Chapter 2 worked solutions – Proof

Solutions to Exercise 2A Foundation questions

1a equality

1b implication

1c equivalence

1d for all

1e there exists

2a If a triangle has two equal angles, then it has two equal sides. True.

2b If the square of a number is odd, then the number is odd. True.

2c If I have four legs, then I am a horse. False.

2d If a number is even, then it ends with the digit 6. False.

2e Every rhombus is a square. False.

2f If $n \geq 0$, then $\sqrt{n} \in R$. True.

3a True

3b False

3c False

© Cambridge University Press 2019

3d False

3e True

3f True

4a Not all cars are red. Alternatively, some cars are not red.

4b $a \leq b$

4c Hillary does not like both steak and pizza, i.e. she doesn't like steak or she doesn't like pizza.

4d Bill and Dave are both incorrect.

4e I live in Tasmania but not in Australia.

4f Nikhil doesn't study and he passes.

4g $x < -3$ or $x > 8$

4h $-5 \leq x < 0$

5a If my plants don't grow, then I haven't watered them.

5b If you live in Melbourne, then you live in Australia.

5c If a triangle doesn't have three equal angles, then it doesn't have three equal sides.

© Cambridge University Press 2019

5d If I like motorists, then I don't like cycling.

5e If a number is not odd (i.e. it is even) then the previous number is not even (i.e. it is odd).

5f If $\frac{1}{a} \geq \frac{1}{b}$ then $a \leq b$ or at least one of a and b is negative.

6a If a number is divisible by both 3 and 5 then it is divisible by 15. Conversely, if a number is divisible by 15 then it is divisible by both 3 and 5.

6b If a triangle has two equal sides, then it has two equal angles. Conversely, if a triangle has two equal angles, then it has two equal sides.

6c If the only divisors of an integer n greater than 1 are 1 and n, then n is prime. Conversely, if n is prime (and implicitly, greater than 1) then its only divisors are 1 and n.

6d If a quadrilateral has a pair of opposite sides that are equal and parallel, then it is a parallelogram. Conversely, if a quadrilateral is a parallelogram, then it has a pair of opposite sides that are equal and parallel.

© Cambridge University Press 2019

Solutions to Exercise 2B Foundation questions

1a Since both are even, let $a = 2m$ and $b = 2n$.

Then $a + b = 2m + 2n$

$= 2(m + n)$

Therefore $a + b$ is even.

1b Since both are odd, let $a = 2m + 1$ and $b = 2n + 1$.

Then $a + b = 2m + 1 + 2n + 1$

$= 2(m + n + 1)$

Therefore $a + b$ is even.

1c Since a is even and b is odd, let $a = 2m$ and $b = 2n + 1$.

Then $a + b = 2m + 2n + 1$

$= 2(m + n) + 1$

Therefore $a + b$ is odd.

2a Since both are even, let $a = 2m$ and $b = 2n$.

Then $ab = 2m \times 2n$

$= 2(2mn)$

Therefore ab is even.

2b Since both are odd, let $a = 2m + 1$ and $b = 2n + 1$.

Then $ab = 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

Therefore ab is odd.

© Cambridge University Press 2019

Chapter 2 worked solutions – Proof

2c Since a is even and b is odd, let $a = 2m$ and $b = 2n + 1$.

Then $ab = 2m(2n + 1)$

$= 2(2mn + m)$

Therefore ab is even.

3a If a is even, let $a = 2m$

Therefore $a^2 = 4m^2$

$= 2(2m^2)$

Therefore a^2 is even.

3b If a is odd, let $a = 2m + 1$

Therefore $a^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

Therefore a^2 is odd.

4a If b and $b + c$ are both divisible by a then let $b = ka$ and $b + c = la$.

$(b + c) - b = la - ka$

$c = (l - k)a$

Therefore c is divisible by a.

4a If b and $b - c$ are both divisible by a then let $b = ka$ and $b - c = la$.

$b - (b - c) = ka - la$

$c = (k - l)a$

Therefore c is divisible by a.

5 If b and c are both divisible by a then let $b = ka$ and $c = la$.

$bx + cy = kax + lay$

$= a(kx + ly)$

Therefore $bx + cy$ is divisible by a.

© Cambridge University Press 2019

6 Note that $b = a + 1, c = a + 2, d = a + 3$.

6a Substituting these identities:

$$a + d = a + a + 3$$
$$= 2a + 3$$
$$b + c = a + 1 + a + 2$$
$$= 2a + 3$$

Therefore $a + d = b + c$.

6b Substituting again:

$$ad = a(a + 3)$$
$$= a^2 + 3a$$
$$bc - 2 = (a + 1)(a + 2) - 2$$
$$= a^2 + 3a + 2 - 2$$
$$= a^2 + 3a$$

Therefore $ad = bc - 2$.

6c Substituting again:

$$a^2 + d^2 = a^2 + (a + 3)^2$$
$$= a^2 + a^2 + 6a + 9$$
$$= 2a^2 + 6a + 9$$
$$b^2 + c^2 + 4 = (a + 1)^2 + (a + 2)^2 + 4$$
$$= a^2 + 2a + 1 + a^2 + 4a + 4 + 4$$
$$= 2a^2 + 6a + 9$$

Therefore $a^2 + d^2 = b^2 + c^2 + 4$.

© Cambridge University Press 2019

Solutions to Exercise 2C Foundation questions

1 Suppose that $\log_7 13$ is rational, so $\log_7 13 = \frac{m}{n}$ where m, n are integers with no common factors other than 1, and n is positive.

Therefore $7^{\frac{m}{n}} = 13$

$7^m = 13^n$

Since $n \geq 1$ it follows that 13 is a factor of the RHS. But clearly 13 is not a factor of the LHS, so there is a contradiction.

Hence $\log_7 13$ is irrational.

2 Suppose that $\sqrt{5}$ is irrational, so $\sqrt{5} = \frac{m}{n}$ where m, n are integers with no common factors other than 1, and n is positive.

Therefore $5 = \frac{m^2}{n^2}$

$5n^2 = m^2$

Therefore m^2 is a multiple of 5, and since 5 is prime, m must be a multiple of 5.

Let $m = 5k$

Therefore $5n^2 = m^2 = 25k^2$

$n^2 = 5k^2$

Therefore n^2 is also a multiple of 5, and since 5 is prime, n must be a multiple of 5.

But this contradicts the assumption that m, n have no common factors.

Hence $\sqrt{5}$ is irrational.

3a "If a is even then a^2 is even".

3b Let $a = 2n$

Then $a^2 = 4n^2 = 2(2n)$

So a^2 is even.

By proving the contrapositive, we have proved that if a^2 is odd then a is odd.

© Cambridge University Press 2019

4 Suppose m is even. Let $m = 2k$.

Therefore $m^2 + 4m + 7 = 4k^2 + 8k + 7$

$$= 2(2k^2 + 4k + 3) + 1$$

which must be odd.

Therefore, if m is even, $m^2 + 4m + 7$ is odd.

Since we have proved the contrapositive, if $m^2 + 4m + 7$ is even, then m is odd.

5 Suppose that a and b are both odd. Let $a = 2m + 1, b = 2n + 1$.

Therefore $ab = (2m + 1)(2n + 1)$

$$= 4mn + 2m + 2n + 1$$

$$= 2(2mn + m + n) + 1$$

which must be odd.

Hence, since we have proved the contrapositive, if ab is even then a and b cannot both be odd, i.e. at least one of them is even.

© Cambridge University Press 2019

Solutions to Exercise 2D Foundation questions

1 Let $a = 1 + h, h > 0$.

Therefore $a^2 - 1 = (1 + h)^2 - 1$

$= 2h + h^2$

Since $2h$ and h^2 are both greater than 0, $a^2 - 1 > 0$ and therefore $a^2 > 1$.

Alternative proof:

$a^2 - 1 = (a + 1)(a - 1)$

Since $a > 1$, $a + 1 > 0$ and $a - 1 > 0$

Therefore $a^2 - 1$ is the product of two positive quantities and so must be positive. From there, the result follows as above.

2a LHS − RHS

$= a^2 + b^2 - 2ab$

$= (a - b)^2$

≥ 0

Therefore LHS ≥ RHS.

2b LHS − RHS

$= \frac{a^2}{b^2} + \frac{b^2}{a^2} - 2$

$= \left(\frac{a}{b} - \frac{b}{a}\right)^2$

≥ 0

Therefore LHS ≥ RHS.

© Cambridge University Press 2019

Chapter 2 worked solutions – Proof

2c LHS − RHS

$$= \frac{a^2 + b^2}{2} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{2a^2 + 2b^2 - a^2 - 2ab - b^2}{4}$$

$$= \frac{a^2 + b^2 - 2ab}{4}$$

$$= \frac{(a-b)^2}{4}$$

$$\geq 0$$

Therefore LHS ≥ RHS.

3 Since $a > 0$, let $a = b^2$

$$a + \frac{1}{a} - 2$$

$$= b^2 - 2 + \frac{1}{b^2}$$

$$= \left(b - \frac{1}{b}\right)^2$$

$$\geq 0$$

Therefore $a + \frac{1}{a} \geq 2$.

4a Since $a, b > 0$, let $a = p^2, b = q^2$ with $p, q > 0$.

$$\frac{1}{2}(a+b) - \sqrt{ab}$$

$$= \frac{p^2 + q^2}{2} - pq$$

$$= \frac{1}{2}(p^2 - 2pq + q^2)$$

$$= \frac{1}{2}(p-q)^2$$

$$\geq 0$$

© Cambridge University Press 2019

Therefore $\frac{1}{2}(a+b) \geq \sqrt{ab}$

4b Let $a = p^2, b = q^2$ with $p, q > 0$

$$\frac{1}{3}a + \frac{3}{4}b - \sqrt{ab}$$

$$= \frac{p^2}{3} + \frac{3q^2}{4} - \sqrt{p^2q^2}$$

$$= \frac{1}{3}p^2 + \frac{3}{4}q^2 - pq$$

$$= \frac{1}{12}(4p^2 - 12pq + 9q^2)$$

$$= \frac{1}{12}(2p - 3q)^2$$

$$\geq 0$$

Therefore $\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$.

5a Let $a = b + h, h > 0$ and $b > 0$

Then $(a^2 - b) - (b^2 - a)$

$= (b^2 + 2bh + h^2 - b) - (b^2 - b - h)$

$= 2bh + h^2 + h$

Since b and h are positive, and therefore $2bh$ and h^2 are also positive,

$(a^2 - b) > (b^2 - a)$

5b $a^3 - b^3 - (a^2b - ab^2)$

$= a^3 - a^2b + ab^2 - b^3$

$= (a - b)(a^2 + b^2)$

Since $a > b, a - b > 0$

Since $a \neq b$, a and b cannot both equal zero.

Therefore at least one of a^2, b^2 is greater than zero and the other is greater than or equal to zero.

Therefore $a^2 + b^2 > 0$

© Cambridge University Press 2019

Therefore $(a-b)(a^2+b^2) > 0$.

Therefore $a^3 - b^3 > (a^2b - ab^2)$.

6a Let $x = a^2, y = b^2$ where $a, b \geq 0$

$x + y - 2\sqrt{xy}$

$= a^2 + b^2 - 2ab$

$= (a-b)^2$

≥ 0

Therefore $x + y \geq 2\sqrt{xy}$.

6b From part a, we have $x + y \geq 2\sqrt{xy}$ and we know $2\sqrt{xy} \geq 0$.

Similarly $x + z \geq 2\sqrt{xz} \geq 0$

and $y + z \geq 2\sqrt{yz} \geq 0$.

Therefore

$(x+y)(x+z)(y+z) \geq \left(2\sqrt{xy}\right)\left(2\sqrt{xz}\right)\left(2\sqrt{yz}\right)$

$(x+y)(x+z)(y+z) \geq 8\sqrt{xy \times xz \times yz}$

$(x+y)(x+z)(y+z) \geq 8\sqrt{x^2y^2z^2}$

$(x+y)(x+z)(y+z) \geq 8xyz$

© Cambridge University Press 2019

Solutions to Exercise 2E Foundation questions

1a A. When $n = 1$,

LHS $= 1$

$$= \frac{1}{2}n(n+1)$$

$$= \text{RHS}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r = \frac{1}{2}(k+1)(k+2)$$

By the induction hypothesis:

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Therefore

$$\sum_{r=1}^{k+1} r = \left(\sum_{r=1}^{k} r\right) + k + 1$$

$$= \frac{1}{2}k(k+1) + k + 1$$

$$= \frac{1}{2}(k^2 + k + 2k + 2)$$

$$= \frac{1}{2}(k+1)(k+2)$$

as required.

© Cambridge University Press 2019

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1b A. When $n = 1$,

$$\text{LHS} = 1(2)$$
$$= 2$$
$$= \frac{1}{3}n(n+1)(n+2)$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} r(r+1) = \left(\sum_{r=1}^{k} r(r+1)\right) + (k+1)(k+2)$$
$$= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)$$
$$= \frac{1}{3}\big(k(k+1)(k+2) + 3(k+1)(k+2)\big)$$
$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

© Cambridge University Press 2019

1c A. When $n = 1$,

$$\text{LHS} = 1^2$$
$$= 1$$
$$= \frac{1}{6}n(n+1)(2n+1)$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} r^2 = \left(\sum_{r=1}^{k} r^2\right) + (k+1)^2$$
$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$
$$= \frac{1}{6}(k(k+1)(2k+1) + 6(k+1)^2)$$
$$= \frac{1}{6}(k+1)\left(2k^2 + k + 6(k+1)\right)$$
$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

as required.

© Cambridge University Press 2019

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1d A. When $n = 1$,

$\text{LHS} = 1$

$= \frac{1}{3}n(2n-1)(2n+1)$

$= \text{RHS}$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Now prove the statement for $n = k+1$. That is, prove that

$$\sum_{r=1}^{k+1}(2r-1)^2 = \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

$$= \frac{1}{3}(k+1)(2k+1)(2k+3)$$

$$= \frac{1}{3}(4k^3+12k^2+11k+3)$$

By the induction hypothesis:

$$\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Therefore

$$\sum_{r=1}^{k+1}(2r-1)^2 = \left(\sum_{r=1}^{k}(2r-1)^2\right) + (2(k+1)-1)^2$$

$$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$$

$$= \frac{1}{3}\big(k(2k-1)(2k+1)\big) + 4k^2 + 4k + 1$$

© Cambridge University Press 2019

$$= \frac{1}{3}(4k^3 - k + 12k^2 + 12k + 3)$$

$$= \frac{1}{3}(4k^3 + 12k^2 + 11k + 3)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1e A. When $n = 1$,

$$\text{LHS} = \frac{1}{2}$$

$$= \frac{n}{n+1}$$

$$= \text{RHS}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \left(\sum_{r=1}^{k} \frac{1}{r(r+1)}\right) + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

© Cambridge University Press 2019

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1f A. When $n = 1$,

$$\text{LHS} = \frac{1}{(2-1)(2+1)}$$

$$= \frac{1}{3}$$

$$= \frac{n}{2n+1}$$

$$= \text{RHS}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2(k+1)+1}$$

$$= \frac{k+1}{2k+3}$$

By the induction hypothesis:

© Cambridge University Press 2019

$$\sum_{r=1}^{k+1}\frac{1}{(2r-1)(2r+1)}=\left(\sum_{r=1}^{k}\frac{1}{(2r-1)(2r+1)}\right)+\frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{2k+3}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2a A. When $n = 1, 5^n + 3 = 8 = 2(4)$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $5^k + 3 = 4a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$5^{k+1} + 3$

is a multiple of 4.

By the induction hypothesis:

$5^k + 3 = 4a$

$5^k = 4a - 3$

$5^{k+1} = 5(4a - 3)$

$= 20a - 15$

$5^{k+1} + 3 = 20a - 12$

© Cambridge University Press 2019

$= 4(5a - 3)$

which is a multiple of 4, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2b A. When $n = 1$, $2^{3n} + 6 = 8 + 6 = 14$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $2^{3k} + 6 = 7a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$2^{3(k+1)} + 6$ is a multiple of 7.

By the induction hypothesis:

$2^{3k} + 6 = 7a$

$2^{3k} = 7a - 6$

$2^{3k+3} = 2^{3k} \times 2^3$

$= 2^{3k} \times 8$

$= 8(7a - 6)$

$= 56a - 48$

$2^{3(k+1)} + 6 = 56a - 48 + 6$

$= 56a - 42$

$= 7(8a - 6)$

which is a multiple of 7, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

© Cambridge University Press 2019

2c A. When $n = 1$, $5^n + 2^{n+1} = 5 + 4 = 9$

which is a multiple of 3, so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$5^k + 2^{k+1} = 3a$ for some integer a

Now prove the statement for $n = k + 1$. That is, prove that

$5^{k+1} + 2^{k+2}$ is a multiple of 3.

By the induction hypothesis:

$5^k + 2^{k+1} = 3a$

$5(5^k + 2^{k+1}) = 15a$

$5^{k+1} + 2 \times 2^{k+1} + 3 \times 2^{k+1} = 15a$

$5^{k+1} + 2^{k+2} = 15a - 3\left(2^{k+1}\right)$

$= 3(5a - 2^{k+1})$

which is a multiple of 3, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2d A. When $n = 1$,

$9^{n+2} - 4^n = 9^3 - 4$

$= 729 - 4$

$= 725$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $9^{k+2} - 4^k = 5a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$9^{k+3} - 4^{k+1}$ is a multiple of 5.

© Cambridge University Press 2019

By the induction hypothesis:

$9^{k+2} - 4^k = 5a$

Therefore

$9(9^{k+2} - 4^k) = 45a$

$9 \times 9^{k+2} - 9 \times 4^k = 45a$

$9^{k+3} - 4 \times 4^k - 5 \times 4^k = 45a$

$9^{k+3} - 4^{k+1} = 45a + 5 \times 4^k$

$= 5(9a + 4^k)$

which is a multiple of 5, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2e A. When $n = 1$, $6^n - 5n + 4 = 6 - 5 + 4 = 5$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $6^k - 5k + 4 = 5a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$6^{k+1} - 5(k + 1) + 4$

is a multiple of 5.

By the induction hypothesis:

$6^k - 5k + 4 = 5a$

$6(6^k - 5k + 4) = 30a$

$6 \times 6^k - 30k + 24 = 30a$

$6^{k+1} - 5k - 5 - 25k + 29 = 30a$

$6^{k+1} - 5(k + 1) + 4 - 25k + 25 = 30a$

$6^{k+1} - 5(k + 1) + 4 = 30a + 25k - 25$

$6^{k+1} - 5(k + 1) + 4 = 5(6a + 5k - 5)$

© Cambridge University Press 2019

which is a multiple of 5, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2f A. When $n = 1$,

$4^n + 6n - 1 = 4 + 6 - 1 = 9$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $4^k + 6k - 1 = 9a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$4^{k+1} + 6(k + 1) - 1$ is a multiple of 9.

By the induction hypothesis:

$4^k + 6k - 1 = 9a$

$4(4^k + 6k - 1) = 36a$

$4 \times 4^k + 24k - 4 = 36a$

$4^{k+1} + 6k + 18k + 6 - 10 = 36a$

$4^{k+1} + 6(k + 1) + 18k - 1 - 9 = 36a$

$4^{k+1} + 6(k + 1) - 1 = 36a - 18k + 9$

$= 9(4a - 2k + 1)$

which is a multiple of 9, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

© Cambridge University Press 2019

Solutions to Exercise 2F Foundation questions

1a Consider triangle AOB, as shown in the diagram.

Angle $AOB = \frac{360^\circ}{12} = 30^\circ$

Therefore

Area of $AOB = \frac{1}{2}|OA||OB| \sin 30^\circ$

$= \frac{1}{4}$ cm^2

Since the inscribed dodecagon is made up of twelve such triangles, its area is

$12 \times \frac{1}{4}\text{cm}^2 = 3$ cm^2.

1b i Let $t = \tan 15^\circ$.

From the double-angle formula,

$$\tan 30^\circ = \frac{2t}{1-t^2}$$

$$\frac{1}{\sqrt{3}} = \frac{2t}{1-t^2}$$

$$1 - t^2 = 2t\sqrt{3}$$

$$t^2 + 2t\sqrt{3} - 1 = 0$$

$$t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2}$$

$$= -\sqrt{3} \pm 2$$

In the first quadrant, tan must be positive.

Therefore $t = 2 - \sqrt{3}$ so $\tan 15^\circ = 2 - \sqrt{3}$.

© Cambridge University Press 2019

Chapter 2 worked solutions – Proof

1b ii Let M be the point where HG touches the circle.

Angle $MOH = 15°$ and $|OM| = 1$ cm

Therefore $|MH| = (2 - \sqrt{3})$ cm

Therefore

Area of $\Delta OGH = 2 \times \frac{1}{2} \times 1 \text{ cm} \times (2 - \sqrt{3}) \text{ cm}$

$= (2 - \sqrt{3}) \text{ cm}^2$

Therefore area of the circumscribed dodecagon is $12(2 - \sqrt{3}) \text{ cm}^2$.

1c Since the circle lies entirely within the circumscribed dodecagon, and the inscribed dodecagon lies entirely within the circle, the circle's area must be between the two dodecagons.

Therefore $3 \text{ cm}^2 < \pi \text{ cm}^2 < 12(2 - \sqrt{3}) \text{ cm}^2$

$3 < \pi < 12(2 - \sqrt{3}) \div 3.22$

2a Let $x_0 = 0, x_1 = \frac{\pi}{6}, x_2 = \frac{\pi}{3}$

By Simpson's rule,

$$\text{Area } \div \frac{\pi}{6} \times \frac{1}{3}\left(0 + 4 \times \sin\frac{\pi}{6} + \sin\frac{\pi}{3}\right)$$

$$= \frac{\pi}{36}(4 + \sqrt{3})$$

2b We can find the exact area by integration:

$$\int_0^{\frac{\pi}{3}} \sin x \ dx$$

$$= [-\cos x]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2} - (-1)$$

$$= \frac{1}{2}$$

© Cambridge University Press 2019

Therefore

$$\frac{\pi}{36}(4+\sqrt{3}) \div \frac{1}{2}$$

$$\pi \div \frac{18}{4+\sqrt{3}}$$

$$= \frac{18(4-\sqrt{3})}{(4+\sqrt{3})(4-\sqrt{3})}$$

$$= \frac{18(4-\sqrt{3})}{16-3}$$

$$= \frac{18}{13}(4-\sqrt{3}) \text{ as required.}$$

Note that:

$$\frac{18}{13}(4-\sqrt{3}) = 3.140\,237\ldots \div 3.14$$

3 Area $ABDC$

$$= \frac{|AC|+|BD|}{2}|CD|$$

$$= \frac{\left(1+\frac{1}{2}\right)}{2} \times 1$$

$$= \frac{\left(\frac{3}{2}\right)}{2}$$

$$= \frac{3}{4} \text{ square units}$$

Let $Q = \left(\frac{3}{2}, 0\right)$ be the foot of the perpendicular drawn from P to the x-axis.

Construct the horizontal line that passes through P. Let E be the point where this line crosses MC and let F be the point where this line crosses the continuation of ND.

$$|EP| = |PF| = \frac{1}{2}$$

$$\angle MPE = \angle NPF$$

$$\angle MEP = \angle NFP = 90°$$

© Cambridge University Press 2019

Therefore $\Delta MEP \equiv \Delta NFP$ (angle-side-angle)

Therefore $|ME| = |NF|$

Therefore $|MC| + |ND| = |EC| + |FD| = 2|PQ|$

Area $MNDC$

$$= \frac{|MC| + |ND|}{2}|CD|$$

$$= \frac{2|PQ|}{2}|CD|$$

$$= |PQ||CD|$$

$$= \frac{2}{3} \times 1$$

$$= \frac{2}{3} \text{ square units}$$

3b The exact area under the curve between $x = 1$ and $x = 2$ can be found by integration:

$$\int_1^2 \frac{1}{x} dx$$

$$= [\ln x]_1^2$$

$$= \ln 2 - \ln 1$$

$$= \ln 2$$

Since this area is completely covered by trapezium $ABDC$, and completely covers $MNDC$, it follows that:

area $MNDC <$ ln 2 $<$ area $ABDC$

That is, $\frac{2}{3} < \ln 2 < \frac{3}{4}$

© Cambridge University Press 2019

4a The exact area is:

$$\int_{-1}^{0} e^x dx$$

$$= [e^x]_{-1}^{0}$$

$$= (1 - e^{-1}) \text{ square units}$$

4b Area of $OABC$

$$= \frac{|AO| + |BC|}{2}|OC|$$

$$= \frac{1 + e^{-1}}{2} \times 1$$

$$= \frac{1}{2}(1 + e^{-1}) \text{ square units}$$

4c Construct the horizontal line that passes through P. Let E be the point where this line crosses MO and let F be the point where this line crosses the continuation of NC.

$$|EP| = |PF| = \frac{1}{2}$$

$$\angle MPE = \angle NPF$$

$$\angle MEP = \angle NFP = 90°$$

Therefore $\Delta MEP \equiv \Delta NFP$ (angle-side-angle)

Therefore $|ME| = |NF|$

Therefore $|MO| + |NC| = |EO| + |FC| = 2|PQ|$

Area $OMNC$

$$= \frac{|MO| + |NC|}{2}|CO|$$

$$= \frac{2|PQ|}{2}|CO|$$

$$= |PQ||CO|$$

$$= e^{-\frac{1}{2}} \times 1$$

© Cambridge University Press 2019

$= e^{-\frac{1}{2}}$ square units

4d The area under the curve is completely covered by $OABC$, and completely covers $MNCO$.

Therefore

area $MNCO < 1 - e^{-1} <$ area $OABC$

$$e^{-\frac{1}{2}} < 1 - e^{-1} < \frac{1}{2}(1 + e^{-1})$$

Looking at the right-hand inequality:

$$1 - e^{-1} < \frac{1}{2}(1 + e^{-1})$$

$2e - 2 < e + 1$ (multiplying both sides by $2e$)

$e < 3$

Looking at the left-hand inequality:

$$e^{-\frac{1}{2}} < 1 - e^{-1}$$

Let $k = e^{-\frac{1}{2}}$

Therefore $k < 1 - k^2$

$k^2 + k - 1 < 0$

$$k^2 + k + \frac{1}{4} < \frac{5}{4}$$

$$\left(k + \frac{1}{2}\right)^2 < \left(\frac{\sqrt{5}}{2}\right)^2$$

$k + \frac{1}{2} < \frac{\sqrt{5}}{2}$ (using the fact that both sides are positive)

$$e^{-\frac{1}{2}} + \frac{1}{2} < \frac{\sqrt{5}}{2}$$

$$e^{-\frac{1}{2}} < \frac{\sqrt{5} - 1}{2}$$

$$e^{-1} < \left(\frac{\sqrt{5} - 1}{2}\right)^2$$

© Cambridge University Press 2019

$$e^{-1} < \frac{6 - 2\sqrt{5}}{4}$$

$$e^{-1} < \frac{3 - \sqrt{5}}{2}$$

$$e > \frac{2}{3 - \sqrt{5}} \quad \text{(taking reciprocal of both sides, noting that both are positive)}$$

$$e > \frac{2(3 + \sqrt{5})}{(3 + \sqrt{5})(3 - \sqrt{5})}$$

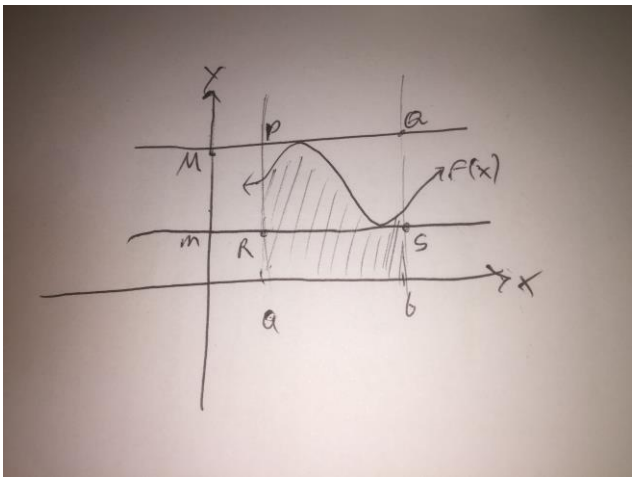
$$e > \frac{2(3 + \sqrt{5})}{9 - 5}$$

$$e > \frac{3 + \sqrt{5}}{2}$$

Therefore:

$$\frac{1}{2}(3 + \sqrt{5}) < e < 3$$

5 Construct points $P = (a, M), Q = (b, M), R = (a, m), S = (b, m)$:



Since $m \leq f(x) \leq M$ in the interval $a \leq x \leq b$, it follows that

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$\int_a^b m\,dx = [mx]_a^b = m(b - a)$$

© Cambridge University Press 2019

$$\int_a^b M\,dx = [Mx]_a^b = M(b-a)$$

Therefore

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

© Cambridge University Press 2019