2\cos\theta)$$

$$= \cos\theta$$

Hence the equation holds true for both results found in part **a.**

11a $z^3 = -1$

$$z^3 + 1 = 0$$

$$z^3 + 1^3 = 0$$

$$(z+1)(z^2 - z(1) + 1^2) = 0$$

$$(z+1)(z^2 - z + 1) = 0$$

Uncorrected proofs

Hence one root is $z = -1$, and using the quadratic formula, we find the roots of

$z^2 - z + 1 = 0$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

So, the roots are $z = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

11b $z^3 + i = 0$

$z^3 - i^3 = 0$

$(z - i)(z^2 + iz + i^2) = 0$

$z = i$ or $z^2 + iz - 1 = 0$

Using the quadratic formula,

$$z = \frac{-i \pm \sqrt{i^2 - 4(1)(-1)}}{2}$$

$$= \frac{-i \pm \sqrt{-1+4}}{2}$$

$$= \frac{-i \pm \sqrt{3}}{2}$$

Hence the roots are $z = i$ or $z = \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

12a Since ω is a root of the equation, it must satisfy the equation when substituted into the above equation. Thus $a\omega^2 + b\omega + c = 0$.

12b $\overline{a\omega^2 + b\omega + c} = 0$

$\overline{a\omega^2} + \overline{b\omega} + c = 0$

$a\overline{\omega^2} + b\overline{\omega} + \overline{\omega} = 0$ (since a and b are real)

Uncorrected proofs

$a(\overline{\omega})^2 + b\overline{\omega} + \overline{\omega} = 0$

12c When a quadratic has real coefficients, for any complex root, the conjugate must also be a root.

13a The conjugate $\overline{\alpha}$ must also be a root since the polynomial has complex coefficients.

13b The equation $z^2 - 2\text{Re}(\alpha)z + \alpha\bar{\alpha} = 0$ has real coefficients.

This is because 1 is real $-2\text{Re}(z)$ is real and $\alpha\bar{\alpha} = |a|^2$ is real.

Substituting α into the left hand side of the equation gives

$\alpha^2 - 2\text{Re}(\alpha)\alpha + \alpha\bar{\alpha}$

$= \alpha(\alpha - 2\text{Re}(\alpha) + \bar{\alpha})$

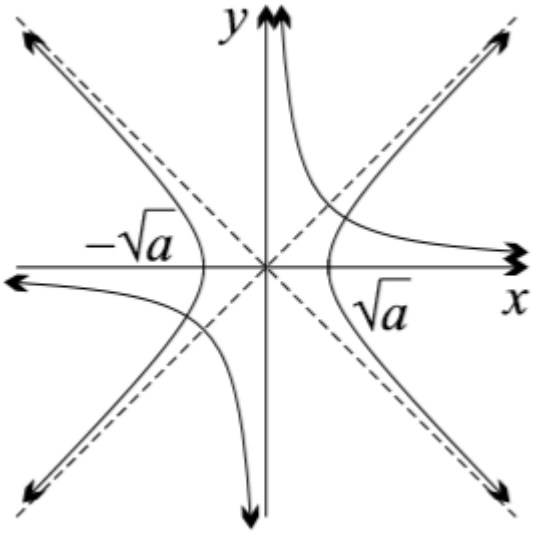
$= \alpha\big(\alpha + \bar{\alpha} - 2\text{Re}(\alpha)\big)$

$= \alpha\big(2\text{Re}(\alpha) - 2\text{Re}(\alpha)\big)$

$= \alpha(0)$

$= 0$

14ai $x^2 - y^2 = a$ is shown by the curve that intersects the x-axis, and $2xy = b$ is given by the curve in the 1st and 3rd quadrants.

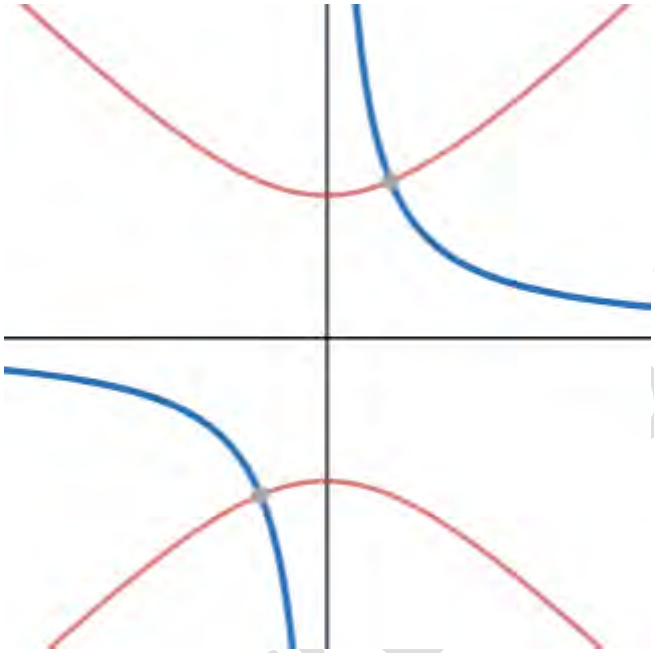


Uncorrected proofs

14aii Since the two graphs have two distinct points of intersection, we can infer that the set of equations $x^2 - y^2 = a$ and $2xy = b$ will have two distinct sets of solutions. Each set of solutions corresponds to a unique value of $z = a + ib$ and hence a distinct square root.

14b Below we sketch each of the possible cases. In each case not that there are always exactly two solutions.

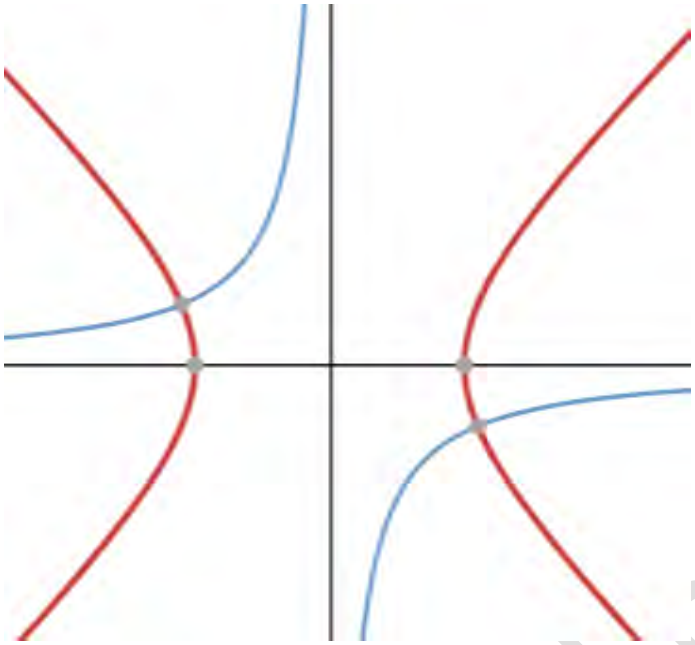
$a < 0, b > 0$



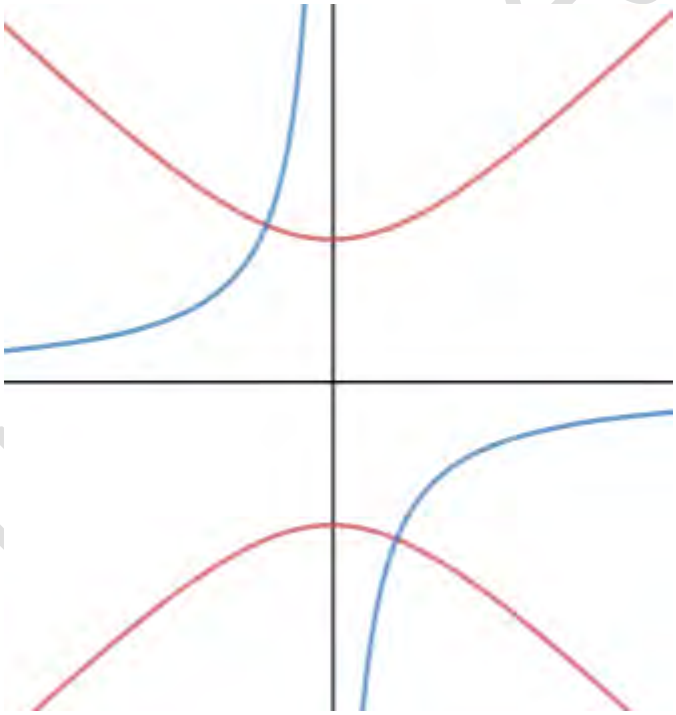
Uncorrected proofs

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$a > 0, b < 0$



$a < 0, b < 0$



ed proofs

15a Let the square root of $-i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = 0$

$a^2 = b^2$

$a = \pm b$ (1)

$ab = -\frac{1}{2}$ (2)

Substituting (1) into (2):

$\pm a^2 = -\frac{1}{2}$

Since a is real, $a^2 > 0$ and hence $a^2 = \frac{1}{2}$ so $a = \pm\frac{1}{\sqrt{2}}$.

Now, using (2):

When $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ and hence $z = \frac{1}{\sqrt{2}}(1 - i)$.

When $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$ and hence $z = -\frac{1}{\sqrt{2}}(1 - i)$.

15b Let the square root of $-6 + 8i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = -6$ (1)

$ab = \frac{1}{2}(8)$ (2)

Substituting (2) into (1):

$a^2 - \left(\frac{4}{a}\right)^2 = -6$

$a^2 - \frac{16}{a^2} = -6$

$a^4 - 16 = -6a^2$

$a^4 + 6a^2 - 16 = 0$

$(a^2 - 2)(a^2 + 8) = 0$

Since a is real, $a^2 > 0$ and hence $a^2 = 2$ so $a = \pm\sqrt{2}$.

Now, using (2):

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When $a = \sqrt{2}, b = 2\sqrt{2}$ and hence $z = \sqrt{2}(1 + 2i)$.

When $a = -\sqrt{2}, b = -2\sqrt{2}$ and hence $z = -\sqrt{2}(1 + 2i)$.

15c Let the square root of $2 + 2i\sqrt{3}$ be $z = a + ib$

Using the result in Box 13,

$a^2 - b^2 = 2 \quad (1)$

$ab = \sqrt{3} \qquad (2)$

Substituting (2) into (1):

$$a^2 - \left(\frac{\sqrt{3}}{a}\right)^2 = 2$$

$$a^2 - \frac{3}{a^2} = 2$$

$$a^4 - 3 = 2a^2$$

$$a^4 - 2a^2 + 3 = 0$$

$$(a^2 - 3)(a^2 + 1) = 0$$

Since a is real, $a^2 > 0$ and hence $a^2 = 3$ so $a = \pm\sqrt{3}$.

Now, using (2):

When $a = \sqrt{3}, b = 1$ and hence $z = (\sqrt{3} + i)$.

When $a = -\sqrt{3}, b = -1$ and hence $z = -(\sqrt{3} + i)$.

15d Let the square root of $10 - 24i$ be $z = a + ib$

Using the result in Box 13,

$a^2 - b^2 = 10 \quad (1)$

$ab = -12 \qquad (2)$

Substituting (2) into (1):

$$a^2 - \left(\frac{-12}{a}\right)^2 = 10$$

$$a^2 - \frac{144}{a^2} = 10$$

Uncorrected proofs

$a^4 - 144 = 10a^2$

$a^4 - 10a^2 - 144 = 0$

$(a^2 - 18)(a^2 + 8) = 0$

$a^2 = 18 \text{ or } -8$

Since a is real, $a^2 > 0$ and hence $a^2 = 18$ so $a = \pm 3\sqrt{2}$.

Now, using (2):

When $a = 3\sqrt{2}$, $b = -2\sqrt{2}$ and hence $z = \sqrt{2}(3 - 2i)$.

When $a = -3\sqrt{2}$, $b = 2\sqrt{2}$ and hence $z = -\sqrt{2}(3 - 2i)$.

15e Let the square root of $2 - 4i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = 2 \quad (1)$

$ab = -2 \quad (2)$

Substituting (2) into (1):

$a^2 - \left(\frac{-2}{a}\right)^2 = 2$

$a^2 - \frac{4}{a^2} = 2$

$a^4 - 4 = 2a^2$

$a^4 - 2a^2 - 4 = 0$

$a^2 = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2}$

$a^2 = \frac{2 \pm \sqrt{20}}{2}$

$a^2 = \frac{2 \pm 2\sqrt{5}}{2}$

$a^2 = 1 \pm \sqrt{5}$

Since a is real, $a^2 > 0$ and hence $a^2 = 1 + \sqrt{5}$ so $a = \pm\sqrt{\sqrt{5} + 1}$.

Now, using (2):

When $a = \sqrt{\sqrt{5} + 1}$,

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$$b = -\frac{2}{\sqrt{\sqrt{5}+1}}$$
$$= \frac{-2\sqrt{\sqrt{5}-1}}{\sqrt{(\sqrt{5}-1)(\sqrt{5}+1)}}$$
$$= \frac{-2\sqrt{\sqrt{5}-1}}{\sqrt{5-1}}$$
$$= \frac{-2\sqrt{\sqrt{5}-1}}{2}$$
$$= -\sqrt{\sqrt{5}-1}$$

and hence $z = (\sqrt{\sqrt{5}+1} - i\sqrt{\sqrt{5}-1})$

When $a = -\sqrt{\sqrt{5}+1}$,

$$b = --\frac{2}{\sqrt{\sqrt{5}+1}}$$
$$= \frac{2\sqrt{\sqrt{5}-1}}{\sqrt{(\sqrt{5}-1)(\sqrt{5}+1)}}$$
$$= \frac{2\sqrt{\sqrt{5}-1}}{\sqrt{5-1}}$$
$$= \frac{2\sqrt{\sqrt{5}-1}}{2}$$
$$= \sqrt{\sqrt{5}-1}$$

and hence $z = -(\sqrt{\sqrt{5}+1} - i\sqrt{\sqrt{5}-1})$

16a $z^2 + (4+2i)z + (1+2i) = 0$

$\Delta = b^2 - 4ac$
$= (4+2i)^2 - 4(1)(1+2i)$
$= 16 + 16i + 4i^2 - 4 - 8i$
$= 16i + 4i^2 - 8i$
$= 8 + 8i$

Let $\lambda = a + bi$ be a square root of Δ.

Uncorrected proofs

Using Box 13,

$a^2 - b^2 = 8$ (1)

$ab = 4$ (2)

Substituting (2) into (1):

$$a^2 - \left(\frac{4}{a}\right)^2 = 8$$

$$a^2 - \frac{16}{a^2} = 8$$

$$a^4 - 16 = 8a^2$$

$$a^4 - 8a^2 - 16 = 0$$

$$a^2 = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-16)}}{2}$$

$$a^2 = \frac{8 \pm \sqrt{128}}{2}$$

$$a^2 = \frac{8 \pm 8\sqrt{2}}{2}$$

$$a^2 = 4 \pm 4\sqrt{2}$$

Since $a^2 \geq 0$ as a is real, it follows that $a^2 = 4 + 4\sqrt{2}$. Thus

$$a = \pm\sqrt{4 + 4\sqrt{2}} = \pm 2\sqrt{1 + \sqrt{2}}$$

When $a = 2\sqrt{1 + \sqrt{2}}$,

$$b = \frac{4}{2\sqrt{1 + \sqrt{2}}}$$
$$= \frac{2}{\sqrt{\sqrt{2} + 1}}$$
$$= \frac{2\sqrt{\sqrt{2} - 1}}{\sqrt{\sqrt{2} + 1}\sqrt{\sqrt{2} - 1}}$$
$$= \frac{2\sqrt{\sqrt{2} - 1}}{\sqrt{2 - 1}}$$
$$= 2\sqrt{\sqrt{2} - 1}$$

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When $a = -2\sqrt{1+\sqrt{2}}$,

$$b = -\frac{4}{2\sqrt{1+\sqrt{2}}}$$
$$= -\frac{2}{\sqrt{\sqrt{2}+1}}$$
$$= -\frac{2\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}+1}\sqrt{\sqrt{2}-1}}$$
$$= -\frac{2\sqrt{\sqrt{2}-1}}{\sqrt{2-1}}$$
$$= -2\sqrt{\sqrt{2}-1}$$

Hence $\lambda = \pm\left(2\sqrt{\sqrt{2}+1} - i2\sqrt{\sqrt{2}-1}\right)$

Hence the roots of the equation are

$$z = \frac{-b \pm \lambda}{2a}$$

$$z = \frac{-(4+2i) \pm \left(2\sqrt{\sqrt{2}+1} - i2\sqrt{\sqrt{2}-1}\right)}{2(1)}$$

$$z = -2 - i \pm \left(\sqrt{\sqrt{2}+1} - i\sqrt{\sqrt{2}-1}\right)$$

16b $z^2 - 2(1+i)z + (2+6i) = 0$

$\Delta = b^2 - 4ac$

$= \left(-2(1+i)\right)^2 - 4(1)(2+6i)$

$= 4(1+2i-1) - 8 - 24i$

$= 8i - 8 - 24i$

$= -8 - 16i$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = -8$ (1)

$ab = -8$ (2)

Substituting (2) into (1):

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$$a^2 - \left(-\frac{8}{a}\right)^2 = -8$$

$$a^2 - \frac{64}{a^2} = -8$$

$$a^4 - 64 = -8a^2$$

$$a^4 + 8a^2 - 64 = 0$$

$$a^2 = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(-64)}}{2}$$

$$a^2 = \frac{-8 \pm \sqrt{320}}{2}$$

$$a^2 = \frac{-8 \pm 8\sqrt{5}}{2}$$

$$a^2 = -4 \pm 4\sqrt{5}$$

Since a is real $a^2 \geq 0$ it follows that $a^2 = -4 + 4\sqrt{5}$ thus

$$a = \pm 2\sqrt{-1 + \sqrt{5}}$$

When $a = 2\sqrt{-1 + \sqrt{5}}$,

$$\begin{aligned} b &= \frac{-8}{2\sqrt{-1 + \sqrt{5}}} \\ &= -\frac{4}{\sqrt{\sqrt{5} - 1}} \\ &= -\frac{4\sqrt{\sqrt{5} + 1}}{\sqrt{\sqrt{5} - 1}\sqrt{\sqrt{5} + 1}} \\ &= -\frac{4\sqrt{\sqrt{5} + 1}}{\sqrt{5 - 1}} \\ &= -\frac{4\sqrt{\sqrt{5} + 1}}{2} \\ &= -2\sqrt{\sqrt{5} + 1} \end{aligned}$$

When $a = -2\sqrt{-1 + \sqrt{5}}$,

$$b = -\frac{-8}{2\sqrt{-1 + \sqrt{5}}}$$

Uncorrected proofs

$$= \frac{4}{\sqrt{\sqrt{5}-1}}$$
$$= \frac{4\sqrt{\sqrt{5}+1}}{\sqrt{\sqrt{5}-1}\sqrt{\sqrt{5}+1}}$$
$$= \frac{4\sqrt{\sqrt{5}+1}}{\sqrt{5-1}}$$
$$= \frac{4\sqrt{\sqrt{5}+1}}{2}$$
$$= 2\sqrt{\sqrt{5}+1}$$

Thus the roots of the equation are

$$z = \frac{-\left(-2(1+i)\right) \pm \left(2\sqrt{\sqrt{5}-1} - 2\sqrt{\sqrt{5}+1}\right)}{2}$$

$$z = 1 + i \pm \left(\sqrt{\sqrt{5}-1} - i\sqrt{\sqrt{5}+1}\right)$$

16c $z^2 + 2\left(1 - i\sqrt{3}\right)z + 2 + 2i\sqrt{3} = 0$

$\Delta = b^2 - 4ac$

$= \left(2\left(1 - i\sqrt{3}\right)\right)^2 - 4(1)\left(2 + 2i\sqrt{3}\right)$

$= 4\left(1 - 2i\sqrt{3} - 3\right) - 8 - 8i\sqrt{3}$

$= 4\left(-2i\sqrt{3} - 2\right) - 8 - 8i\sqrt{3}$

$= -8i\sqrt{3} - 8 - 8 - 8i\sqrt{3}$

$= -16 - 16i\sqrt{3}$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = -16$ (1)

$ab = -8\sqrt{3}$ (2)

Substituting (2) into (1):

$$a^2 - \left(-\frac{8\sqrt{3}}{a}\right)^2 = -16$$

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$$a^2 - \frac{192}{a^2} = -16$$

$$a^4 - 192 = -16a^2$$

$$a^4 + 16a^2 - 192 = 0$$

$$a^2 = \frac{-(16) \pm \sqrt{(-16)^2 - 4(1)(-192)}}{2}$$

$$a^2 = \frac{-(16) \pm \sqrt{1024}}{2}$$

$$a^2 = \frac{-16 \pm 32}{2}$$

$a^2 = 8$ (since $a^2 > 0$)

$$a = \pm 2\sqrt{2}$$

When $a = 2\sqrt{2}$,

$$b = -\frac{8\sqrt{3}}{2\sqrt{2}}$$
$$= -\frac{4\sqrt{3}}{\sqrt{2}}$$
$$= -\frac{4\sqrt{6}}{2}$$
$$= -2\sqrt{6}$$

When $a = -2\sqrt{2}$,

$$b = -\left(-\frac{8\sqrt{3}}{2\sqrt{2}}\right)$$
$$= \frac{4\sqrt{3}}{\sqrt{2}}$$
$$= \frac{4\sqrt{6}}{2}$$
$$= 2\sqrt{6}$$

So $\lambda = \pm\left(2\sqrt{2} - 2\sqrt{6}i\right)$

Hence the roots of the equation are

$$z = \frac{-2\left(1 - i\sqrt{3}\right) \pm \left(2\sqrt{2} - 2\sqrt{6}i\right)}{2(1)}$$

$$= -1 + i\sqrt{3} \pm \left(\sqrt{2} - i\sqrt{6}\right)$$

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16d $z^2 + (1-i)z + (i-1) = 0$

$\Delta = b^2 - 4ac$

$= (1-i)^2 - 4(1)(i-1)$

$= (1 - 2i - 1) - 4i + 4$

$= 4 - 6i$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = 4 \quad (1)$

$ab = -3 \quad (2)$

Substituting (2) into (1):

$a^2 - \left(\frac{-3}{a}\right)^2 = 4$

$a^2 - \frac{9}{a^2} = 4$

$a^4 - 9 = 4a^2$

$a^4 - 4a^2 - 9 = 0$

$a^2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-9)}}{2(1)}$

$a^2 = \frac{4 \pm \sqrt{52}}{2}$

$a^2 = \frac{4 \pm 2\sqrt{13}}{2}$

$a^2 = 2 \pm \sqrt{13}$

Since $a^2 \geq 0$, $a^2 = 2 + \sqrt{13}$

$a = \pm\sqrt{\sqrt{13} + 2}$

When $a = \sqrt{\sqrt{13} + 2}$,

$b = \frac{-3}{\sqrt{\sqrt{13} + 2}}$

$= \frac{-3\sqrt{\sqrt{13} - 2}}{\sqrt{\sqrt{13} - 2}\sqrt{\sqrt{13} + 2}}$

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$$= \frac{-3\sqrt{\sqrt{13}-2}}{\sqrt{13-4}}$$

$$= -\sqrt{\sqrt{13}-2}$$

When $a = -\sqrt{\sqrt{13}+2}$,

$$b = -\frac{-3}{\sqrt{\sqrt{13}-2}}$$

$$= -\frac{-3\sqrt{\sqrt{13}-2}}{\sqrt{\sqrt{13}-2}\sqrt{\sqrt{13}+2}}$$

$$= \frac{3\sqrt{\sqrt{13}-2}}{\sqrt{13-4}}$$

$$= \sqrt{\sqrt{13}-2}$$

So $\lambda = \pm\left(\sqrt{\sqrt{13}+2} - i\sqrt{\sqrt{13}-2}\right)$

Hence the roots of the equation are

$$z = \frac{-(1-i) \pm \left(\sqrt{\sqrt{13}+2} - i\sqrt{\sqrt{13}-2}\right)}{2(1)}$$

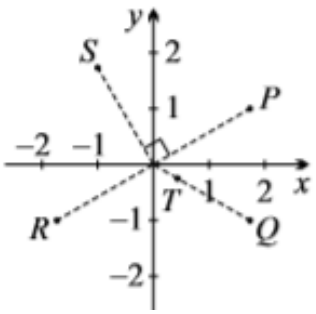
$$= \frac{1}{2}\left(-(1-i) \pm \left(\sqrt{\sqrt{13}+2} - i\sqrt{\sqrt{13}-2}\right)\right)$$

$$= \frac{1}{2}\left(-1+i \pm \left(\sqrt{\sqrt{13}+2} - i\sqrt{\sqrt{13}-2}\right)\right)$$

Uncorrected proofs

Solutions to Exercise 1C Development questions

6 P will be 2 units from the origin making an angle of $\frac{\pi}{6}$ with the x-axis. Taking the conjugate of a complex number makes the imaginary part negative and leaves the real part unaffected. Hence, we make the y-component of z negative. This is equivalent to reflecting P about the x-axis to obtain Q. Taking the negative of a complex number is equivalent to a 180° rotation or a reflection about the x and then y-axes, hence this gives the position of R. Multiplying by i is equivalent to a rotation by $\frac{\pi}{2}$ about the origin. Noting that $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{|z|^2}\bar{z}$, T must have the same angle as Q. Since $|z| > 1, \frac{1}{|z|^2} < 1$ and hence T will be closer to the origin than Q.

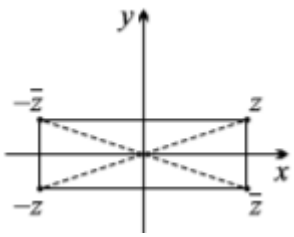


7 Let $z = x + iy$

$$\begin{aligned} -\bar{z} &= -(x - iy) \\ &= -x + iy \end{aligned}$$

Hence $-\bar{z}$ is equivalent to the transformation $(x, y) \to (-x, y)$ which is a reflection of the point representing z in the y-axis.

8



Let $z = x + iy$ so $\bar{z} = x - iy$, $-z = -x - iy$ and $-\bar{z} = -x + iy$

Consider the diagonal between $-z$ and z. It has length

$$\sqrt{(-x - x)^2 + \big(y - (-y)\big)^2}$$

Uncorrected proofs

$= \sqrt{(2x)^2 + (2y)^2}$
$= 2\sqrt{x^2 + y^2}$

The midpoint is

$$\left(\frac{-x + x}{2}, \frac{y - y}{2}\right) = (0, 0)$$

Now, the diagonal between $-\bar{z}$ and $\bar{z}$. It has length

$$\sqrt{(-x - x)^2 + \big(y - (-y)\big)^2}$$

$= \sqrt{(2x)^2 + (2y)^2}$
$= 2\sqrt{x^2 + y^2}$

The midpoint is

$$\left(\frac{-x + x}{2}, \frac{y - y}{2}\right) = (0, 0)$$

Thus, both diagonals have the same length, and as they have the same midpoint they must bisect each other. Hence the points form a rectangle.

8b Geometry of opposites: Since z and $-z$ are rotations of each other by π about the origin the midpoint of the two points must be the origin. Similarly, the midpoint of $\bar{z}$ and $-\bar{z}$ must be the origin. Thus, the two diagonals bisect one another.

Geometry of conjugates: Since z and $\bar{z}$ are reflections of one another about the real axis they must have the same length. Similarly, $-z$ and $-\bar{z}$ must have a common length. Hence it follows that both diagonals are of the same length.

Thus, the diagonals are equal and bisect one another so the points form a rectangle.

9 Note that these results follow immediately from the result in the book as z being real and z being imaginary are just special cases of z being complex.

9a Consider x where x is real, this lies on the x-axis. Multiplying by i gives ix. This lies on the y-axis. If $x > 0$ then x and ix are on the positive x- and y-axes

Uncorrected proofs

respectively. Hence this is a rotation by $\frac{\pi}{2}$. If $x < 0$ then x and ix are on the negative x- and y-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

9b Consider ix where x is real, this lies on the x-axis.

Multiplying by i gives $i^2x = -x$ this lies on the y-axis.

If $x > 0$ then ix and $-x$ are on the positive y and negative x-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

If $x < 0$ then ix and $-x$ are on the negative y and positive x-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

10a

$$m_{OA} = \frac{b-0}{a-0} = \frac{b}{a}$$

$$\begin{aligned} w &= i(a+ib) \\ &= ai - b \\ &= -b + ai \end{aligned}$$

$$\begin{aligned} m_{OB} &= \frac{a-0}{-b-0} \\ &= -\frac{a}{b} \\ &= -\frac{1}{m_{OA}} \end{aligned}$$

Hence it follows that $OA \perp OB$.

10b $OA = \sqrt{a^2 + b^2}$

$$\begin{aligned} OB &= \sqrt{a^2 + (-b)^2} \\ &= \sqrt{a^2 + b^2} \\ &= OA \end{aligned}$$

10c It has a right angle and two sides are of the same length. Hence it is right-isosceles triangle.

Uncorrected proofs

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11

$$\frac{1}{z}-\frac{1}{\bar{z}}=i$$

Multiplying through by $z\bar{z}$ (note that $z \neq 0$),

$$\bar{z}-z=iz\bar{z}$$

Let $z=x+iy$.

$$(x-iy)-(x+iy)=i|z|^2$$

$$-2iy=i|z|^2$$

$$-2y=|z|^2$$

$$-2y=(x^2+y^2)$$

$$x^2+y^2+2y=0$$

$$x^2+y^2+2y+1=1$$

$$x^2+(y+1)^2=1$$

This is the circle with centre $(0,-1)$ and radius 1 unit, omitting the origin.

12 $\text{Re}\left(\frac{z-6}{z}\right)=0$

$$\text{Re}\left(1-\frac{6}{z}\right)=0$$

$$\text{Re}\left(1-\frac{6\bar{z}}{z\bar{z}}\right)=0$$

$$\text{Re}\left(1-\frac{6\bar{z}}{|z|^2}\right)=0$$

Let $z=x+iy$

$$\text{Re}\left(1-\frac{6}{|z|^2}(x-iy)\right)=0$$

$$\text{Re}\left(1-\frac{6}{|z|^2}x+i\frac{6y}{|z|^2}\right)=0$$

$$1-\frac{6}{|z|^2}x=0$$

$6x=|z|^2$ (note $z \neq 0$)

$$6x=x^2+y^2$$

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$x^2 - 6x + y^2 = 0$

$x^2 - 6x + 9 + y^2 = 9$

$(x - 3)^2 + y^2 = 9$

This is the circle with centre $(3, 0)$ and radius 3 units, omitting the origin.

13 $(z - 2)\overline{(z - 2)} = 9$

$(z - 2)(\bar{z} - \bar{2}) = 9$

$(z - 2)(\bar{z} - 2) = 9$

$z\bar{z} - 2z - 2\bar{z} + 4 = 9$

$|z|^2 - 2(z + \bar{z}) = 5$

Let $z = x + iy$

$x^2 + y^2 - 2(x + iy + x - iy) = 5$

$x^2 + y^2 - 4x = 5$

$x^2 - 4x + 4 + y^2 = 9$

$(x - 2)^2 + y^2 = 3$

This is the equation of a circle with centre $(2, 0)$ and radius $\sqrt{3}$ units.

14 $z\bar{z} = \big(\text{Re}(z - 1 + 3i)\big)^2$

$|z|^2 = \big(\text{Re}(z - 1 + 3i)\big)^2$

Let $z = x + iy$

$x^2 + y^2 = \big(\text{Re}(x + iy - 1 + 3i)\big)^2$

$x^2 + y^2 = (x - 1)^2$

$x^2 + y^2 = x^2 - 2x + 1$

$2x = 1 - y^2$

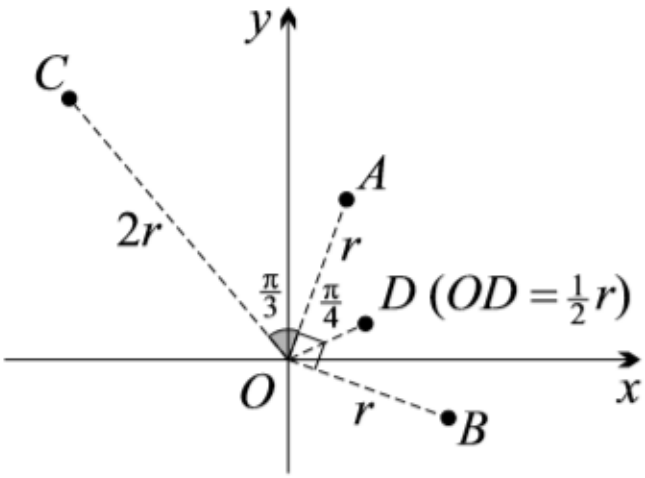
$x = \frac{1}{2}(1 - y^2)$

This is the parabola with focus the origin and directrix $x = 1$.

Uncorrected proofs

Solutions to Exercise 1D Development questions

11 Since $0 < \arg z < \frac{\pi}{2}$, we know that A must be in the first quadrant. Now, multiplication by i represents a $\frac{\pi}{2}$ anticlockwise rotation around the origin and so multiplication by $-i$ represents a $\frac{\pi}{2}$ clockwise rotation. Thus, B is found by rotating A $\frac{\pi}{2}$ radians around the origin. Multiplying by $2 \text{ cis}\left(\frac{\pi}{3}\right) z$ doubles the distance of the point from the origin and rotates it by $\frac{\pi}{3}$ radians. Multiplying by $\frac{1}{2} \text{ cis}\left(-\frac{\pi}{4}\right)$ halves the distance from the origin and rotates the point $\frac{\pi}{4}$ radians clockwise.



12a From Box 22,

$|wz| = |w||z|$

Replacing z with $z \div w$,

$|w(z \div w)| = |w||z \div w|$

$\left|w\frac{z}{w}\right| = |w|\left|\frac{z}{w}\right|$

$|z| = |w|\left|\frac{z}{w}\right|$

$\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$

Uncorrected proofs

12b From Box 22,

$\arg(zw) = \arg(w) + \arg(z)$

Replacing z with $z \div w$,

$\arg\big((z \div w)w\big) = \arg(z \div w) + \arg(w)$

$\arg\left(\frac{z}{w}w\right) = \arg\left(\frac{z}{w}\right) + \arg(w)$

$\arg(z) = \arg\left(\frac{z}{w}\right) + \arg(w)$

Hence $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$.

13a $z_1 = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}\,\text{cis}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$ (first quadrant)

$= \sqrt{4}\,\text{cis}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$= 2\,\text{cis}\frac{\pi}{6}$

$z_2 = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}\,\text{cis}\left(\tan^{-1}\frac{2\sqrt{2}}{2\sqrt{2}}\right)$ (first quadrant)

$= \sqrt{16}\,\text{cis}\,(\tan^{-1}1)$

$= 4\,\text{cis}\frac{\pi}{4}$

13b $z_1 z_2 = 2\,\text{cis}\frac{\pi}{6} \times 4\,\text{cis}\frac{\pi}{4}$

$= (2 \times 4)\,\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= 8\,\text{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= 8\,\text{cis}\frac{5\pi}{12}$

$\frac{z_2}{z_1} = \frac{4\,\text{cis}\frac{\pi}{4}}{2\,\text{cis}\frac{\pi}{6}}$

$= \frac{4}{2}\text{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= 2\,\text{cis}\frac{\pi}{12}$

Uncorrected proofs

14 $z_1 = \sqrt{(-\sqrt{3}) + (1)^2}\ \text{cis}\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$ (second quadrant)

$= \sqrt{4}\ \text{cis}\left(\pi - \frac{\pi}{6}\right)$

$= 2\ \text{cis}\frac{5\pi}{6}$

$z_2 = \sqrt{(-1)^2 + (-1)^2}\ \text{cis}\left(\tan^{-1}\left(\frac{-1}{-1}\right)\right)$ (third quadrant)

$= \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

$z_1 z_2 = 2\ \text{cis}\frac{5\pi}{6} \times \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

$= 2\sqrt{2}\ \text{cis}\left(\frac{5\pi}{6} - \frac{3\pi}{4}\right)$

$= 2\sqrt{2}\ \text{cis}\frac{\pi}{12}$

$\frac{z_2}{z_1} = \frac{\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)}{2\ \text{cis}\left(\frac{5\pi}{6}\right)}$

$= \frac{\sqrt{2}}{2}\ \text{cis}\left(-\frac{3\pi}{4} - \frac{5\pi}{6}\right)$

$= \frac{\sqrt{2}}{2}\ \text{cis}\left(-\frac{19\pi}{12}\right)$

$= \frac{\sqrt{2}}{2}\ \text{cis}\left(2\pi - \frac{19\pi}{12}\right)$

$= \frac{\sqrt{2}}{2}\ \text{cis}\left(\frac{24\pi - 19\pi}{12}\right)$

$= \frac{\sqrt{2}}{2}\ \text{cis}\frac{5\pi}{12}$

15a

$\frac{\left(1 + i\sqrt{3}\right)}{1 + i}$

$= \frac{\left(1 + i\sqrt{3}\right)(1 - i)}{(1 + i)(1 - i)}$

$= \frac{1 - i + i\sqrt{3} + i^2\sqrt{3}}{(1 + i)(1 - i)}$

$= \frac{1 - i + i\sqrt{3} - \sqrt{3}}{1^2 - i^2}$

Uncorrected proofs

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$$= \frac{(\sqrt{3}+1)+i(\sqrt{3}-1)}{2}$$
$$= \frac{1}{2}\left((\sqrt{3}+1)+i(\sqrt{3}-1)\right)$$

15b

$$\frac{1+i\sqrt{3}}{1+i}$$
$$= \frac{2\,\text{cis}\,\frac{\pi}{3}}{\sqrt{2}\,\text{cis}\,\frac{\pi}{4}}$$
$$= \frac{2}{\sqrt{2}}\text{cis}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$
$$= \sqrt{2}\,\text{cis}\,\frac{\pi}{12}$$

15c $\sqrt{2}\,\text{cis}\,\frac{\pi}{12}$

$$= \frac{1+i\sqrt{3}}{1+i}$$
$$= \frac{1}{2}\left((\sqrt{3}+1)+i(\sqrt{3}-1)\right)$$

Hence $\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right) = \frac{1}{2}\left((\sqrt{3}+1)+i(\sqrt{3}-1)\right)$

Equating the real parts in the above equation,

$$\sqrt{2}\cos\frac{\pi}{12} = \frac{1}{2}(\sqrt{3}+1)$$

Hence

$$\cos\frac{\pi}{12} = \frac{1}{2\sqrt{2}}(\sqrt{3}+1)$$

16a $|z| = \left|\frac{z_1}{z_2}\right|$

$$= \frac{|z_1|}{|z_2|}$$
$$= \frac{\sqrt{1+5^2}}{\sqrt{3^2+2^2}}$$

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$$= \frac{\sqrt{1+25}}{\sqrt{9+4}}$$
$$= \frac{\sqrt{26}}{\sqrt{13}}$$
$$= \sqrt{2}$$

16b $\tan\left(\tan^{-1}5 - \tan^{-1}\frac{2}{3}\right)$

$$= \frac{\tan(\tan^{-1}5) - \tan\left(\tan^{-1}\frac{2}{3}\right)}{1 + \tan(\tan^{-1}5)\tan\left(\tan^{-1}\frac{2}{3}\right)}$$
$$= \frac{5 - \frac{2}{3}}{1 + 5\left(\frac{2}{3}\right)}$$
$$= \frac{\frac{13}{3}}{\frac{13}{3}}$$
$$= 1$$

$\arg z = \arg z_1 - \arg z_2$
$= \tan^{-1}5 - \tan^{-1}\frac{2}{3}$

$\tan(\arg z)$

$= \tan\left(\tan^{-1}5 - \tan^{-1}\frac{2}{3}\right)$
$= 1$

$\arg z$

$= \tan^{-1}1$
$= \frac{\pi}{4}$

16c

$z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$
$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$
$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$
$= 1 + i$

Uncorrected proofs

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17a $z\bar{z} = (r \text{ cis } \theta)\big(r \text{ cis}(-\theta)\big)$
$= r^2 \text{ cis}(\theta - \theta)$
$= r^2 \text{cis } 0$
$= r^2$
$= |z|^2$

17b $\arg(z^2)$
$= \arg((r \text{ cis } \theta)^2)$
$= \arg\big(r^2 \text{ cis}(\theta + \theta)\big)$
$= \arg(r^2 \text{ cis } 2\theta)$
$= 2\theta$
$= 2 \arg(r \text{ cis } \theta)$
$= 2 \arg(z)$

17c $|z| = 1$
$z\bar{z} = 1$
$\bar{z} = \dfrac{1}{z}$ (as z is non zero)
$\bar{z} = z^{-1}$

18a Note that $|z|^2$ is a real number and as such its argument must be zero, that is
$\arg(|z|^2) = 0$
$\arg z\bar{z} = 0$
$\arg z + \arg \bar{z} = 0$
$\arg \bar{z} = -\arg z$

19a, b $z^2 = (\cos\theta + i \sin\theta)^2$
$= \cos^2\theta + 2i \sin\theta \cos\theta + i^2 \sin^2\theta$
$= \cos^2\theta + 2i \sin\theta \cos\theta - \sin^2\theta$
$= \cos^2\theta - \sin^2\theta + 2i \sin\theta \cos\theta$

$z^2 = (\cos\theta + i \sin\theta)^2$
$= (\text{cis}\,\theta)^2$
$= \text{cis } 2\theta$

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$= \cos 2\theta + i \sin 2\theta$

Thus $\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$

Equating the real and imaginary coefficients in the above equation,

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

20a $|z-1| = 1$

Let $z = x + iy$.

$|x + iy - 1| = 1$

$|x - 1 + iy| = 1$

$|x - 1 + iy|^2 = 1$

$(x-1)^2 + y^2 = 1$

$x^2 - 2x + 1 + y^2 = 1$

$x^2 + y^2 = 2x$

$|z|^2 = 2x$

$|z|^2 = 2\text{Re}(z)$

20b $|z-1| = 1$

$|z-1|^2 = 1$

$(z-1)\overline{z-1} = 1$

$(z-1)(\bar{z}-\bar{1}) = 1$

$(z-1)(\bar{z}-1) = 1$

$z\bar{z} - z - \bar{z} + 1 = 1$

$z\bar{z} - z - \bar{z} = 0$

$z\bar{z} = z + \bar{z}$

$|z|^2 = z + \bar{z}$

$|z|^2 = x + iy + x - iy$

$|z|^2 = 2x$

$|z|^2 = 2\text{Re(z)}$

Uncorrected proofs

21a $|2z-1| = |z-2|$

Let $z = x + iy$.

$|2(x+iy)-1| = |(x+iy)-2|$

$|(2x-1)+iy| = |(x-2)+iy|$

$|(2x-1)+iy|^2 = |(x-2)+iy|^2$

$(2x-1)^2 + y^2 = (x-2)^2 + y^2$

$(2x-1)^2 = (x-2)^2$

$4x^2 - 4x + 1 = x^2 - 4x + 4$

$3x^2 = 3$

$x^2 = 1$

$x = \pm 1$

21b $|2z-1|^2 = |z-2|^2$

$(2z-1)\overline{2z-1} = (z-2)\overline{z-2}$

$(2z-1)(2\bar{z}-1) = (z-2)(\bar{z}-2)$

$4z\bar{z} - 2z - 2\bar{z} + 1 = z\bar{z} - 2z - 2\bar{z} + 4$

$3z\bar{z} = 3$

$z\bar{z} = 1$

$|z|^2 = 1$

22a $z = 1 + \cos\theta + i\sin\theta$

$$\begin{aligned}|z| &= |1 + \cos\theta + i\sin\theta| \\ &= \sqrt{(1+\cos\theta)^2 + (\sin\theta)^2} \\ &= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \\ &= \sqrt{1 + 2\cos\theta + 1} \\ &= \sqrt{2 + 2\cos\theta} \\ &= \sqrt{2 + 2\cos 2\left(\frac{\theta}{2}\right)}\end{aligned}$$

Uncorrected proofs

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$$= \sqrt{2 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)}$$
$$= \sqrt{4\cos^2\frac{\theta}{2}}$$
$$= 2\cos\frac{\theta}{2}$$

$\arg z$

$= \arg(1 + \cos\theta + i\sin\theta)$
$= \tan^{-1}\left(\frac{\sin\theta}{1 + \cos\theta}\right)$
$= \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)}\right)$
$= \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$
$= \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$
$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$
$= \frac{\theta}{2}$

22b

$z = 2\cos\frac{\theta}{2}\text{cis}\frac{\theta}{2}$

$z^{-1} = \frac{1}{z}$
$= \frac{1}{2\cos\frac{\theta}{2}\text{cis}\frac{\theta}{2}}$
$= \frac{1}{2\cos\frac{\theta}{2}}\text{cis}\left(-\frac{\theta}{2}\right)$
$= \frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$

Uncorrected proofs

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$$= \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right)$$
$$= \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}$$

Uncorrected proofs

Solutions to Exercise 1E Development questions

7 w can be obtained by rotating z by $\frac{\pi}{2}$ radians. Hence it follows that $w = iz$.

Hence

$z^2 + w^2$

$= z^2 + (iz)^2$

$= z^2 - z^2$

$= 0$

8 $E = OE$

$= AB$

$= OB - OA$

$= w_2 - w_1$

F is a $\frac{\pi}{2}$ rotation from E, hence $F = i(w_2 - w_1)$

$C = OC$

$= OB + BC$

$= OB + OF$

$= w_2 + i(w_2 - w_1)$

$D = OD$

$= OA + AD$

$= OA + OF$

$= w_1 + i(w_2 - w_1)$

9a $z_1 - z_2$ represents the vector BA and $z_3 - z_2$ represents the vector BC.

Since BA is $\frac{\pi}{2}$ radians anticlockwise of BC it follows that $(z_1 - z_2) = i(z_3 - z_2)$.

$(z_1 - z_2)^2 + (z_3 - z_2)^2$

$= \big(i(z_3 - z_2)\big)^2 + (z_3 - z_2)^2$

$= -(z_3 - z_2)^2 + (z_3 - z_2)^2$

$= 0$

Uncorrected proofs

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Vectors BA and BC represent $z_1 - z_2$ and $z_3 - z_2$ repsectivly, and BA is the anticlockwise otation of BC through $\frac{\pi}{2}$ about B. So $z_1 - z_2 = i(z_3 - z_2)$. Squaring both sides gives the result.

9b $D = OD$

$= OA + AD$

$= z_1 + BC$

$= z_1 + (z_3 - z_2)$

$= z_1 - z_2 + z_3$

10a OC is formed by taking the vector OA, rotating it by $\frac{\pi}{2}$ (multiplying ω by i) and then doubling the length of the rotated vector. This gives $2i\omega$.

10b The diagonals of a rectangle bisect one another. Hence the point of intersection will be at the midpoint of A and C.

This is at

$$\frac{\omega + 2i\omega}{2} = \frac{1}{2}\omega(1 + 2i)$$

11 The angle subtended by the vertices will be $\frac{2\pi}{3}$ radians (they divide a full rotation, 2π, into 3 equal segments). Hence the other vertices are obtained by rotating by $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$.

This is equivalent to multiplying $1 + \sqrt{3}i$ by

$$\text{cis}\left(\pm\frac{2\pi}{3}\right) = \cos\left(\pm\frac{2\pi}{3}\right) + i\sin\left(\pm\frac{2\pi}{3}\right) = \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

So the other two vertices are given by

$$\left(1 + \sqrt{3}i\right)\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2}\left(1 + \sqrt{3}i\right)\left(1 \pm \sqrt{3}i\right)$$

$$= -\tfrac{1}{2}(1 - 3i^2) \text{ or } -\tfrac{1}{2}\left(1 + 2\sqrt{3}i + 3i^2\right)$$

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$= -2$ or $1 - i\sqrt{3}$

$= -2$ and $1 - \sqrt{3}i$

12a This is done by simply rotating by $\pm\frac{\pi}{2}$ about the origin. To do this, multiply z by $\pm i$ to give

$\pm iz$

$= \pm i(3 + 4i)$

$= \pm(3i - 4)$

$= \pm(4 - 3i)$

Hence $w = -4 + 3i$ or $4 - 3i$.

12b To make the right angle at z itself we simply need to translate the two vectors from part (a) by z. This gives

$3 + 4i \pm (4 - 3i) = 7 + i$ or $-1 + 7i$

Hence $w = -1 + 7i$ or $7 + i$

12c In this case the line OW and WZ must be perpendicular, hence

$w = \pm i(w - z)$

$w = iw - iz$ or $w = -iw + iz$

$(1 - i)w = -iz$ or $(1 + i)w = iz$

$w = \left(-\frac{i}{1-i}\right)z$ or $w = \left(\frac{i}{1+i}\right)z$

$w = \left(-\frac{i(1+i)}{2}\right)z$ or $w = \left(\frac{i(1-i)}{2}\right)z$

$w = \left(\frac{1-i}{2}\right)z$ or $w = \left(\frac{1+i}{2}\right)z$

$w = \left(\frac{1-i}{2}\right)(3 + 4i)$ or $w = \left(\frac{1+i}{2}\right)(3 + 4i)$

$w = \frac{1}{2}(3 + 4i - 3i - 4i^2)$ or $w = \frac{1}{2}(3 + 4i + 3i + 4i^2)$

Hence $w = \frac{1}{2}(7 + i)$ or $\frac{1}{2}(-1 + 7i)$

Uncorrected proofs

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13 In order to be parallelogram, the opposite sides must be parallel and of equal length. Another way of saying this is that they must be the same vector. Hence, we must solve for the following equations:

$z_4 - z_3 = z_2 - z_1$

$z_4 = z_3 + z_2 - z_1$

$= -1 + 7i \ + \ 2 + 6i - (1 + i)$

$= 12i$

$z_4 - z_3 = z_1 - z_2$

$z_4 = z_3 + z_1 - z_2$

$= -1 + 7i + 1 + i - (2 + 6i)$

$= -2 + 2i$

$z_4 - z_1 = z_2 - z_3$

$z_4 = z_2 + z_1 - z_3$

$= 2 + 6i + 1 + i - (-1 + 7i)$

$= 4$

Three possible values of z_4 are $-2 + 2i$, $12i$, 4.

14 Since z has a modulus of 1, the quadrilateral formed by the points O, 1, z and $1 + z$ has all sides of equal length. The diagonal of a rhombus bisects the angle at the corner of the rhombus. Thus, the line between 0 and $(1 + z)$ bisects the angle between the line from 0 to 1 and the line from 0 to z.

That is, $\text{Arg}(z + 1) = \frac{1}{2}\text{Arg}\, z$ or $2\,\text{Arg}(z + 1) = \text{Arg}\, z$.

15a $z_1 - z_2$ represents AB and $z_4 - z_3$ represents CD hence $AB = CD$ so the opposite sides are both equal in length and parallel. Thus, $ABCD$ is a parallelogram.

Uncorrected proofs

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15b $z_1 - z_3$ represents AC whilst $z_4 - z_2$ represents DB. Since $z_1 - z_3 = i(z_4 - z_2)$ it follows that

$$\arg(z_1 - z_3)$$

$$= \arg i(z_4 - z_2)$$

$$= \arg i + \arg(z_4 - z_2)$$

$$= \frac{\pi}{2} + \arg(z_4 - z_2)$$

This means that AC and DB are perpendicular. Since the diagonals of the parallelogram are perpendicular, it must be a square.

16

$$\frac{z_2 - z_1}{z_3 - z_1} = \text{cis}\frac{\pi}{3}$$

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = \left|\text{cis}\frac{\pi}{3}\right|$$

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = 1$$

$$|z_2 - z_1| = |z_3 - z_2|$$

This means that the triangle is isosceles.

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\text{cis}\frac{\pi}{3}\right)$$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{3}$$

The angle between the two equal sides is $\frac{\pi}{3}$.

Hence the triangle is equilateral.

17

$$\frac{z_2}{z_1} = \frac{z_3}{z_2}$$

$$\arg\left(\frac{z_2}{z_1}\right) = \arg\left(\frac{z_3}{z_2}\right)$$

$$\arg z_2 - \arg z_1 = \arg z_3 - \arg z_2$$

Hence $\angle P_2OP_1 = \angle P_3OP_2$.

Uncorrected proofs

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So OP_2 bisects P_1OP_3.

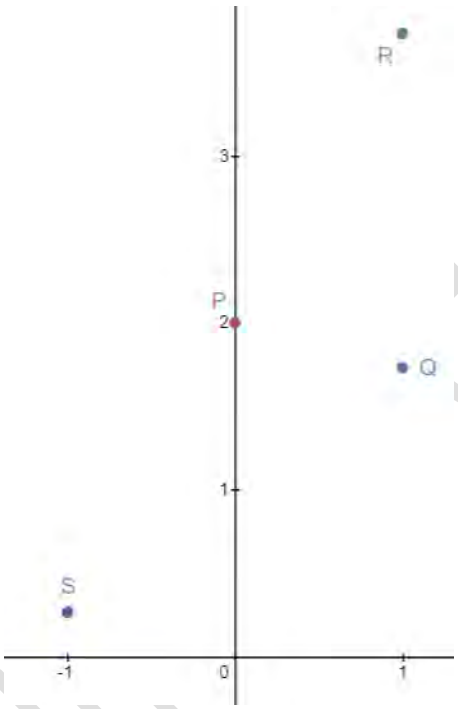
18a $i = \text{cis}\frac{\pi}{2}$ so $z_1 = 2\,\text{cis}\frac{\pi}{2}$

$z_2 = \sqrt{1+\left(\sqrt{3}\right)^2}\,\text{cis}(\tan^{-1}\sqrt{3})$

$= 2\,\text{cis}\,\frac{\pi}{3}$

$z_1 = 2\,\text{cis}\frac{\pi}{2}, z_2 = 2\,\text{cis}\frac{\pi}{3}$

18b



18c i $\arg(z_1 + z_2)$

$= \arg\left(2i + 1 + \sqrt{3}i\right)$

$= \arg\left(1 + \left(2 + \sqrt{3}\right)i\right)$

$= \tan^{-1}\left(\frac{2+\sqrt{3}}{1}\right)$

$= \tan^{-1}\left(2 + \sqrt{3}\right)$

Uncorrected proofs

$= \dfrac{5\pi}{12}$

18c ii $\arg(z_1 - z_2)$

$= \arg(2i - 1 - \sqrt{3}i)$

$= \arg(-1 + (2 - \sqrt{3})i)$

$= \tan^{-1}\left(\dfrac{2 - \sqrt{3}}{-1}\right)$

$= \tan^{-1}(\sqrt{3} - 2)$

$= \dfrac{11\pi}{12}$

19a $|z|^2 = |x + iy|$

$= x^2 + y^2$

$= x^2 - i^2y^2$

$= (x + iy)(x - iy)$

$= z\bar{z}$

19b $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$

$= (z_1 + z_2)(\bar{z_1} + \bar{z_2}) + (z_1 - z_2)(\bar{z_1} - \bar{z_2})$

$= z_1\bar{z_1} + z_2\bar{z_1} + z_1\bar{z_2} + z_2\bar{z_2} + z_1\bar{z_1} - z_2\bar{z_1} - z_1\bar{z_2} + z_2\bar{z_2}$

$= z_1\bar{z_1} + z_2\bar{z_2} + z_1\bar{z_1} + z_2\bar{z_2}$

$= 2(z_1\bar{z_1} + z_2\bar{z_2})$

$= 2(|z_1|^2 + |z_2|^2)$

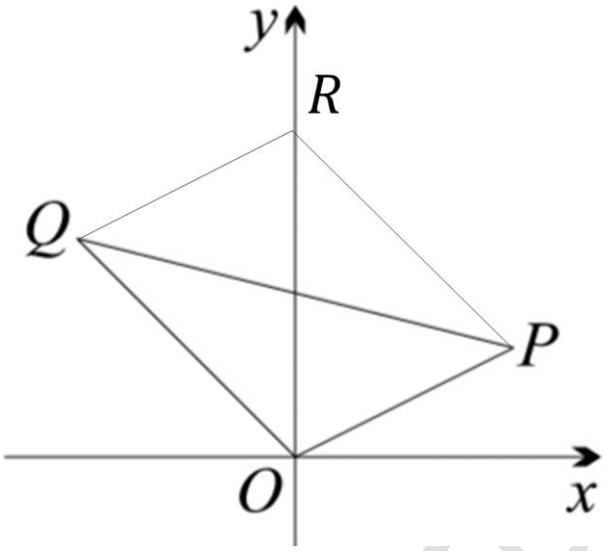
19c The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

Uncorrected proofs

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20a $|z-w|$ represents the length PQ, $|z|$ is the length of OP and $|w|$ is the length of OQ. The length of any one side of a triangle must be less than or equal to the sum of the lengths of the other two sides of the triangle (this is the triangle inequality) and hence $|z-w| \leq |z| + |w|$.

20b



b

20c This is a parallelogram as all opposing sides are parallel.

20d This implies that the diagonals are of equal length and thus $OPRQ$ is a square, this means that OP and OQ are perpendicular hence $\frac{w}{z}$ is a multiple of i and purely imaginary.

$\arg\frac{z}{w} = \frac{\pi}{2}$, so $\frac{w}{z}$ is purely imaginary.

21a

If $\frac{z_3-z_1}{z_2-z_1}$ is real then

$$\arg\left(\frac{z_3-z_1}{z_2-z_1}\right) = 0$$

$$\arg(z_3-z_1) - \arg(z_2-z_1) = 0$$

So $z_3 - z_1$ and $z_2 - z_1$ are parallel and thus z_1, z_2 and z_3 are collinear.

Uncorrected proofs

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Since z_1, z_2 and z_3 are collinear, it follows that the vectors $z_3 - z_1$ and $z_2 - z_1$ must be parallel. This means that $z_3 - z_1$ must be a scalar multiple of $z_2 - z_1$. Hence $z_3 - z_1 = \alpha(z_2 - z_1)$ where α is real. Thus $\frac{z_3 - z_1}{z_2 - z_1} = \frac{\alpha(z_2 - z_1)}{z_2 - z_1} = \alpha$ which is real, as required.

21b

$$\left(\frac{19 + 29i - (5 + 8i)}{(13 + 20i) - (5 + 8i)}\right)$$

$$= \frac{14 + 21i}{8 + 12i}$$

$$= \frac{7(2 + 3i)}{4(2 + 3i)}$$

$$= \frac{7}{4}$$

Thus $\frac{z_3 - z_1}{z_2 - z_1}$ is real and hence it follows that the three points are collinear.

Uncorrected proofs

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Solutions to Exercise 1F Development questions

5a $|z - 2 + i| \leq 2$

$|z - 2 + i| = 2$ is a circle with centre $2 - i$ or $(2, -1)$ and radius 2 units.

Alternatively, let $z = x + iy$.

$|(x + iy) - 2 + i| \leq 2$

$|(x - 2) + (y + 1)i| \leq 2$

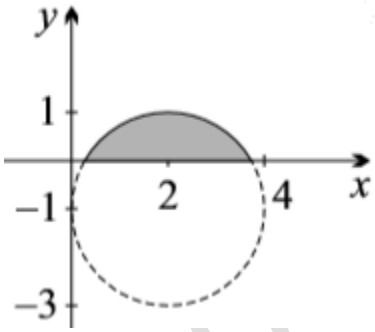
$|(x - 2) + (y + 1)i|^2 \leq 4$

$(x - 2)^2 + (y + 1)^2 \leq 4$

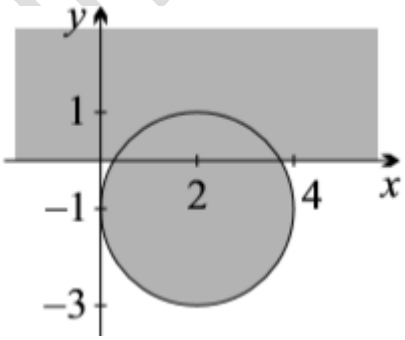
$\text{Im}(z) \geq 0$

$y \geq 0$

5a i



5a ii



Uncorrected proofs

5b Let $z = x + iy$.

$0 \leq \text{Re}(z) \leq 2$

$0 \leq x \leq 2$

$|z - 1 + i| \leq 2$

$|z - 1 + i| = 2$ is a circle with centre $1 - i$ or $(1, -1)$ and radius 2 units.

Alternatively,

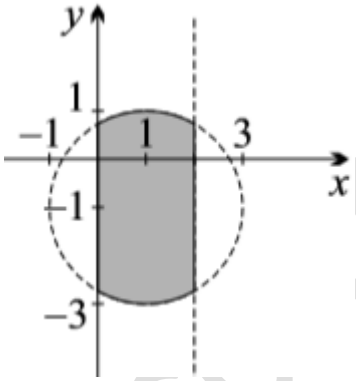
$|x + iy - 1 + i| \leq 2$

$|(x - 1) + i(y - 1)| \leq 2$

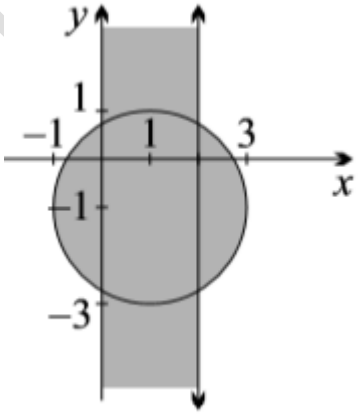
$|(x - 1) + i(y - 1)|^2 \leq 4$

$(x - 1)^2 + (y - 1)^2 \leq 4$

5b i



5b ii



Uncorrected proofs

5c Let $z = x + iy$

$|z - \bar{z}| < 2$

$|(x + iy) - (x - iy)| < 2$

$|2iy| < 2$

$2|y| < 2$

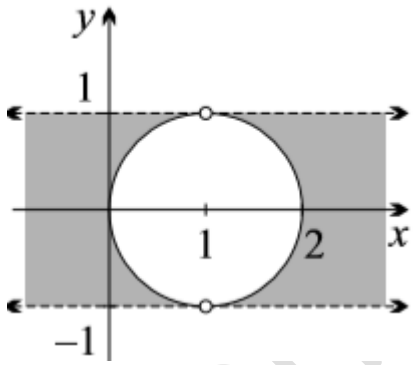
$|y| < 1$

$-1 < y < 1$

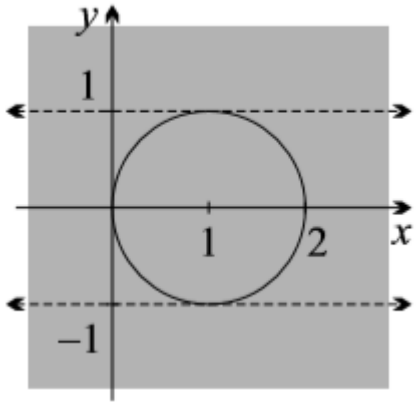
$|z - 1| \geq 1$

$|z - 1| = 1$ is a circle with centre $(1, 0)$ and radius 1 unit.

5c i



5c ii



Uncorrected proofs

5d Let $z = x + iy$

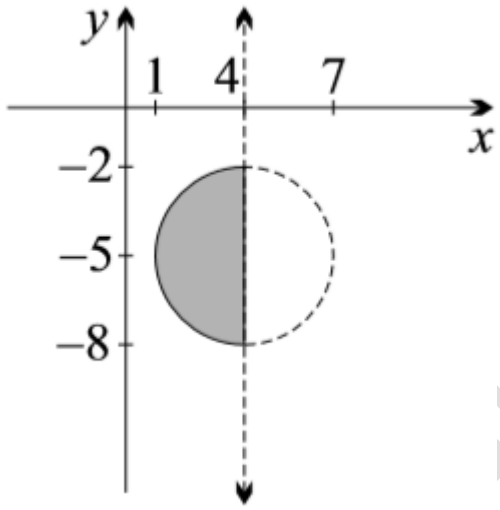
$\text{Re}(z) \leq 4$

$x \leq 4$

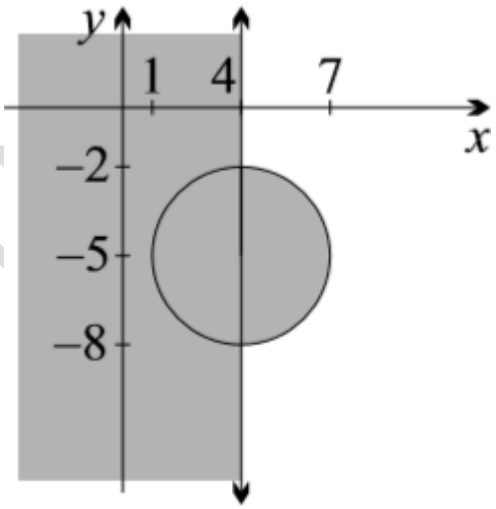
$|z - 4 + 5i| \leq 3$

$|z - 4 + 5i| = 3$ is a circle with centre $4 - 5i$ or $(4, -5)$ and radius 3 units.

5d i



5d ii



Unedited proofs

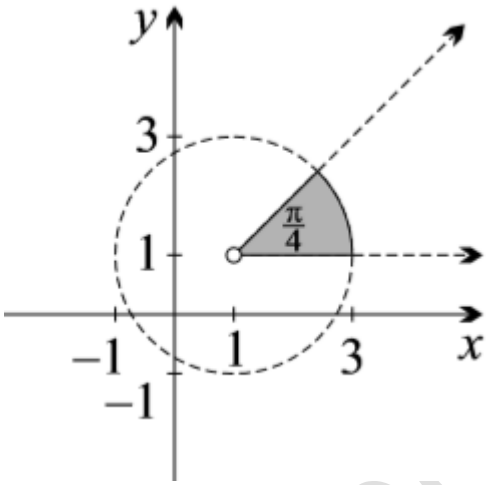
5e $|z - 1 - i| \leq 2$

$|z - 1 - i| = 2$ is a circle with centre $1 + i$ or $(1, 1)$ and radius 2 units.

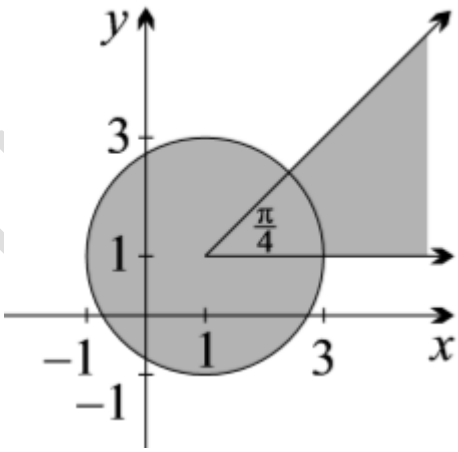
$0 \leq \arg(z - 1 - i) \leq \frac{\pi}{4}$

Vertex of angle at $1 + i$ or $(1, 1)$.

5e i



5e ii



Uncorrected proofs

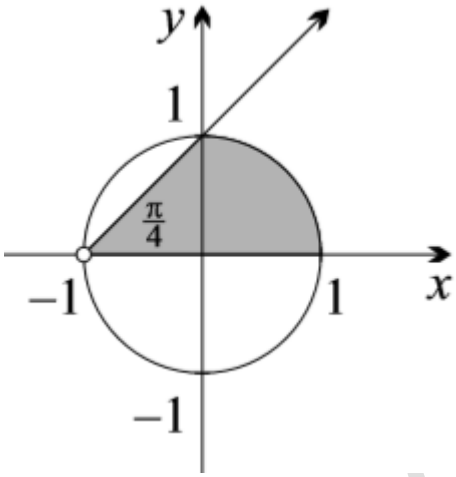
5f $|z| \leq 1$

$|z| = 1$ is a circle with centre $(0, 0)$ and radius 1 unit.

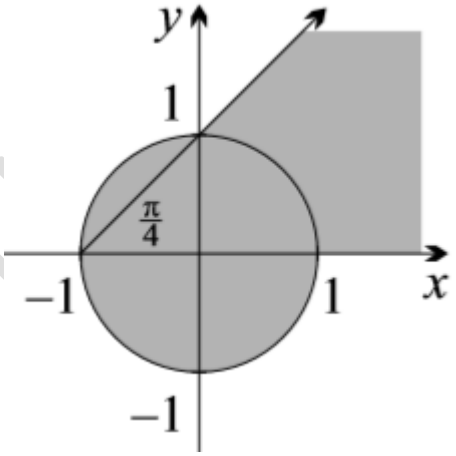
$0 \leq \arg(z + 1) \leq \frac{\pi}{4}$

Vertex of angle at $-1 + 0i$ or $(-1, 0)$.

5f i



5f ii



Uncorrected proofs

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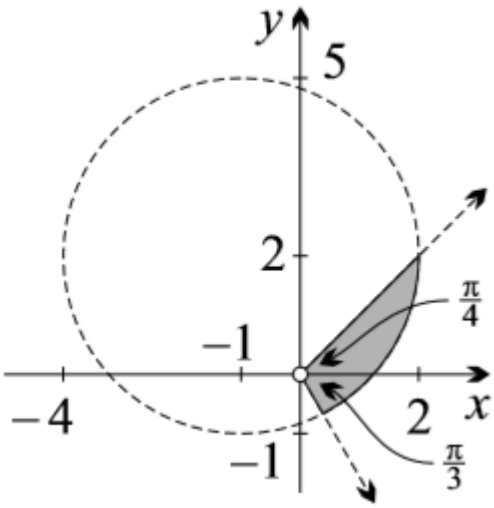
5g $|z + 1 - 2i| \leq 3$

$|z + 1 - 2i| = 3$ is a circle with centre $-1 + 2i$ or $(-1, 2)$ and radius 3 units.

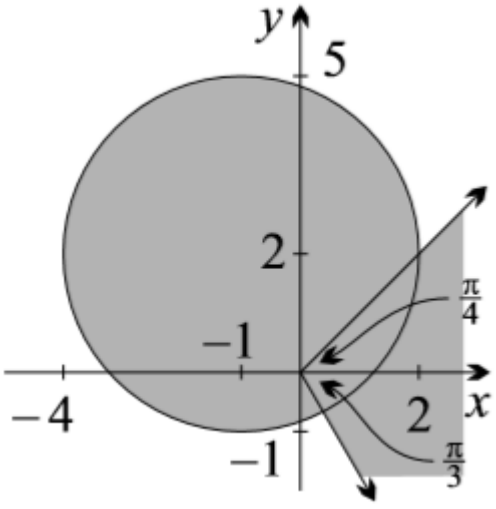
$$-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{4}$$

Vertex of angle at $(0, 0)$.

5g i



5g ii



Uncorrected proofs

5h $|z-3-i| \leq 5$

$|z-3-i| = 5$ is a circle with centre $3+i$ or $(3, 1)$ and radius 5 units.

Let $z = x + iy$.

$|z+1| \leq |z-1|$

$|(x+iy)+1| \leq |(x+iy)-1|$

$|(x+1)+iy| \leq |(x-1)+iy|$

$|(x+1)+iy| \leq |(x-1)+iy|$

$(x+1)^2 + y^2 \leq (x-1)^2 + y^2$

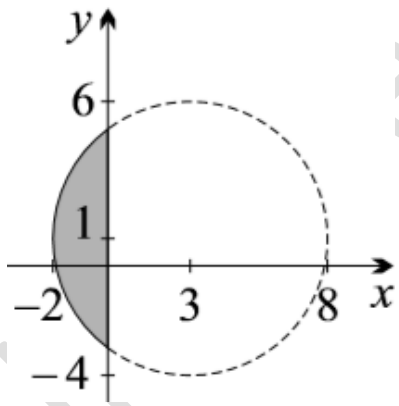
$(x+1)^2 \leq (x-1)^2$

$x^2 + 2x + 1 \leq x^2 - 2x + 1$

$4x \leq 0$

$x \leq 0$

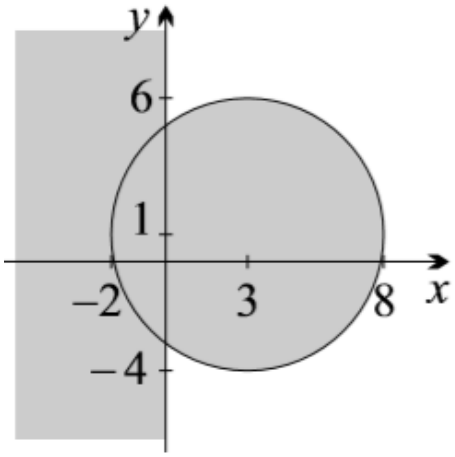
5h i



Uncorrected proofs

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5h ii



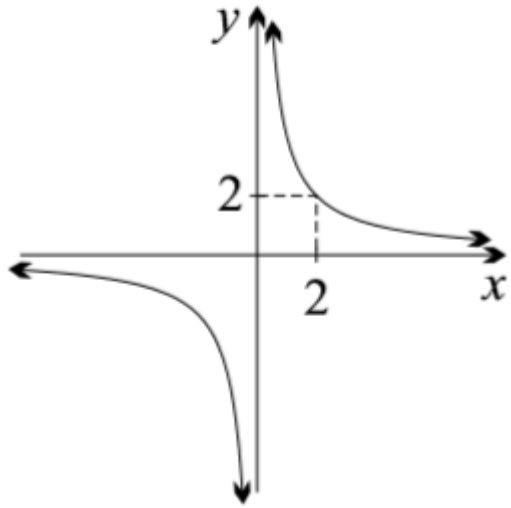
6a $(x + iy)^2 - (x - iy)^2 = 16i$

$x^2 + 2ixy - y^2 - (x^2 - 2ixy - y^2) = 16i$

$4ixy = 16i$

$4xy = 16$

$y = \frac{4}{x}$



Uncorrected proofs

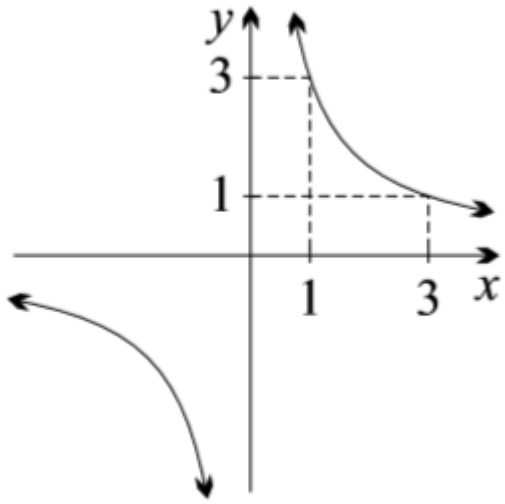
6b $(x+iy)^2-(x-iy)^2=12i$

$x^2+2ixy-y^2-(x^2-2ixy-y^2)=12i$

$4ixy=12i$

$4xy=12$

$y=\dfrac{3}{x}$



7a $|z-3i|=\text{Im}(z)$

$|x+iy-3i|=y$

$|x+i(y-3)|=y$

$|x+i(y-3)|^2=y^2$

$x^2+(y-3)^2=y^2$

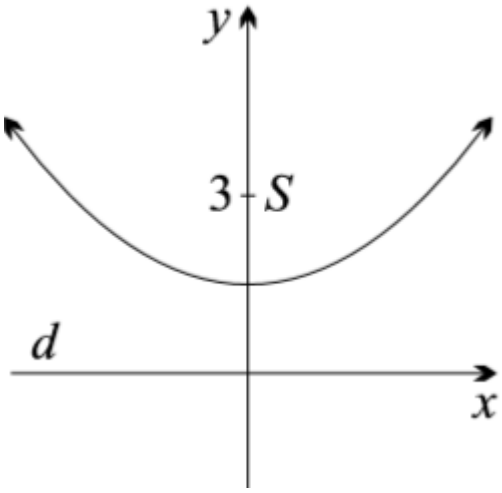
$x^2+y^2-6y+9=y^2$

$x^2-6y+9=0$

$y=\dfrac{1}{6}(x^2+9)$

Uncorrected proofs

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7b $|z + 2| = -\text{Re}(z)$

$|x + iy + 2| = -x$

$|(x + 2) + iy| = -x$

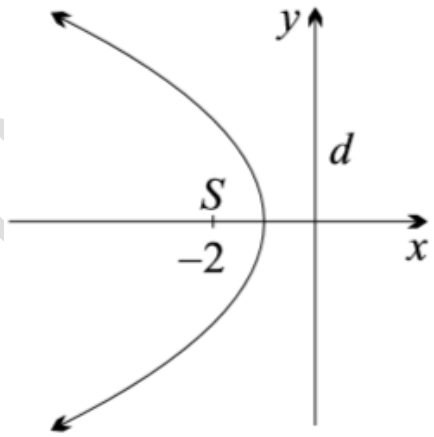
$|(x + 2) + iy|^2 = x^2$

$(x + 2)^2 + y^2 = x^2$

$x^2 + 4x + 4 + y^2 = x^2$

$4x + 4 + y^2 = 0$

$x = -\left(1 + \frac{1}{4}y^2\right)$



Uncorrected proofs

7c $|z| = \text{Re}(z + 2)$

$|x + iy| = \text{Re}(x + iy + 2)$

$|x + iy| = \text{Re}\big((x + 2) + iy\big)$

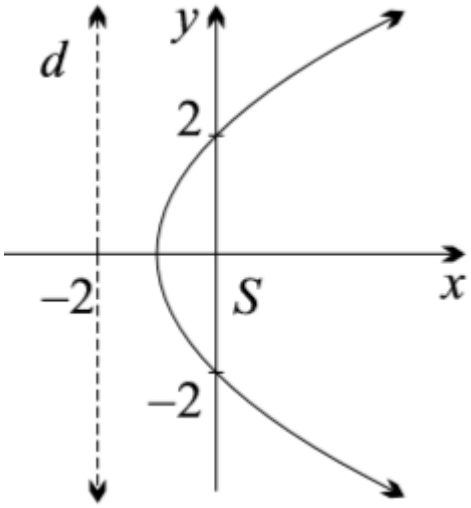
$|x + iy| = x + 2$

$|x + iy|^2 = (x + 2)^2$

$x^2 + y^2 = x^2 + 4x + 4$

$y^2 = 4x + 4$

$x = \frac{1}{4}y^2 - 1$



7d $|z - i| = \text{Im}(z + i)$

$|x + iy - i| = \text{Im}(x + iy + i)$

$|x + iy - i| = \text{Im}(x + i(y + 1))$

$|x + i(y - 1)| = y + 1$

$|x + i(y - 1)|^2 = (y + 1)^2$

$x^2 + (y - 1)^2 = y^2 + 2y + 1$

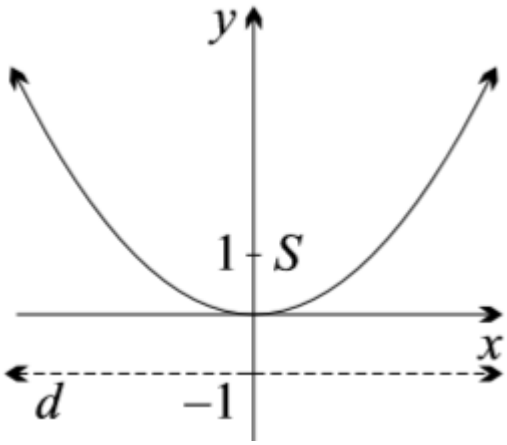
$x^2 + y^2 - 2y + 1 = y^2 + 2y + 1$

$x^2 = 4y$

$y = \frac{1}{4}x^2$

Uncorrected proofs

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8a $\text{Im}(z) = |z|$

$\text{Im}(x + iy) = |x + iy|$

$y = |x + iy|$

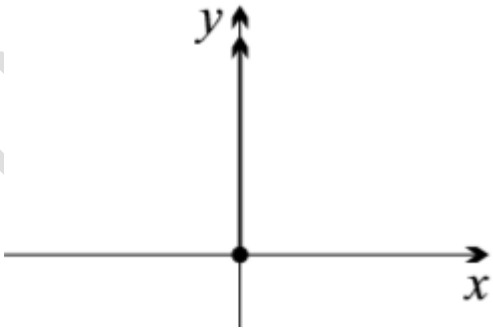
$y^2 = |x + iy|^2$

$y^2 = x^2 + y^2$

$x^2 = 0$

$x = 0$

Since we have taken the square, we must check if this works in the case where $y < 0$. We can show it does not by considering $y = -1$ and $x = 0$. In this case, $\text{Im(z)} = -1$ but $|z| = 1$. Thus, the solution is $x = 0$ for $y \geq 0$.



Uncorrected proofs

8b $\text{Re}\left(1-\frac{4}{z}\right)=0$

$\text{Re}\left(1-\frac{4\bar{z}}{z\bar{z}}\right)=0$

$\text{Re}\left(1-\frac{4\bar{z}}{|z|^2}\right)=0$

$\text{Re}\left(1-\frac{4(x-iy)}{|z|^2}\right)=0$

$\text{Re}\left(1-\frac{4x}{|z|^2}+i\frac{4y}{|z|^2}\right)=0$

$1-\frac{4x}{|z|^2}=0$

$1=\frac{4x}{|z|^2}$

$4x=|z|^2 \qquad (z\neq 0)$

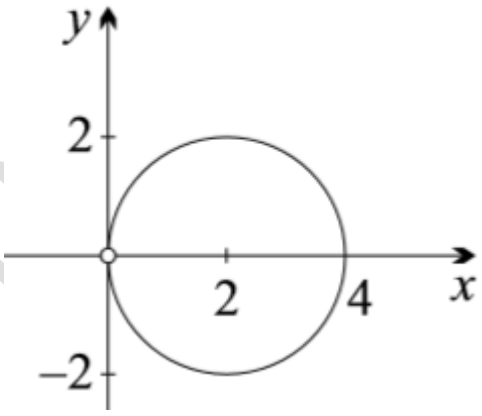
$4x=x^2+y^2$

$x^2-4x+y^2=0$

$x^2-4x+4+y^2=4$

$(x-2)^2+y^2=4$

Circle with centre $(2, 0)$ and radius 2 units, omitting the origin.



Uncorrected proofs

8c $\text{Re}\left(z - \frac{1}{z}\right) = 0$

$\text{Re}\left(z - \frac{1\bar{z}}{z\bar{z}}\right) = 0$

$\text{Re}\left(z - \frac{\bar{z}}{|z|^2}\right) = 0$

$\text{Re}\left(x + iy - \frac{(x - iy)}{|z|^2}\right) = 0$

$\text{Re}\left(x - \frac{x}{|z|^2} + i\left(y + \frac{y}{|z|^2}\right)\right) = 0$

$x - \frac{x}{|z|^2} = 0$

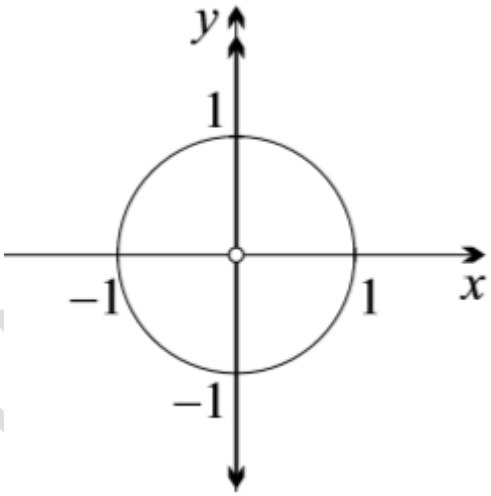
$x = \frac{x}{|z|^2}$

$x = x|z|^2 \qquad (z \neq 0)$

$1 = |z|^2 \qquad (\text{since } z \neq 0)$

$1 = x^2 + y^2$

Circle with centre $(0, 0)$ and radius 1 unit (omitting the origin).



ted proofs

9a

$$\arg\left(\frac{z-2}{z}\right)=\frac{\pi}{2}$$

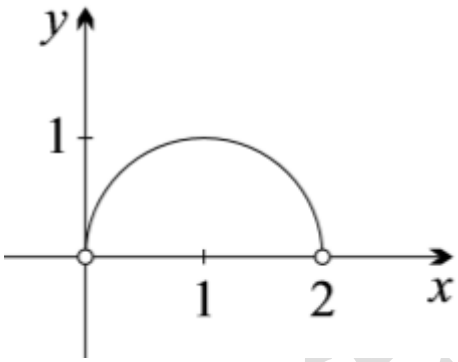
$$\arg(z-2)-\arg z=\frac{\pi}{2}$$

Consider point $A=2+0i=2$ and point $B=0+0i=0$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$$\angle APB=\frac{\pi}{2}$$

Hence the chord AB forms the diameter of the circle so $C=1+0i$ and radius is 1 unit.



9b

$$\arg\left(\frac{z-1+i}{z-1-i}\right)=\frac{\pi}{2}$$

$$\arg(z-1+i)-\arg(z-1-i)=\frac{\pi}{2}$$

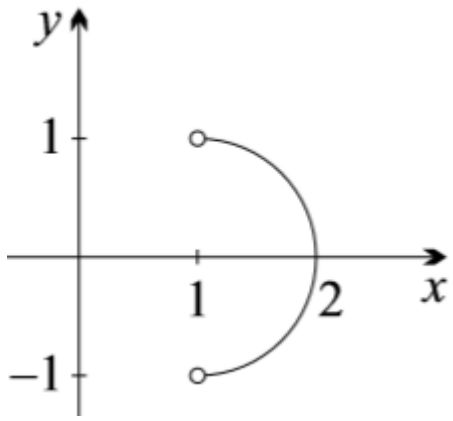
Consider point $A=1-i$ and point $B=1+i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$$\angle APB=\frac{\pi}{2}$$

Hence the chord AB forms the diameter of the circle so $C=1+0i$ and radius is 1 unit.

Uncorrected proofs



9c

$\arg\left(\dfrac{z-i}{z+i}\right)=\dfrac{\pi}{4}$

$\arg(z-i)-\arg(z+i)=\dfrac{\pi}{4}$

Consider point $A=i$ and point $B=-i$.

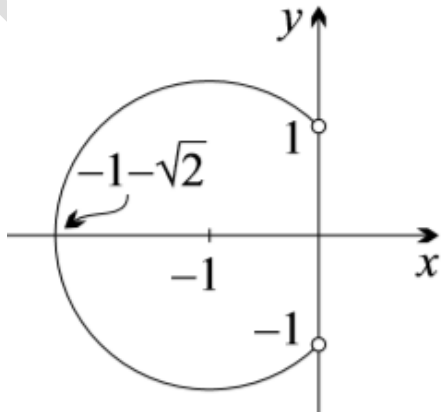
The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$\angle APB=\dfrac{\pi}{4}$ and $\angle ACB=\dfrac{\pi}{2}$ (angles at the centre and circumference)

So $\angle CAB=\dfrac{\pi}{4}$ (base angles of isosceles triangle)

Using ΔAOC, $\tan\dfrac{\pi}{4}=\dfrac{OC}{OA}=\dfrac{OC}{1}$ so $OC=1$.

Hence $C=-1+0i$ and $AC=\sqrt{1^2+1^2}=\sqrt{2}$ is its radius.



Uncorrected proofs

9d

$$\arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$$

$$\arg(z+1) - \arg(z-3) = \frac{\pi}{3}$$

Consider point $A = -1 + 0i$ and point $B = 3 + 0i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

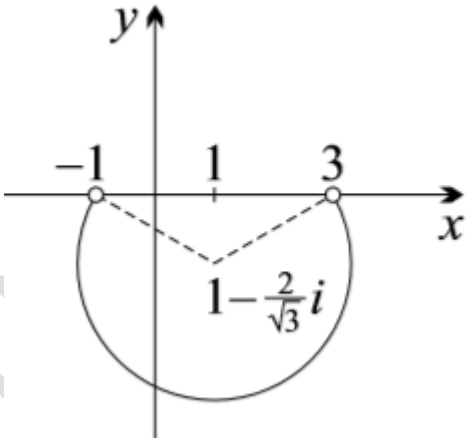
$\angle APB = \frac{\pi}{3}$ and $\angle ACB = \frac{2\pi}{3}$ (angles at the centre and circumference)

So $\angle CAB = \frac{\pi}{6}$ (base angles of isosceles triangle)

Let D be the point at $1 + 0i$ (midpoint of AB).

Using ΔADC, $\tan\frac{\pi}{6} = \frac{CD}{AD} = \frac{CD}{2}$ so $CD = 2 \times \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$.

Hence $C = 1 - \frac{2}{\sqrt{3}}i$ and $AC = \sqrt{2^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{4}{\sqrt{3}}$ is its radius.



Unedited proofs

9e

$$\arg\left(\frac{z-2i}{z+2i}\right)=\frac{\pi}{6}$$

$$\arg(z-2i)-\arg(z+2i)=\frac{\pi}{6}$$

Consider point $A=2i$ and point $B=-2i$.

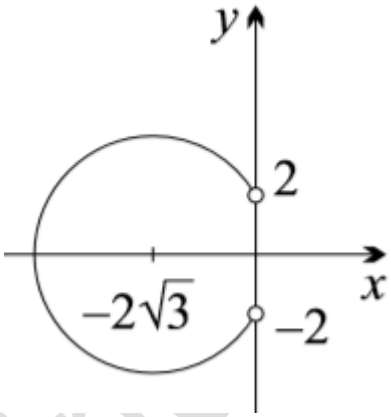
The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$\angle APB=\frac{\pi}{6}$ and $\angle ACB=\frac{\pi}{3}$ (angles at the centre and circumference)

So $\angle CAB=\frac{\pi}{3}$ (base angles of isosceles triangle)

Using ΔAOC, $\tan\frac{\pi}{3}=\frac{OC}{OA}=\frac{OC}{2}$ so $OC=2\times\sqrt{3}=2\sqrt{3}$.

Hence $C=-2\sqrt{3}+0i$ and $AC=\sqrt{2^2+(2\sqrt{3})^2}=4$ is its radius.



9f

$$\arg\left(\frac{z}{z+4}\right)=\frac{3\pi}{4}$$

$$\arg(z)-\arg(z+4)=\frac{3\pi}{4}$$

Consider point $A=0+0i$ and point $B=-4+0i$.

Uncorrected proofs

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

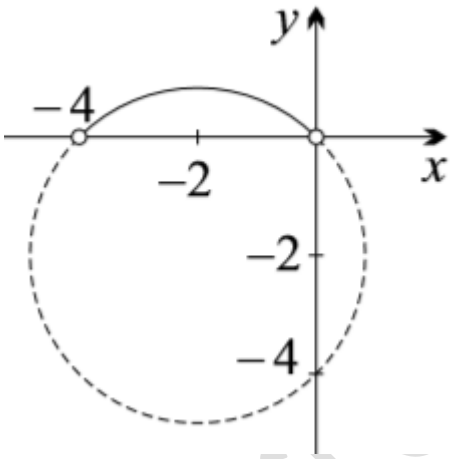
$\angle APB = \frac{3\pi}{4}$ and $\angle ACB = \frac{\pi}{2}$ (angles at the centre and circumference)

So $\angle CAB = \frac{\pi}{4}$ (base angles of isosceles triangle)

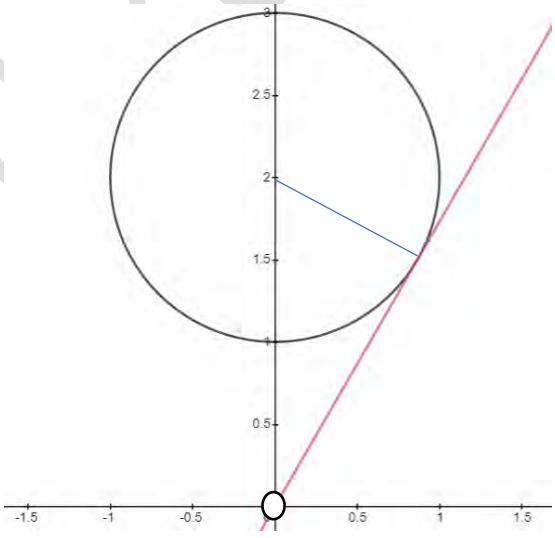
Let D be the point at $-2 + 0i$ (midpoint of AB).

Using ΔADC, $\tan\frac{\pi}{4} = \frac{CD}{AD} = \frac{CD}{2}$ so $CD = 2 \times 1 = 2$.

Hence $C = -2 - 2i$ and $AC = \sqrt{2^2 + 2^2} = \sqrt{8}$ is its radius.



10a



Corrected proofs

z lies somewhere on the red line (as $\arg z = \frac{\pi}{3}$). Now, $|z - 2i| \geq 1$ on and outside the black circle which has centre $2i$ and radius 1. Hence, we can see that $|z - 2i| \geq 1$.

10b To find the solution we must find the point where the line and the circle in the above diagram touch.

The distance between the origin and the point where the line touches the circle is $\sqrt{2^2 - 1^2} = \sqrt{3}$ (using the Pythagorean theorem)

Since $\arg z = \frac{\pi}{3}$ and $|z| = \sqrt{3}$, $z = \sqrt{3}\text{cis}\frac{\pi}{3}$.

11a $\arg(z + 3) = \tan^{-1}\frac{3\sqrt{3}}{3}$

$$\arg(z + 3) = \frac{\pi}{3}$$

11b z will take on the minimum value when it is perpendicular to the line in the argand diagram, hence $\arg z = \frac{5\pi}{6}$. $\cos\frac{\pi}{6} = \frac{|z|}{3}$, thus $|z| = 3\cos\frac{\pi}{6} = 3\left(\frac{\sqrt{3}}{2}\right)$.

Thus $|z| = \frac{3\sqrt{3}}{2}$, $\arg z = \frac{5\pi}{6}$.

11c

$$|z| = \frac{3\sqrt{3}}{2}\text{cis}\frac{5\pi}{6}$$
$$= \frac{3\sqrt{3}}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$
$$= -\frac{9}{4} + \frac{3\sqrt{3}}{4}i$$

12a $|z - 1| = 2$

Circle with centre at $(1, 0)$ and radius 2 units.

Alternatively,

$|x + iy - 1| = 2$

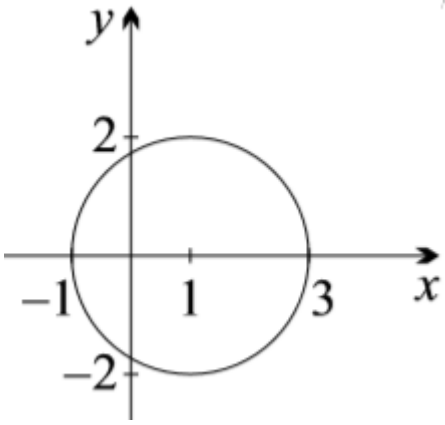
Uncorrected proofs

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$|(x-1)+iy| = 2$

$|(x-1)+iy|^2 = 4$

$(x-1)^2 + y^2 = 4$



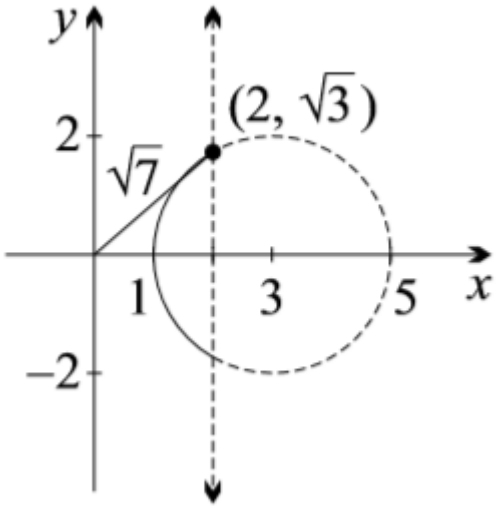
The maximum and minimum values of $|z|$ are given by the maximum and minimum distance of a point on the circle to the origin. From the diagram, the farthest point is at $z = 3$ whilst the closest is at $z = -1$. Thus, the maximum and minimum values for $|z|$ are 3 and 1 respectively.

12b $\text{Re}(z) \leq 2$

$x \leq 2$

$|z-3| = 2$

Circle with centre $(3, 0)$ and radius 2 units.



Uncorrected proofs

When $x = 2$,

$(2-3)^2 + y^2 = 4$

$1 + y^2 = 4$

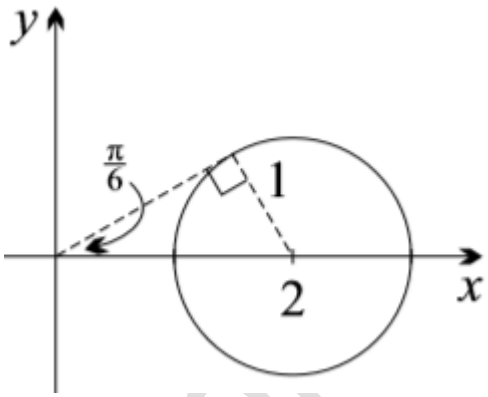
$y^2 = 3$

$y = \pm\sqrt{3}$

So the point with distance on the arc furthest from the origin is $(2, \pm\sqrt{3})$. Hence $|z| = \sqrt{2^2 + \left(\pm\sqrt{3}\right)^2} = \sqrt{4+3} = \sqrt{7}$ so the maximum value is $\sqrt{7}$. Observe from the diagram that the minimum value is when the arc intersects the x-axis. This is when $|z| = 1$.

So $1 \leq |z| \leq \sqrt{7}$.

13a i Circle with centre $(2, 0)$ and radius 1 unit.



13a ii As is shown in the diagram, the maximum value of $\arg z$ occurs when the vector formed by z is a tangent to the circle. This is when $\arg z = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

Hence $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$.

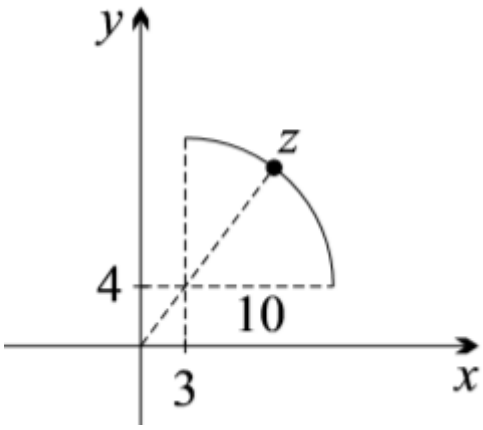
13b This is simply part (a) shifted left by 2 units.

Uncorrected proofs

14a $|w| = 10$

$0 \leq \arg w \leq \frac{\pi}{2}$

These two constraints give that w lies on a circle of radius 10, centred at the origin restricted to the first quadrant. z is w translated by $3 + 4i$, hence it is a circle of radius 10, centred $(3, 4)$ restricted to the upper right-hand quarter.



14b The maximum value of z occurs when the vector for z passes through the origin. The distance from the origin to the centre of the circle is $\sqrt{4^2 + 3^2} = 5$. The distance from the centre to the outside of the circle is 10 units. Hence the maximum value of $|z|$ is 15.

14c $\arg z = \tan^{-1}\frac{4}{3}$

Hence the value of z for which this occurs is

$15 \text{ cis}\left(\tan^{-1}\frac{4}{3}\right)$

$= 15\cos\left(\tan^{-1}\frac{4}{3}\right) + 15i\sin\left(\tan^{-1}\frac{4}{3}\right)$

$= 15 \times \frac{3}{\sqrt{3^2 + 4^2}} + 15i \times \frac{4}{\sqrt{3^2 + 4^2}}$

$= 9 + 12i$

Uncorrected proofs

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15a $|z - z_0| = r$

$|z - z_0|^2 = r^2$

$(z - z_0)\overline{(z - z_0)} = r^2$

$(z - z_0)(\bar{z} - \overline{z_0}) = r^2$

$z\bar{z} - (z\overline{z_0} + \bar{z}z_0) + z_0\overline{z_0} - r^2 = 0$

15b i $z\bar{z} - \left(z\overline{(-2)} + \bar{z}(-2)\right) + (-2)\overline{-2} - 2^2 = 0$

Hence using the above result

$|z + 2| = 2$

This is a circle with centre $-2 + 0i$ or $(-2, 0)$ and radius 2 units.

15b ii $z\bar{z} - \left(z\overline{(1 + \imath)} + \bar{z}(1 + i)\right) + (1 + i)(1 - i) - 2 + 1 = 0$

$z\bar{z} - \left(z\overline{(1 + \imath)} + \bar{z}(1 + i)\right) + (1 + i)(1 - i) - 1 = 0$

$|z - (1 + i)| = 1$

This is a circle with centre $1 + i$ or $(1, 1)$ and radius 1 unit.

15b iii $\frac{1}{z} + \frac{1}{\bar{z}} = 1$

$\frac{z\bar{z}}{z} + \frac{z\bar{z}}{z} = z\bar{z}$

$z + \bar{z} = z\bar{z}$

$z\bar{z} - (z + \bar{z}) + (1)(1) - 1 = 0$

$|z - 1| = 1$

This is a circle with centre $1 + 0i$ or $(1, 0)$ and radius 1 unit.

Uncorrected proofs

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16a The line through 1 and i, omitting i.

$$\frac{z-1}{z-i}$$

$$= \frac{x+iy-1}{x+iy-i}$$

$$= \frac{(x-1)+iy}{x+i(y-1)}$$

$$= \frac{((x-1)+iy)\big(x-i(y-1)\big)}{\big(x+i(y-1)\big)\big(x-i(y-1)\big)}$$

$$= \frac{x(x-1)+i(-(x-1)(y-1)+xy-1)+y(y-1)}{x^2+(y-1)^2}$$

$$= \frac{x(x-1)+y(y-1)+i(-xy+x+y-1+xy)}{x^2+(y-1)^2}$$

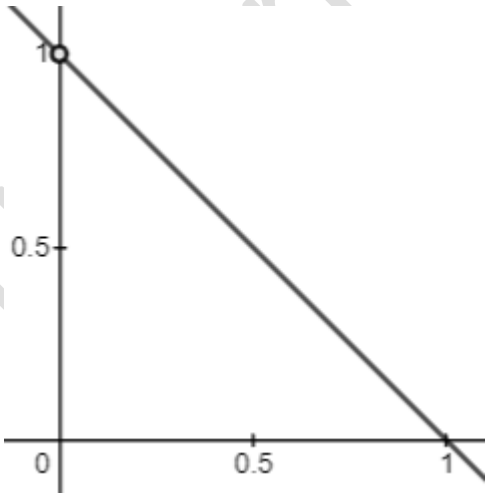
$$= \frac{x(x-1)+y(y-1)+i(x+y-1)}{x^2+(y-1)^2}$$

If the fraction is real, then the imaginary part must be equal to zero, hence

$$\frac{(x+y-1)}{x^2+(y-1)^2}$$

$$x+y-1=0 \qquad (x^2+(y-1)^2 \neq 0)$$

$$x+y=1 \qquad (x,y) \neq (1,0)$$



Uncorrected proofs

16b The circle with diameter joining 1 and i, omitting these two points.

Using the above equation, if the fraction is to be purely imaginary, it follows that the real component is equal to zero.

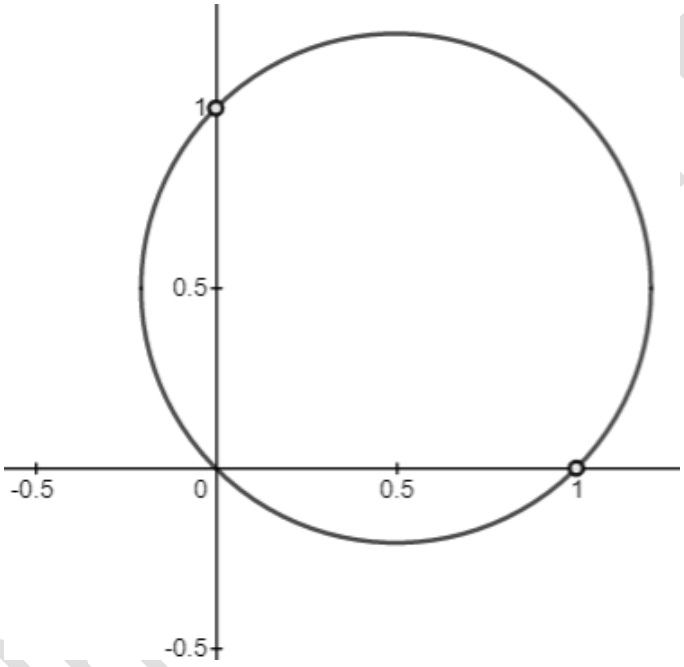
$$\frac{x(x-1)+y(y-1)}{x^2+(y-1)^2}=0$$

$$\frac{x^2-x+y^2-y}{x^2+(y-1)^2}=0$$

$$x^2-x+y^2-y=0 \qquad (x^2+(y-1)^2\neq 0)$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}$$

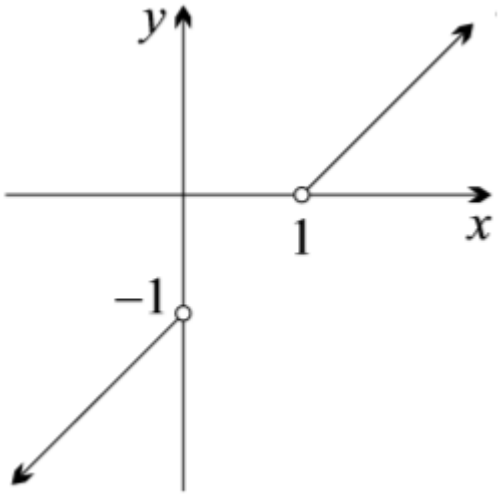
$$\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{2} \qquad (x,y)\neq(1,0),(0,1)$$



17a $\arg\left(\frac{z+i}{z-1}\right)=0$

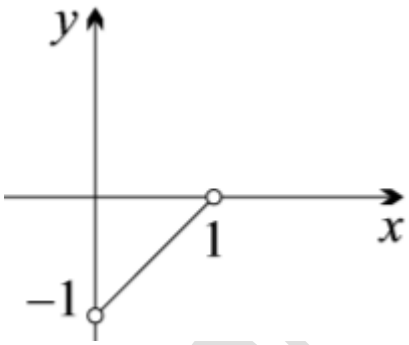
<solution to come>

Uncorrected proofs

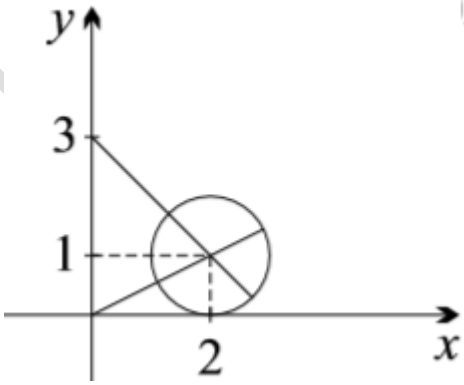


17b $\arg\left(\frac{z+i}{z-1}\right) = \pi$

<solution to come>



18a $|z - 2 - i| = 1$ is a circle with centre $(2, 1)$ and radius 1 unit.



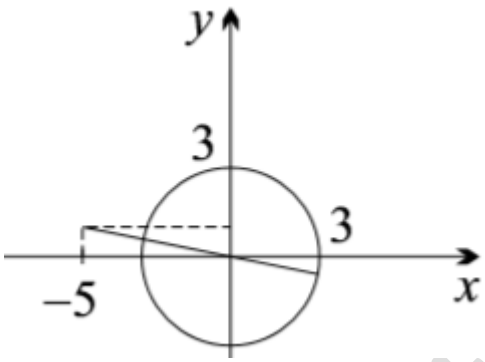
Observe that the vector passing through the origin and the centre of the circle intersects the circle at the maximum and minimum values. The distance to the

Uncorrected proofs

centre of the circle is $\sqrt{2^2 + 1^2} = \sqrt{5}$. The distance from the centre of the circle to the circumference is 1 unit. Hence the minimum value of $|z|$ is $\sqrt{5} - 1$ and the maximum is $\sqrt{5} + 1$.

To consider the maximum and minimum values of $|z - 3i|$ now consider the vector passing through the point $(0, 3)$ and the centre of the circle. The distance between these two points is $\sqrt{(3-1)^2 + 2^2} = 2\sqrt{2}$. Hence the minimum value of $|z - 3i|$ is $2\sqrt{2} - 1$ and the maximum is $2\sqrt{2} + 1$.

18b The curve representing $|z| = 3$ is a circle with centre at $(0, 0)$ and radius 3 units.



To consider the maximum and minimum values of $|z + 5 - i|$ consider a line passing through the point $(-5, 1)$ and the centre of the circle. The distance between these two points is $\sqrt{(-5)^2 + 1^2} = \sqrt{26}$. Hence the minimum value of $|z + 5 - i|$ is $\sqrt{26} - 3$ and the maximum is $\sqrt{26} + 3$.

18c i Like before, $|z - z_0| = r$ represents a circle of centre z_0 and radius r. The distance to the centre of the circle from the origin is $|z|$, thus the minimum distance from the origin to the circle circumference is $\big||z_0| - r\big|$ and the maximum distance is $\big||z_0| - r\big|$. Hence $\big||z_0| - r\big| \leq |z| \leq |z_0| + r$.

18c ii Like before, $|z - z_0| = r$ represents a circle of centre z_0 and radius r. The distance to the centre of the circle from the origin is $|z|$, thus the minimum distance from z_1 to the circle circumference is $\big||z_0 - z_1| - r\big|$ and the maximum distance is $\big||z_0 - z_1| + r\big|$. Hence, $\big||z_0 - z_1| - r\big| \leq |z - z_1| \leq |z_0 - z_1| + r$.

18d In each case we let $w = r$ and $|z|$ be $|z|$ or $|z - z_0|$ and observe that the above inequalities are produced.

Uncorrected proofs

Solutions to Exercise 1G Development questions

6a The coefficients of $Q(x)$ are real, so complex zeroes occur in conjugate pairs.

6b $\alpha + \beta + \delta + \gamma = 6$

$2i - 2i + \delta + \gamma = 6$

$\delta + \gamma = 6$

$\alpha\beta\delta\gamma = 16$

$(2i)(-2i)\delta\gamma = 16$

$4\delta\gamma = 16$

$\delta\gamma = 4$

$\delta = \frac{4}{\gamma}$

$\frac{4}{\gamma} + \gamma = 6$

$4 + \gamma^2 = 6\gamma$

$\gamma^2 - 6\gamma + 4 = 0$

$\gamma = \frac{6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}$

$= \frac{6 \pm \sqrt{20}}{2}$

$= \frac{6 \pm 2\sqrt{5}}{2}$

$= 3 \pm \sqrt{5}$

6c i $(x - 2i)(x + 2i)\left(x - 3 - \sqrt{5}\right)\left(x - 3 + \sqrt{5}\right)$

6c ii $(x^2 + 4)(x - 3 - \sqrt{5})(x - 3 + \sqrt{5})$

6c iii $(x^2 + 4)(x^2 - 6x + 4)$

Uncorrected proofs

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7a Since all the coefficients are real, and since $1 + 3i$ is a root, its complex conjugate, $1 - 3i$ must also be a root. The sum of the roots is

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -(-3)$$

$$1 + 3i + 1 - 3i + \gamma + \delta = 3$$

$$2 + \gamma + \delta = 3$$

$$\gamma + \delta = 1 \qquad (1)$$

The product of the roots is

$$\alpha\beta\delta\gamma = \frac{d}{a} = -60$$

$$(1 - 3i)(1 + 3i)\delta\gamma = -60$$

$$10\delta\gamma = -60$$

$$\delta\gamma = -6$$

$$\delta = -\frac{6}{\gamma} \qquad (2)$$

Substituting (2) into (1):

$$\gamma - \frac{6}{\gamma} = 1$$

$$\gamma^2 - 6 = \gamma$$

$$\gamma^2 - \gamma - 6 = 0$$

$$(\gamma - 3)(\gamma + 2) = 0$$

$\gamma = -2$ or 3

Hence $x = 1 \pm 3i$, 3 or -2.

7b Since all the coefficients are real, and since $1 - i$ is a root, its complex conjugate, $1 + i$ must also be a root. The sum of the roots is

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -(-6)$$

$$1 - i + 1 + i + \gamma + \delta = 6$$

$$2 + \gamma + \delta = 6$$

$$\gamma + \delta = 4 \qquad (1)$$

The product of the roots is

Uncorrected proofs

$\alpha\beta\delta\gamma = \frac{d}{a} = 10$

$(1 - i)(1 + i)\delta\gamma = 10$

$2\delta\gamma = 10$

$\delta\gamma = 5$

$\delta = \frac{5}{\gamma}$ (2)

Substituting (2) into (1):

$\gamma + \frac{5}{\gamma} = 4$

$\gamma^2 + 5 = 4\gamma$

$\gamma^2 - 4\gamma + 5 = 0$

$\gamma = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2}$

$= \frac{4 \pm \sqrt{-4}}{2}$

$= \frac{4 \pm 2i}{2}$

$= 2 \pm i$

Hence $x = 2 \pm i, 1 \pm i$.

8a $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

$P'(x) = 4x^3 - 15x^2 + 8x + 3$

$P'(3) = 4(3)^3 - 15(3)^2 + 8(3) + 3 = 0$

Hence, $x = 3$ is a double root.

8b The sum of the roots of the equation is

$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -\frac{-5}{1}$

$3 + 3 + \gamma + \delta = 5$

$\gamma + \delta = -1$ (1)

$\alpha\beta\gamma\delta = 9$

Uncorrected proofs

$(3)(3)\gamma\delta = 9$

$\gamma\delta = 1$

$\delta = \frac{1}{\gamma}$ (2)

Substituting (2) into (1):

$\gamma + \frac{1}{\gamma} = -1$

$\gamma^2 + 1 = -\gamma$

$\gamma^2 + \gamma + 1 = 0$

$\gamma = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$

$= \frac{-1 \pm \sqrt{-3}}{2}$

$= \frac{-1 \pm 3i}{2}$

Hence $x = 3, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

9a Since $a + ib$ and $a + 2ib$ are roots, and since the coefficients of the equation are real, their complex conjugates, $a - ib$ and $a - 2ib$ must also be roots.

The sum of the roots is

$\alpha + \beta + \gamma + \delta = -\frac{b}{a}$

$a + ib + a + 2ib + a - ib + a - 2ib = -\frac{-12}{1}$

$4a = 12$

$a = 3$

9b The product of the roots is

$\alpha\beta\gamma\delta = \frac{d}{a}$

$\alpha\beta\gamma\delta = 130$

$(a + ib)(a + 2ib)(a - ib)(a - 2ib) = 130$

Uncorrected proofs

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$(a + ib)(a - ib)(a + 2ib)(a - 2ib) = 130$

$(a^2 + b^2)(a^2 + 4b^2) = 130$

Since $a = 3$

$(3^2 + b^2)(3^2 + 4b^2) = 130$

$(9 + b^2)(9 + 4b^2) = 130$

$81 + 36b^2 + 9b^2 + 4b^4 = 130$

$4b^4 + 45b^2 - 49 = 0$

$4(b^2)^2 + 45b^2 - 49 = 0$

$(4b^2 + 49)(b^2 - 1) = 0$

$b^2 = -\frac{49}{4}$ or $b^2 = 1$

Since b is real, $b^2 \geq 0$, thus $b^2 = 1$.

Hence $b = \pm 1$

Since $b > 0$, $b = 1$.

9c The roots are $3 \pm i$, $3 \pm 2i$, hence as a product of roots, the polynomial is

$(3 - i)(3 + i)(3 - 2i)(3 + 2i)$

which is

$(z^2 - 6z + 10)(z^2 - 6z + 13)$

10a $P(2i) = (2i)^3 + k(2i)^2 + 6$

$= -8i - 4k + 6$

$= (6 - 4k) - 8i$

10b $P(x) = (x^2 + 4)Q(x) - 4x - 6$

$P(2i) = ((2i)^2 + 4)Q(2i) - 4(2i) - 6$

$= (-4 + 4)Q(2i) - 8i - 6$

$= 0 - 8i - 6$

$= -8i - 6$

Uncorrected proofs

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Equating this to the result in part **a** gives,

$(6 - 4k) - 8i = -8i - 6$

$4k - 6 = 6$

$4k = 12$

$k = 3$

11a $P(-i) = (-i)^3 - (-i)^2 + m(-i) + n$

$= i + 1 - im + n$

$= (1 + n) + i(1 - m)$

11b $P(x) = (x^2 + 1)Q(x) + 6x - 3$

$P(-i) = ((-i)^2 + 1)Q(-i) + 6(-i) - 3$

$= (-1 + 1)Q(-i) - 6i - 3$

$= -6i - 3$

Equating this to the result in part **a** gives,

$(1 + n) + i(1 - m) = -6i - 3$

Equating real parts,

$1 + n = -3$

$n = -4$

Equating imaginary parts,

$1 - m = -6$

$m = 7$

Hence $m = 7, n = -4$.

12a By the remainder theorem the remainder is

$P(-2i) = (-2i)^3 + (-2i)^2 + 6(-2i) - 3$

$= 8i - 4 - 12i - 3$

$= -7 - 4i$

Uncorrected proofs

12b i By the remainder theorem the remainder is

$P(2i) = (2i)^3 + (2i)^2 + 6(2i) - 3$

$= -8i - 4 + 12i - 3$

$= -7 + 4i$

12b ii Since we are dividing by a degree 2 polynomial, the remainder will be of degree 1, hence

$P(x) = (x^2 + 4)Q(x) + ax + b$

$P(2i) = ((2i)^2 + 4)Q(x) + a(2i) + b$

$-7 + 4i = (-4 + 4)Q(x) + a(2i) + b$

$-7 + 4i = (0)Q(x) + a(2i) + b$

$-7 + 4i = 2ai + b \qquad (1)$

$P(-2i) = ((-2i)^2 + 4)Q(x) + a(-2i) + b$

$-7 - 4i = (-4 + 4)Q(x) + a(-2i) + b$

$-7 - 4i = (0)Q(x) + a(-2i) + b$

$-7 - 4i = -2ai + b \qquad (2)$

Adding (1) and (2):

$-14 = 2b$

$b = -7$

Subtracting (2) from (1):

$8i = 4ai$

$a = 2$

Hence the remainder is $2x - 7$.

13a $P(iw) = (iw)^8 - \frac{5}{2}(iw)^4 + 1$

$= (i^8w^8) - \frac{5}{2}(i^4w^4) + 1$

$= w^8 - \frac{5}{2}w^4 + 1$

Uncorrected proofs

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$= P(w)$

$= 0$ (since w is a root)

$$P\left(\frac{1}{w}\right) = \left(\frac{1}{w}\right)^8 - \frac{5}{2}\left(\frac{1}{w}\right)^4 + 1$$

$$= \frac{1}{w^8} - \frac{5}{2w^4} + 1$$

$$= \frac{1}{w^8}\left(1 - \frac{5}{2}w^4 + w^8\right)$$

$$= \frac{1}{w^8}\left(w^8 - \frac{5}{2}w^4 + 1\right)$$

$$= \frac{1}{w^8}P(w)$$

$$= \frac{1}{w^8}(0) \qquad \text{(since } w \text{ is a root)}$$

$$= 0$$

Hence w and $\frac{1}{w}$ are both roots of the equation.

13b Factorising the equation (treating it as a quadratic with z^4 as the variable) gives $P(z) = \frac{1}{2}(z^4 - 2)(2z^4 - 1)$ so one root is $z = \sqrt[4]{2}$.

13c z^8 equation, so, 8 roots exist.

$z^4 - 2 = 0$

$z = \pm\sqrt[4]{2}$

From parts(a) and (b),

$\pm i\sqrt[4]{2}$; $\pm\frac{1}{\sqrt[4]{2}}$ and $\pm\frac{i}{\sqrt[4]{2}}$ are also the roots of the equation

14a $P(x)$ had minimum value B, when $x = 0$. We know that this is a minimum as since $A > 0$ for all real x it will be the case that $x^4 + Ax^2 \geq 0$.

Since $B > 0$, it follows that $P(x) > 0$ for all real values of x.

14b $-ic, -id$; the coefficients of $P(x)$ are real, so zeroes occur in conjugate pairs.

Uncorrected proofs

14c Since ic is a root,

$P(ic) = 0$

$(ic)^4 + A(ic)^2 + B = 0$

$c^4 - Ac^2 + B = 0$

Similarly, since id is a root,

$d^4 - Ad^2 + B = 0$

Adding these two equations together gives

$c^4 - Ac^2 + B + d^4 - Ad^2 + B = 0$

$c^4 + d^4 = A(c^2 + d^2) - 2B$

Now, the product of the pairs of the roots will be

$(ic)(-ic) + (id)(-id) + (ic)(id) + (-ic)(id) + (ic)(-id) + (-ic)(-id) = \frac{c}{a}$

$c^2 + d^2 = \frac{A}{1}$

$c^2 + d^2 = A$

Hence

$c^4 + d^4 = A^2 - 2B$

15a They form a conjugate pair, since $P(x)$ has real coefficients.

15b $P'(x) = 3x^2 + c$

Since the graph has turning points, there must be real values of x such that $P'(x) = 0$. That is for

$3x^2 + c = 0$

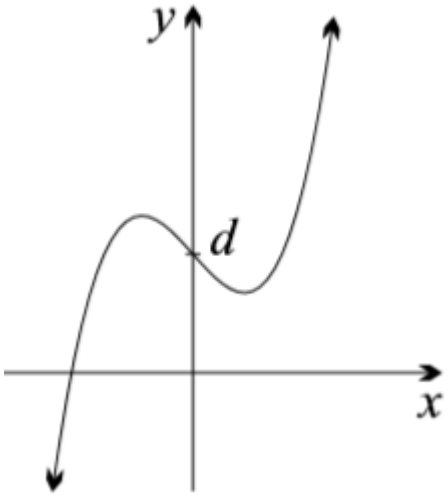
$c = -3x^2$

Since x and c are real in this case, it follows that $c < 0$

Uncorrected proofs

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15c



15d The sum of the products of the roots will be

$(a+ib)(a-ib)+k(a+ib)+k(a-ib)=c$

$a^2+b^2+2ka=c$

$2ka=c-a^2-b^2$

$k=\frac{1}{2a}(c-a^2-b^2)$

Now since $a^2+b^2\geq 0$ it follows that $-a^2-b^2\leq 0$ and hence from part (b), $c-a^2-b^2\leq 0$. Thus $2ka<0$. Since $k<0$ it follows that $a>0$ as required.

15e Roots of the equation are: $k, a\pm ib$

Sum of the roots:

$0=k+(a+ib)+(a-ib)$

$k=-2a \qquad (1)$

Sum of products of two roots:

$c=k(a+ib)+k(a-ib)+(a^2+b^2)$

$=2ka+a^2+b^2$

Substituting (1):

$c=2(-2a)+a^2+b^2$

$=b^2-3a^2$

Uncorrected proofs

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$b^2 = c + 3a^2$ (2)

Product of the roots:

$-d = k(a + ib)(a - ib)$

Substituting (1):

$-d = (-2a)(a^2 + b^2)$

$d = 2a(a^2 + b^2)$

Substituting (2):

$d = 2a(a^2 + 3a^2 + c)$

$= 8a^3 + 2ac$

16a The minimum stationary point is at $x = 1$. $f(1) = k - 2 > 0$. Hence the graph of $f(x)$ has only one x-intercept which lies to the left of the maximum stationary point at $x = -1$.

16b $f(x)$ has real coefficients.

16c Let α be the third root of the equation, the sum of the roots is

$a + ib + a - ib + \alpha = 0$

Hence the third root is

$\alpha = -2a$

The sum of the products of pairs gives:

$-2a(a + ib) - 2a(a - ib) + (a + ib)(a - ib) = -3$

$-4a^2 + a^2 + b^2 = -3$

$b^2 = 3a^2 - 3$

$b^2 = 3(a^2 - 1)$

Uncorrected proofs

16d The product of the roots is

$-2a(a+ib)(a-ib) = -2702$

$-2a(a^2+b^2) = -2702$

$-2a\big(a^2+3(a^2-1)\big) = -2702$

$-2a(4a^2-1) = -2702$

$-8a^3+2a = -2702$

$8a^3-2a+2702 = 0$

$4a^3-a+1351 = 0$

Since a is an integer, it must also be a factor of 1351. The smallest factor greater than 1 is 7. Substituting 7 into the equation solves it, hence $a = -2\times 7 = -14$ and $b^2 = 3(7^2-1) = 144$ so $b = \pm 12$.

Thus the roots of the equation are $-14, 7\pm 12i$.

17 $P'(x) = 0+1+\frac{2x}{2!}+\frac{3x^2}{3!}+\cdots+\frac{nx^{n-1}}{n!}$

$P'(x) = 1+x+\frac{x^2}{2!}+\cdots+\frac{x^{n-1}}{(n-1)!}$

Double roots will occur if any root occurs at maxima or minima. That is,

$P(a) = 0$ for a such that $P'(a) = 0$

$P(X)-P'(X) = 0$

For this case:

$P(X)-P'(X) = \frac{x^n}{n!}$,

so the only point where double roots can occur is $x = 0$,

but $P(X=0) = 1$, hence no double roots.

18a $P(-1+2i)$

$= (-1+2i)^4+4(-1+2i)^3+14(-1+2i)^2+20(-1+2i)+25$

$= -7+24i+44-8i-42-56i-20+40i+25$

$= 0$

$P'(x) = 4z^3+12z^2+28z+20$

Uncorrected proofs

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$$P'(-1+2i)$$
$$= 4(-1+2i)^3 + 12(-1+2i)^2 + 28(-1+2i) + 20$$
$$= 44 - 8i - 36 - 48i - 28 + 56i + 20$$
$$= 0$$

18b $-1+2i$ is a double zero of $P(z)$.

18c The coefficients of $P(z)$ are real and $-1+2i$ counts as two of the zeroes of $P(z)$ so its conjugate $-1-2i$ must also count as two zeroes.

18d $P(z) = (z+1-2i)^2(z+1+2i)^2$
$= (z^2+2z+5)^2$

Uncorrected proofs

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Solutions to Exercise 1H Chapter Review

1a $6z - \bar{w}$

$= 6(3 - i) - \overline{(17 + i)}$

$= 18 - 6i - (17 - i)$

$= 1 - 5i$

1b z^3

$= (3 - i)^3$

$= (3 - i)(3 - i)^2$

$= (3 - i)(9 - 6i + i^2)$

$= (3 - i)(8 - 6i)$

$= 24 - 18i - 8i + 6i^2$

$= 18 - 26i$

1c

$\frac{w}{z}$

$= \frac{17 + i}{3 - i}$

$= \frac{(17 + i)(3 + i)}{(3 - i)(3 + i)}$

$= \frac{51 + 20i + i^2}{9 - i^2}$

$= \frac{51 + 20i - 1}{9 + 1}$

$= \frac{50 + 20i}{10}$

$= 5 + 2i$

Uncorrected proofs

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Chapter 1 worked solutions – Complex numbers I

2a $z^2 + 100$

$= z^2 - 100i^2$

$= z^2 - (10i)^2$

$= (z + 10i)(z - 10i)$

2b $z^2 + 10z + 34$

Using the quadratic formula,

$$z = \frac{-10 \pm \sqrt{10^2 - 4(1)(34)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{-36}}{2}$$

$$= \frac{-10 \pm 6i}{2}$$

$$= -5 \pm 3i$$

Hence, the equation can be written as

$\big(z - (-5 - 3i)\big)\big(z - (-5 + 3i)\big)$

$= (z + 5 + 3i)(z + 5 - 3i)$

3a $z^2 - 8z + 25 = 0$

Using the quadratic formula,

$$z = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2 \times 1}$$

$$= \frac{8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{8 \pm \sqrt{-36}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

$$= 4 \pm 3i$$

Uncorrected proofs

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3b $16z^2 + 16z + 13 = 0$

Using the quadratic formula,

$$z = \frac{-16 \pm \sqrt{16^2 - 4(16)(13)}}{2(16)}$$

$$= \frac{-16 \pm \sqrt{16^2 - 4(16)(13)}}{2(16)}$$

$$= \frac{-16 \pm \sqrt{256 - 832}}{32}$$

$$= \frac{-16 \pm \sqrt{-576}}{32}$$

$$= \frac{-16 \pm 24i}{32}$$

$$= -\frac{1}{2} \pm \frac{3}{4}i$$

4a Let z be a square root of $5 - 12i$, then

$$z^2 = (a + bi)^2$$
$$= a^2 + 2abi + b^2i^2$$
$$= a^2 - b^2 + 2abi$$

Since $z^2 = 5 - 12i$

$5 - 12i = a^2 - b^2 + 2abi$

Equating real parts,

$5 = a^2 - b^2$ (1)

Equating imaginary parts,

$-12 = 2ab$

$b = -\frac{6}{a}$ (2)

Substituting (2) into (1):

$$5 = a^2 - \left(-\frac{6}{a}\right)^2$$

$$5 = a^2 - \frac{36}{a^2}$$

Uncorrected proofs

$5a^2 = a^4 - 36$

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2 + 4) = 0$

Since a is real, $a^2 \geq 0$, hence $a^2 - 9 = 0$. Thus $(a - 3)(a + 3) = 0$ so $a = \pm 3$.

When $a = 3, b = -2$ and when $a = -3, b = 2$ so the roots are $z = \pm(3 - 2i)$.

4b Let z be a square root of $7 + 6\sqrt{2}i$, then

$$\begin{aligned} z^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

Since $z^2 = 7 + 6\sqrt{2}i$

$7 + 6\sqrt{2}i = a^2 - b^2 + 2abi$

Equating real parts,

$7 = a^2 - b^2 \quad (1)$

Equating imaginary parts,

$6\sqrt{2} = 2ab$

$b = \dfrac{3\sqrt{2}}{a} \quad (2)$

Substituting (2) into (1):

$7 = a^2 - \left(\dfrac{3\sqrt{2}}{a}\right)^2$

$7 = a^2 - \dfrac{18}{a^2}$

$7a^2 = a^4 - 18$

$a^4 - 7a^2 - 18 = 0$

$(a^2 - 9)(a^2 + 2) = 0$

Since a is real, $a^2 \geq 0$, hence $a^2 - 9 = 0$. Thus $(a + 3)(a - 3) = 0$ so $a = \pm 3$.

When $a = 3, b = \frac{3}{2}\sqrt{2}$ and when $a = -3, b = -\sqrt{2}$

so the roots are $z = \pm(3 + \sqrt{2}i)$.

Uncorrected proofs

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5a $z^2 - 5z + (7 + i) = 0$

$$\left(z^2 - 5z + \frac{25}{4}\right) - \frac{25}{4} + (7 + i) = 0$$

$$\left(z - \frac{5}{2}\right)^2 + \frac{3}{4} + i = 0$$

$$\left(z - \frac{5}{2}\right)^2 = -\frac{3}{4} - i$$

Let λ^2 be a square root of $-\frac{3}{4} - i$,

$$\begin{aligned}\lambda^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ &= a^2 - b^2 + 2abi\end{aligned}$$

Equating real parts,

$$a^2 - b^2 = -\frac{3}{4} \qquad (1)$$

Equating imaginary parts,

$$2ab = -1$$

$$b = -\frac{1}{2a} \qquad (2)$$

Substituting (2) into (1):

$$a^2 - \left(-\frac{1}{2a}\right)^2 = -\frac{3}{4}$$

$$a^2 - \frac{1}{4a^2} = -\frac{3}{4}$$

$$4a^4 - 1 = -3a^2$$

$$4a^4 + 3a^2 - 1 = 0$$

$$a^2 = \frac{-3 \pm \sqrt{(-3)^2 - 4(4)(-1)}}{2(4)}$$

$$= \frac{-3 \pm \sqrt{25}}{8}$$

$$= \frac{-3 \pm 5}{8}$$

$$= -1 \text{ or } \frac{1}{4}$$

Uncorrected proofs

Since a is real, $a^2 \geq 0$ and hence $a^2 = \frac{1}{4}$ so $a = \pm\frac{1}{2}$.

When $a = \frac{1}{2}$, $b = -1$ and when $a = -\frac{1}{2}$, $b = 1$ so $\lambda = \pm\left(\frac{1}{2} - i\right)$

$$z - \frac{5}{2} = \pm\left(\frac{1}{2} - i\right)$$

$z = 2 + i$ or $3 - i$

5b $z^2 - (6 + i)z + (14 + 8i) = 0$

$\Delta = b^2 - 4ac$

$= \left(-(6 + i)\right)^2 - 4(1)(14 + 8i)$

$= 36 + 12i - 1 - 56 - 32i$

$= -21 - 20i$

Let λ^2 be a square root of $-21 - 12i$,

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

Equating real parts,

$a^2 - b^2 = -21$ (1)

Equating imaginary parts,

$ab = -10$

$b = \frac{-10}{a}$ (2)

Substituting (2) into (1):

$$a^2 - \left(-\frac{10}{a}\right)^2 = -21$$

$$a^2 - \frac{100}{a^2} = -21$$

$a^4 - 100 = -21a^2$

$a^4 + 21a^2 - 100 = 0$

$(a^2 + 25)(a^2 - 4) = 0$

Since $a^2 \geq 0$, $a^2 = 4$ and hence $a = \pm 2$

Uncorrected proofs

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When $a = 2, b = -\frac{10}{2} = -5$

When $a = -2, b = -\frac{10}{-2} = 5$

Thus $\lambda = \pm(2 - 5i)$

Hence the roots of the equation are

$$z = \frac{-b \pm \lambda}{2a}$$

$$= \frac{-\big(-(6 + i)\big) \pm (2 - 5i)}{2(1)}$$

$$= \frac{6 + i \pm (2 - 5i)}{2}$$

$z = 2 + 3i$ or $4 - 2i$

6 $\overline{3i} = -3i$ is also a zero, so $(z - 3i)(z + 3i) = z^2 + 9$ is a factor.

7a The coefficients of $P(z)$ are real and hence the conjugate is also a root.

7b The sum of the zeroes is

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha + \beta + \gamma = -\frac{-8}{1}$$

$$\alpha + \beta + \gamma = 8$$

Hence

$2 - 5i + 2 + 5i + \gamma = 8$

$4 + \gamma = 8$

$\gamma = 4$

So the third root is 4.

Uncorrected proofs

7c $P(z) = (z-4)\big(z-(2-5i)\big)\big(z-(2+5i)\big)$

$= (z-4)(z^2-4z+29)$

8a $z = \sqrt{1^2+1^2}\,\text{cis}\left(\tan^{-1}\left(-\frac{1}{1}\right)\right)$ (4th quadrant)

$= \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

8b $z = \sqrt{\left(3\sqrt{3}\right)^2+3^2}\,\text{cis}\,\left(\tan^{-1}\left(\frac{3}{-3\sqrt{3}}\right)\right)$ (2nd quadrant)

$= 6\,\text{cis}\left(\frac{5\pi}{6}\right)$

9a $4\,\text{cis}\frac{\pi}{2}$

$= 4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

$= 4(0+i\times 1)$

$= 4i$

9b $\sqrt{6}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

$= \sqrt{6}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$

$= \sqrt{6}\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$

$= -\sqrt{3}-\sqrt{3}i$

10a $2\,\text{cis}\frac{\pi}{2}\times 3\,\text{cis}\frac{\pi}{3}$

$= (3\times 2)\,\text{cis}\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$

$= 6\,\text{cis}\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)$

$= 6\,\text{cis}\left(\frac{5\pi}{6}\right)$

Uncorrected proofs

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10b

$$\frac{10\text{cis}10\theta}{5\text{cis}5\theta}$$

$$= \frac{10}{5}\text{cis}(10\theta - 5\theta)$$

$$= 2\text{cis}(5\theta)$$

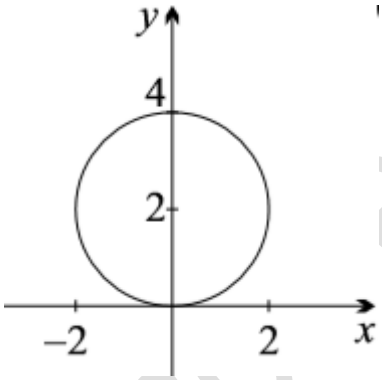
10c $(3\text{cis}3\alpha)^2$

$$= 3^2\text{cis}(3\alpha \times 2)$$

$$= 9\text{cis}(6\alpha)$$

11a $|z - 2i| = 2$

Circle with centre $2i$ or $(0, 2)$ and radius 2 units.



11b $|z| = |z - 2 - 2i|$

Line equidistant from $z = 0$ and $z = 2 + 2i$.

Alternatively,

$|x + iy| = |x + iy - 2 - 2i|$

$|x + iy| = |(x - 2) + (y - 2)i|$

$|x + iy|^2 = |(x - 2) + (y - 2)i|^2$

$x^2 + y^2 = (x - 2)^2 + (y - 2)^2$

$x^2 + y^2 = x^2 - 4x + 4 + y^2 - 4y + 4$

Uncorrected proofs

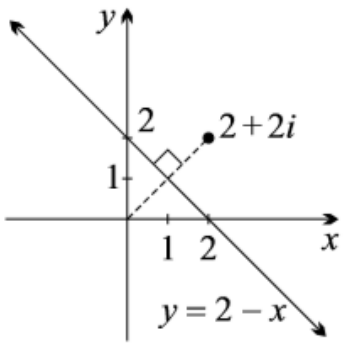
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$0 = -4x + 4 - 4y + 4$

$4x + 4y = 8$

$x + y = 2$

$y = 2 - x$



11c

$\arg(z + 2) = -\frac{\pi}{4}$

Endpoint of line at $-2 + 0i$ (not included) and line extending with angle $-\frac{\pi}{4}$.

Alternatively,

$\arg(x + iy + 2) = -\frac{\pi}{4}$

$\arg(x + 2 + iy) = -\frac{\pi}{4}$

$\tan^{-1}\left(\frac{y}{x+2}\right) = -\frac{\pi}{4}$ $\quad (x \neq -2)$

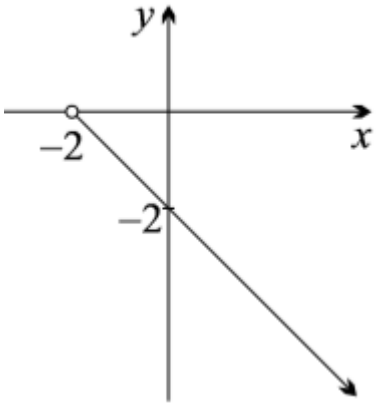
$\frac{y}{x+2} = \tan\left(-\frac{\pi}{4}\right)$

$\frac{y}{x+2} = -1$

$y = -(x + 2)$ $\quad$ (For $x < -2, y > 0$ and the slope cannot be $-\frac{\pi}{4}$, so $x > -2$)

Uncorrected proofs

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11d

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$$

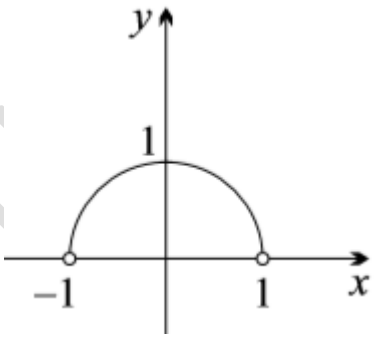
$$\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$$

Consider point $A = 1 + 0i$ and point $B = -1 + 0i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$$\angle APB = \frac{\pi}{2}$$

Hence the chord AB forms the diameter of the circle so $C = 0 + 0i$ and radius is 1 unit.

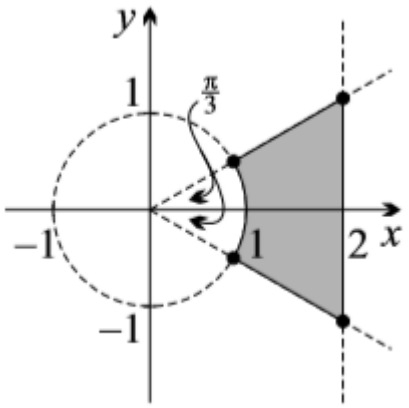


Uncorrected proofs

12 From $|z| \geq 1$ we get that $|z|^2 \geq 1^2$ and hence $x^2 + y^2 \geq 1$

From Re(z) ≤ 2 we get that $x \leq 2$

From $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$ we get that $-\frac{\pi}{3} \leq \tan^{-1}\frac{y}{x} \leq \frac{\pi}{3}$ and hence $-\sqrt{3} \leq \frac{y}{x} \leq \sqrt{3}$ so $-\sqrt{3}x \leq y \leq \sqrt{3}x$.



13a

$$\frac{z}{w}$$

$$= \frac{-1+\sqrt{3}i}{1+i}$$

$$= \frac{(-1+\sqrt{3}i)(1-i)}{(1+i)(1-i)}$$

$$= \frac{-1+(1+\sqrt{3})i-\sqrt{3}i^2}{1-i^2}$$

$$= \frac{-1+(1+\sqrt{3})i+\sqrt{3}}{2}$$

$$= \frac{1}{2}(\sqrt{3}-1)+\frac{1}{2}(\sqrt{3}+1)i$$

13b $z = \sqrt{(-1)^2+(\sqrt{3})^2}\,\text{cis}\left(\tan^{-1}\frac{\sqrt{3}}{-1}\right)$ (2nd quadrant)

$$= \sqrt{4}\,\text{cis}\left(\pi-\frac{\pi}{3}\right)$$

$$= 2\,\text{cis}\frac{2\pi}{3}$$

Uncorrected proofs

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$$w = \sqrt{1^2 + 1^2}\ \text{cis}\left(\tan^{-1}\frac{1}{1}\right) \qquad \text{(1st quadrant)}$$

$$= \sqrt{2}\ \text{cis}\frac{\pi}{4}$$

13c

$$\frac{z}{w} = \frac{2\ \text{cis}\frac{2\pi}{3}}{\sqrt{2}\ \text{cis}\frac{\pi}{4}}$$

$$= \frac{2}{\sqrt{2}}\ \text{cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sqrt{2}\ \text{cis}\left(\frac{8\pi - 3\pi}{12}\right)$$

$$= \sqrt{2}\ \text{cis}\frac{5\pi}{12}$$

13d Equating parts (a) and (c),

$$\sqrt{2}\ \text{cis}\frac{5\pi}{12} = \frac{1}{2}(\sqrt{3} - 1) + \frac{1}{2}(\sqrt{3} + 1)i$$

$$\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) = \frac{1}{2}(\sqrt{3} - 1) + \frac{1}{2}(\sqrt{3} + 1)i$$

Equating the real parts,

$$\sqrt{2}\cos\frac{5\pi}{12} = \frac{1}{2}(\sqrt{3} - 1)$$

$$\cos\frac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3} - 1)\sqrt{2}}{2(2)}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

14a $z\bar{z} = z + \bar{z}$

$$(x + iy)(x - iy) = (x + iy) + (x - iy)$$

$$x^2 - i^2y^2 = 2x$$

Uncorrected proofs

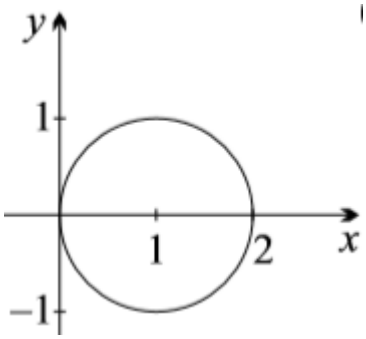
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$x^2 + y^2 = 2x$

$x^2 - 2x + 1 + y^2 = 1$

$(x - 1)^2 + y^2 = 1$

Circle with centre $(1, 0)$ and radius 1 unit.



14b $\bar{z} = iz$

$x - iy = i(x + iy)$

$x - iy = ix + i^2y$

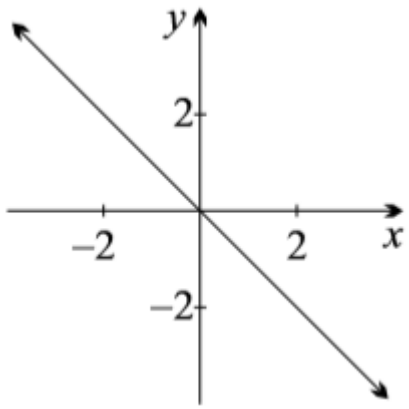
$x - iy = ix - y$

$(x + y) - i(x + y) = 0$

The real and imaginary parts must both be equal to zero, hence

$x + y = 0$

$y = -x$



Uncorrected proofs

14c $|z+2| = 2|z-4|$

$|(x+iy)+2| = 2|(x+iy)-4|$

$|(x+2)+iy| = 2|(x-4)+iy|$

$|(x+2)+iy|^2 = 4|(x-4)+iy|^2$

$(x+2)^2 + y^2 = 4((x-4)^2 + y^2)$

$x^2 + 4x + 4 + y^2 = 4(x^2 - 8x + 16 + y^2)$

$x^2 + 4x + 4 + y^2 = 4x^2 - 32x + 64 + 4y^2$

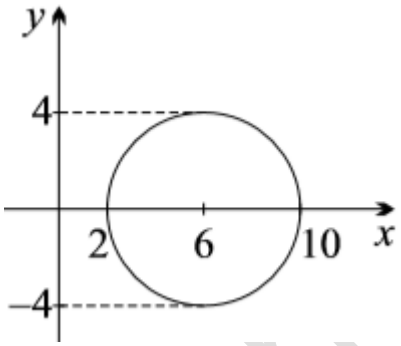
$0 = 3x^2 - 36x + 60 + 3y^2$

$0 = x^2 - 12x + 20 + y^2$

$0 = (x-6)^2 - 36 + 20 + y^2$

$(x-6)^2 + y^2 = 16$

Circle with centre $(6, 0)$ and radius 4 units.



15a $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -(4-2i) + (7+3i)$

$= 3 + 5i$

15b $\overrightarrow{PR} = i\overrightarrow{PQ}$ (since PQR is isosceles $|PR| = |PQ|$ and since $\angle P = 90°$ and hence $\arg PR = \arg PQ + \frac{\pi}{2} = i \arg PQ$). Hence

$\overrightarrow{PR} = i(3+5i) = 3i + 5i^2 = -5 + 3i$

Uncorrected proofs

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15c $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= 4 - 2i + (-5 + 3i)$

$= -1 + i$

16a Since there is a right angle at z_1 and since two sides of the triangle are of equal length, the sides $z_3 - z_1$ and $z_2 - z_1$ must satisfy the equation.

$(z_3 - z_1) = \pm i(z_2 - z_1)$

$\big(z_3 - (4 - i)\big) = \pm i(2i - (4 - i))$

$z_3 - 4 + i = \pm i(3i - 4)$

$z_3 = (4 - i) \pm i(3i - 4)$

$z_3 = (4 - i) \pm (-3 - 4i)$

$z = 1 - 5i$ or $7 + 3i$

16b Since there is a right angle at z_2 and since two sides of the triangle are of equal length, the sides $z_3 - z_2$ and $z_1 - z_2$ must satisfy the equation.

$(z_3 - z_2) = \pm i(z_1 - z_2)$

$(z_3 - 2i) = \pm i((4 - i) - 2i)$

$z_3 - 2i = \pm i(4 - 3i)$

$z_3 = 2i \pm i(4 - 3i)$

$z_3 = 2i \pm (4i + 3)$

$z = 3 + 6i$ or $-3 - 2i$

17 <solution to come>

18a $\arg(z_1 + z_2)$

$= \arg\left(2 \operatorname{cis} \frac{\pi}{12} + 2i\right)$

$= \arg\left(2 \operatorname{cis} \frac{\pi}{12} + 2 \operatorname{cis} \frac{\pi}{2}\right)$

$= \arg\left(\operatorname{cis} \frac{\pi}{12} + \operatorname{cis} \frac{\pi}{2}\right)$

Uncorrected proofs

$$= \arg\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \arg\left(\cos\frac{\pi}{12} + \cos\frac{\pi}{2} + i\left(\sin\frac{\pi}{12} + \sin\frac{\pi}{2}\right)\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{\pi}{12} + \sin\frac{\pi}{2}}{\cos\frac{\pi}{12} + \cos\frac{\pi}{2}}\right)$$

$$= \tan^{-1}\left(\frac{2\sin\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{12} - \frac{\pi}{2}}{2}\right)}{2\cos\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{12} - \frac{\pi}{2}}{2}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\sin\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)}{\cos\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)\right)$$

$$= \tan^{-1}\left(\tan\frac{7\pi}{24}\right)$$

$$= \frac{7\pi}{24}$$

18b $\arg(z_2 - z_1)$

$$= \arg\left(2i - 2\,\text{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(2\,\text{cis}\frac{\pi}{2} - 2\,\text{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(\text{cis}\frac{\pi}{2} - \text{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

$$= \arg\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{12} + i\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{12}\right)\right)$$

Uncorrected proofs

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$$= \tan^{-1}\left(\frac{\sin\frac{\pi}{2} - \sin\frac{\pi}{12}}{\cos\frac{\pi}{2} - \cos\frac{\pi}{12}}\right)$$

$$= \tan^{-1}\left(\frac{2\cos\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{\pi}{12}}{2}\right)}{-2\sin\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{\pi}{12}}{2}\right)}\right)$$

$$= \tan^{-1}\left(-\cot\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\right)$$

$$= \tan^{-1}\left(-\cot\frac{7\pi}{24}\right)$$

$$= \tan^{-1}\left(\cot\left(-\frac{7\pi}{24}\right)\right)$$

$$= \tan^{-1}\left(\cot\left(\pi - \frac{7\pi}{24}\right)\right)$$

$$= \tan^{-1}\left(\cot\frac{17\pi}{24}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{17\pi}{24}\right)\right)$$

$$= \tan^{-1}\left(\tan\left(-\frac{5\pi}{24}\right)\right)$$

$$= \tan^{-1}\left(\tan\left(\pi - \frac{5\pi}{24}\right)\right)$$

$$= \tan^{-1}\left(\tan\frac{19\pi}{24}\right)$$

$$= \frac{19\pi}{24}$$

Uncorrected proofs

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19 $|z_1| = |z_2|$

$\arg((z_1 + z_2)^2) = 2\arg(z_1 + z_2)$

Now $z_1 + z_2$ bisects the angle between z_1 and z_2 as z_1 and z_2 represent two sides of a rhombus whilst $z_1 + z_2$ is the diagonal, thus

$\arg(z_1 + z_2)$

$= \frac{1}{2}(\arg(z_1) + \arg(z_2))$

$= \frac{1}{2}\arg(z_1 z_2)$

Hence $\arg((z_1 + z_2)^2)$

$= 2\arg(z_1 + z_2)$

$= 2\left(\frac{1}{2}\arg(z_1 z_2)\right)$

$= \arg(z_1 z_2)$

20

$$\frac{z^2 - 1}{z^2 + 1}$$

$$= \frac{(\text{cis}\theta)^2 - 1}{(\text{cis}\theta)^2 + 1}$$

$$= \frac{(\cos\theta + i\sin\theta)^2 - 1}{(\cos\theta + i\sin\theta)^2 + 1}$$

$$= \frac{\cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta - 1}{\cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta + 1}$$

$$= \frac{\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta - 1}{\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta + 1}$$

$$= \frac{-(1 - \cos^2\theta) + 2i\sin\theta\cos\theta - \sin^2\theta}{\cos^2\theta + 2i\sin\theta\cos\theta + 1 - \sin^2\theta}$$

$$= \frac{-\sin^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta}{\cos^2\theta + 2i\sin\theta\cos\theta + \cos^2\theta}$$

$$= \frac{2i\sin\theta\cos\theta - 2\sin^2\theta}{2\cos^2\theta + 2i\sin\theta\cos\theta}$$

Uncorrected proofs

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$$= \frac{i \sin\theta \cos\theta - \sin^2\theta}{\cos^2\theta + i \sin\theta \cos\theta}$$

$$= \frac{\sin\theta\,(i\cos\theta - \sin\theta)}{\cos\theta\,(\cos\theta + i\sin\theta)}$$

$$= \tan\theta\left(\frac{i\cos\theta - \sin\theta}{\cos\theta + i\sin\theta}\right)$$

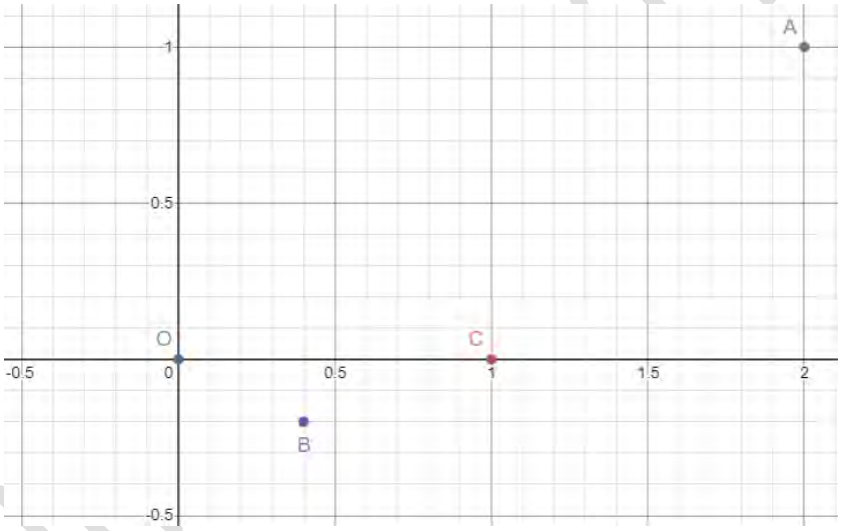
$$= i\tan\theta\left(\frac{i\cos\theta - \sin\theta}{i(\cos\theta + i\sin\theta)}\right)$$

$$= i\tan\theta\left(\frac{i\cos\theta - \sin\theta}{i\cos\theta + i^2\sin\theta}\right)$$

$$= i\tan\theta\left(\frac{i\cos\theta - \sin\theta}{i\cos\theta - \sin\theta}\right)$$

$$= i\tan\theta$$

21a



$\angle OAC$

$= \arg(\overrightarrow{AO}) - \arg(\overrightarrow{AC})$

$= \arg(\overrightarrow{AO}) - \arg(\overrightarrow{AO} + \overrightarrow{OC})$

$= \arg(-\overrightarrow{OA}) - \arg(-\overrightarrow{OA} + \overrightarrow{OC})$

$= \arg(-z) - \arg(-z + 1)$

$= \arg\left(\frac{-z}{-z+1}\right)$

Unedited proofs

$= \arg\left(\frac{z}{z-1}\right)$

$\angle OCB$

$= \arg \overrightarrow{CO} - \arg \overrightarrow{CB}$

$= \arg(-\overrightarrow{OC}) - \arg(\overrightarrow{CO} + \overrightarrow{OB})$

$= \arg(-\overrightarrow{OC}) - \arg(-\overrightarrow{OC} + \overrightarrow{OB})$

$= \arg 1 - \arg(-1 + \frac{1}{z})$

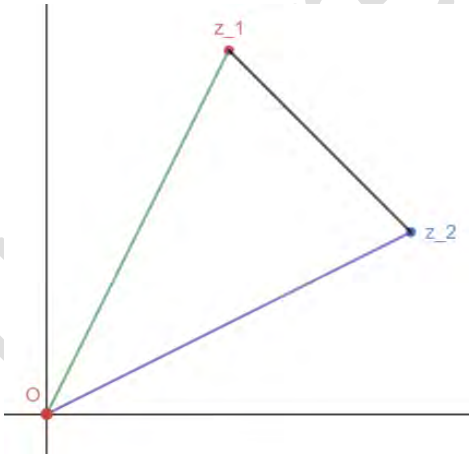
$= \arg\left(\frac{1}{-1+\frac{1}{z}}\right)$

$= \arg\left(\frac{z}{-z+1}\right)$

$= \arg\left(\frac{z}{1-z}\right)$

Hence $\angle OAC = \angle OCB$.

22a



$|z_1 - z_2|$ denotes the length of the black line in the diagram above. On the other hand $|z_1| - |z_2|$ is the difference in length between the green and blue lines. The latter distance must be shorter (unless the blue and green lines overlap in which case the two distances are equal). Thus $|z_1 - z_2| \geq |z_1| - |z_2|$.

Uncorrected proofs

22b

$$\left|z-\frac{4}{z}\right|=2$$

$$\left|\frac{z^2-4}{z}\right|=2$$

$$\frac{|z^2-4|}{|z|}=2$$

$$|z^2-4|=2|z|$$

$$|z^2|-|4|\leq 2|z|$$

$$|z|^2-4\leq 2|z|$$

$$|z|^2-2|z|-4\leq 0$$

The maximum value of $|z|$ will occur when $|z|^2-2|z|-4=0$.

$$|z|=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}$$

$$=\frac{2\pm\sqrt{4+16}}{2}$$

$$=\frac{2\pm\sqrt{20}}{2}$$

$$=\frac{2\pm 2\sqrt{5}}{2}$$

$$=1\pm\sqrt{5}$$

So the maximum value is $\sqrt{5}+1$.

Uncorrected proofs

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