(1)

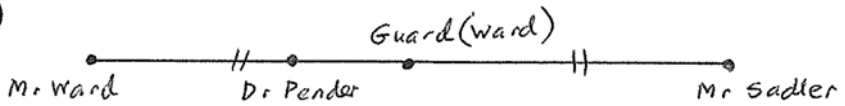
Enrichment Solutions - Extension 2

Ch. 2 Proof

2A

(15) If either Anna or Bryan passed then Chris passed.
Since the statement is false, Chris failed.

(16)



Mr Ward earns \$100000, which is not divisible by 3.
So the guard's nearest neighbour is not Mr Ward.
So Mr Ward lives in Melbourne and so Dr Pender is the guard's nearest neighbour. So the guard is Ward.
Pender is neither the fireman nor the guard.
So the driver is Pender.

2B

(20)(a) $a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$

(b) Put $a = 545$ and $b = 4^{136}$ into (a):

$545^4 + 4 \times (4^{136})^4$

$= (545^2 - 2 \times 545 \times 4^{136} + 2 \times 4^{272})(545^2 + 2 \times 545 \times 4^{136} + 2 \times 4^{272})$

The first factor cannot be 1, since the second factor is obviously less than $545^4 + 4^{545}$.
So $545^4 + 4^{545}$ is not prime.

(21)(a) Let n be an even number.

Then $n = 2k$, for some $k \in \mathbb{Z}$.

So $n^2 = 4k^2$, which is divisible by 4 since $k^2 \in \mathbb{Z}$.

(b) Let m be an odd number.

Then $m = 2l+1$, for some $l \in \mathbb{Z}$.

So $m^2 = (2l+1)^2$

$= 4l^2 + 4l + 1$

$= 4l(l+1) + 1$ (*)

Now, $4l(l+1)$ is divisible by 8, since either l or $l+1$ is even.

So, from (*), it follows that the remainder is 1 when m^2 is divided by 8.

(c) Let $a = 2p+1$ and $b = 2q+1$, where $p, q \in \mathbb{Z}$.

Then $a^2+b^2 = (4p(p+1)+1) + (4q(q+1)+1)$

$= 4p(p+1) + 4q(q+1) + 2$.

So the remainder is 2 when a^2+b^2 is divided by 4.

Since a^2+b^2 is even, it follows from part (a) that a^2+b^2 is not a square.

(Even if a^2+b^2 was odd, part (b) tells us that it couldn't be a square because the remainder is 2 when it is divided by 8.)

(3)

(22) Let p be a prime greater than 30.

Then, when p is divided by $30 = 2\times3\times5$, we have $p = 30q + r$, for some quotient q and remainder r, where $0 \le r < 30$.

If r is composite, then it is divisible by 2, 3 or 5, since all the even composite numbers less than 30 are obviously divisible by 2, and 9, 15, 21, 27 are divisible by 3, and 25 is divisible by 5.

So if r is composite, then $30q + r = p$ is composite, since 30 is divisible by 2, 3 and 5, while r is divisible by 2, 3 or 5.

But we know that p is prime, so r cannot be composite, so r is 1 or prime.

(23) (a) The factors of 6 excluding itself are 1, 2, 3, and $1+2+3=6$.

The factors of 28 excluding itself are 1, 2, 4, 7, 14 and $1+2+4+7+14=28$.

(b) The sum of the factors that are powers of 2 is

$$S_1 = 2^0 + 2^1 + \cdots + 2^{n-1} \quad \text{(geometric series)}$$
$$= 2^n - 1.$$

The sum of the factors that include $2^n - 1$ is

$$S_2 = (2^n - 1)(2^0 + 2^1 + \cdots + 2^{n-2}),$$ noting that we must exclude the number itself,

$$= (2^n - 1)(2^{n-1} - 1)$$
$$= 2^{n-1}(2^n - 1) - (2^n - 1)$$

So the sum of all the factors excluding the number itself is $S_1 + S_2 = 2^{n-1}(2^n - 1)$, so $2^{n-1}(2^n - 1)$ is perfect.

(24) $1 = 2^0 \times 1$

$2 = 2^1 \times 1$

$3 = 2^0 \times 3$

$4 = 2^2 \times 1$

$5 = 2^0 \times 5$

$6 = 2^1 \times 3$

$7 = 2^0 \times 7$

$8 = 2^3 \times 1$

$9 = 2^0 \times 9$

$10 = 2^1 \times 5$

(4)

We will now use the pigeonhole principle, where the odd numbers 1, 3, 5, 7, 9 in the products above represent the pigeonholes.

Since we are choosing 6 numbers, there must be a pigeonhole with 2 (or more) numbers in it.

These 2 numbers have the same odd number in their product, so they only differ in their power of 2.

It follows that the number with the lower power of 2 is a divisor (factor) of the number with the higher power of 2, and so our proof is complete.

2C

(5)

(16)(a) If p is not prime then $(\exists\, a, b \in \mathbb{Z}^+,$ $p \mid ab \Rightarrow p \nmid a$ and $p \nmid b)$.

(b) Let $a = p_1$, where p_1 is a prime factor of p, and let $b = \frac{p}{p_1}$.

Then $p \mid ab$, since $ab = p_1 \cdot \frac{p}{p_1}$

$= p$,

and $p \nmid p_1 = a$, since $p_1 < p$,

and $p \nmid \frac{p}{p_1} = b$, since $\frac{p}{p_1} < p$.

So the theorem is proven.

(17)(a) $\frac{a}{b+\sqrt{c}} + \frac{d}{\sqrt{c}} = \frac{a(b-\sqrt{c})}{b^2-c} + \frac{d\sqrt{c}}{c}$

$= \frac{ac(b-\sqrt{c}) + d(b^2-c)\sqrt{c}}{c(b^2-c)}$

$= \frac{(abc) + (b^2d - ac - cd)\sqrt{c}}{c(b^2-c)}$

If this is rational, then $b^2d - ac - cd = 0$

$b^2d = c(a+d)$.

(b) Let $b = 1$, and suppose that $\frac{a}{1+\sqrt{c}} + \frac{d}{\sqrt{c}}$ is rational.

Then, from (a), $d = c(a+d)$, and so $a+d$ is a factor of d.

This is impossible, since $a, d \in \mathbb{Z}^+$.

So $\frac{a}{1+\sqrt{c}} + \frac{d}{\sqrt{c}}$ is not rational.

(6)

(18) Assume that no digit occurs more than twice.
Then, because p^n is a 20-digit number, each of the digits $0, 1, \ldots, 9$ occurs exactly twice.
So the sum of the 20 digits is
$2(0+1+2+\cdots+9) = 2(45)$
$= 90$, which is divisible by 3.
From the divisibility test for 3, it follows that p^n is divisible by 3.
This is impossible, since p is a prime greater than 3, and so p^n is not divisible by 3.
We have a contradiction, so there must be a digit that occurs at least 3 times.

(19) Assume that 2^k is the smallest power of 2 equal to $(36a+b)(36b+a)$.
It follows that $36a+b$ and $36b+a$ are both powers of 2, so a and b are both even, since $36a$ and $36b$ are even.
Let $36a+b = 2^m$, where $m \in \mathbb{Z}^+$, and $36b+a = 2^n$, where $n \in \mathbb{Z}^+$.
Since a, b are even, we can put $a = 2p$ and $b = 2q$, where $p, q \in \mathbb{Z}^+$.
Then $36(2p) + (2q) = 2^m$, and so $36p + q = 2^{m-1}$, and similarly, $36q + p = 2^{n-1}$.
So $(36p+q)(36q+p) = 2^{m-1} \cdot 2^{n-1}$
$= 2^{m+n-2}$
$= 2^{k-2}$.
This is a contradiction, since 2^{k-2} is a smaller power of 2 than 2^k.
(Note that, since $p, q \in \mathbb{Z}^+$, they are possible values of a and b respectively.)

2D

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(19) $\dfrac{1+ab}{1+a}+\dfrac{1+bc}{1+b}+\dfrac{1+ca}{1+c}$

$=\dfrac{1+\frac{1}{c}}{1+a}+\dfrac{1+\frac{1}{a}}{1+b}+\dfrac{1+\frac{1}{b}}{1+c}$, since $abc=1$

$=\dfrac{1+c}{c(1+a)}+\dfrac{1+a}{a(1+b)}+\dfrac{1+b}{b(1+c)}$

$\geqslant 3\sqrt[3]{\dfrac{(1+c)(1+a)(1+b)}{abc(1+a)(1+b)(1+c)}}$, using AM/GM with $n=3$

$=3\sqrt[3]{1}$

$=3$

(20)(a)(i) RHS $-$ LHS $= a^2-(a+b-c)(a+c-b)$

$=a^2-(a^2+ac-ab+ab+bc-b^2-ac-c^2+bc)$

$=a^2-(a^2-b^2-c^2+2bc)$

$=b^2+c^2-2bc$

$=(b-c)^2$

$\geqslant 0$, since a square cannot be negative.

So LHS $\leqslant$ RHS.

So $(a+b-c)(a+c-b)\leqslant a^2$.

(ii) From (i), we have

$(a+b-c)(a+c-b)\leqslant a^2$

and $(b+c-a)(b+a-c)\leqslant b^2$

and $(c+a-b)(c+b-a)\leqslant c^2$.

Multiplying, it follows that

$(a+b-c)^2(b+c-a)^2(a+c-b)^2\leqslant a^2b^2c^2$

so $(a+b-c)(b+c-a)(a+c-b)\leqslant abc$, since both sides are positive.

(8)

(20)(b)(i) By the cosine rule,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

so $1 - 2\sin^2\frac{A}{2} = \frac{b^2+c^2-a^2}{2bc}$ (using the double angle result $\cos 2\theta = 1-2\sin^2\theta$)

$$2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{2bc-b^2-c^2+a^2}{4bc}$$

$$= \frac{a^2-(b-c)^2}{4bc}$$

$$= \frac{(a+b-c)(a+c-b)}{4bc}.$$

(ii) Using part (i), we have

$$\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{(a+b-c)(a+c-b)}{4bc}\cdot\frac{(b+a-c)(b+c-a)}{4ac}\cdot\frac{(c+a-b)(c+b-a)}{4ab}$$

$$= \frac{(a+b-c)^2(a+c-b)^2(b+c-a)^2}{64a^2b^2c^2}$$

so $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{(a+b-c)(a+c-b)(b+c-a)}{8abc}$,

since LHS, RHS > 0

so $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{abc}{8abc} = \frac{1}{8}$, using (a)(ii).

Note: We can use the results in (a), because $a, b, c > 0$, being the sides of a triangle, and $a+b>c$, $a+c>b$, $b+c>a$ by the triangle inequality.

(9)

(21) $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$

$= \left(\frac{a}{b+c}+1\right) + \left(\frac{b}{a+c}+1\right) + \left(\frac{c}{a+b}+1\right) - 3$

$= \frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3$

$= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3$

$= \frac{1}{2}\left((b+c)+(a+c)+(a+b)\right)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3$

$\geq \frac{1}{2} \cdot 3.\sqrt[3]{(b+c)(a+c)(a+b)} \cdot 3.\sqrt[3]{\frac{1}{(b+c)(a+c)(a+b)}} - 3,$

(by AM/GM with $n=3$)

$= \frac{9}{2} \cdot \sqrt[3]{1} - 3$

$= \frac{3}{2}$

2E (22)(b) Confirm the result for $n=1$ and $n=2$. (10)

$T_1 = 5^1 + 3^1 = 8$

$T_2 = 5^2 + 3^2 = 34$

So the result is true for $n=1$ and $n=2$.

Assume that the result is true for $n=k$ and $n=k+1$, where $k \geq 3$.

i.e. assume that $T_k = 5^k + 3^k$ and $T_{k+1} = 5^{k+1} + 3^{k+1}$. (*)

Prove that the result is true for $n=k+2$.

i.e. prove that $T_{k+2} = 5^{k+2} + 3^{k+2}$.

$LHS = T_{k+2}$

$= 8T_{k+1} - 15T_k$ (from the definition of the sequence)

$= 8(5^{k+1} + 3^{k+1}) - 15(5^k + 3^k)$

$= 8 \times 5 \times 5^k + 8 \times 3 \times 3^k - 15 \times 5^k - 15 \times 3^k$

$= 25 \times 5^k + 9 \times 3^k$

$= 5^{k+2} + 3^{k+2}$

$= RHS$

So the result is true for $n=k+2$ if it is true for $n=k$ and $n=k+1$.

But the result is true for $n=1$ and $n=2$.

So, by induction, the result is true for all positive integer values of n.

The other parts of Q22 have very similar solutions.

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(11)

(23)(a) $LHS = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)$

$= \frac{1}{1\times 2}+\frac{1}{3\times 4}+\cdots+\frac{1}{2k(2k-1)}$

$> \frac{1}{(2k+1)(2k+2)}+\frac{1}{(2k+1)(2k+2)}+\cdots+\frac{1}{(2k+1)(2k+2)}$ (k terms)

(since $1\times 2, 3\times 4, \ldots, 2k(2k-1)$ are all less than $(2k+1)(2k+2)$)

$= \frac{k}{(2k+1)(2k+2)} = RHS$

(b) When $n=2$, $LHS = 2\left(1+\frac{1}{3}\right) = \frac{8}{3}$

$RHS = 3\left(\frac{1}{2}+\frac{1}{4}\right) = \frac{9}{4} < LHS$,

so the result is true for $n=2$.

Assume that the result is true for $n=k$, where $k \geq 2$.

i.e. assume that $k\left(1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2k}\right)$ (*)

Prove that the result is true for $n=k+1$.

i.e. prove that

$$(k+1)\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}+\frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}+\frac{1}{2k+2}\right).$$

$LHS = (k+1)\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}+\frac{1}{2k+1}\right)$

$= k\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}\right)+1\left(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$

$> \underbrace{(k+1)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\right)}_{\text{(by (*))}} + \underbrace{\frac{k}{(2k+1)(2k+2)}+\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\right)}_{\text{(by part (a))}}+\frac{k+1}{2k+1}$

$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\right)+\frac{k+(k+1)(2k+2)}{(2k+1)(2k+2)}$

$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\right)+\frac{2k^2+5k+2}{(2k+1)(2k+2)}$

$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}\right)+\frac{(2k+1)(k+2)}{(2k+1)(2k+2)}$

$= (k+2)\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2k}+\frac{1}{2k+2}\right)$

$= RHS$, so, by induction, the result is true for all $n \geq 2$.

(24) $S_2 = 1 \times 2 = 2$ (12)

and, using the formula for S_n,

$S_2 = \frac{1}{24}(1)(2)(3)(8) = 2$.

So the result is true for $n = 2$.

Assume that the result is true for $n = k$, where $k \geq 2$.

i.e. assume that $S_k = \frac{1}{24}k(k-1)(k+1)(3k+2)$. (*)

Prove that the result is true for $n = k+1$.

i.e. prove that $S_{k+1} = \frac{1}{24}k(k+1)(k+2)(3k+5)$.

$$S_{k+1} = S_k + (k+1)(1+2+\cdots+k)$$

$$= \frac{1}{24}k(k-1)(k+1)(3k+2) + (k+1)\cdot\frac{1}{2}k(k+1) \quad \text{(by *)}$$

$$= \frac{1}{24}k(k+1)\left((k-1)(3k+2) + 12(k+1)\right)$$

$$= \frac{1}{24}k(k+1)(3k^2+2k-3k-2+12k+12)$$

$$= \frac{1}{24}k(k+1)(3k^2+11k+10)$$

$$= \frac{1}{24}k(k+1)(k+2)(3k+5), \text{ as required.}$$

So, by induction, the result is true for all integers $n \geq 2$.

(13)

(25)(a) There are $(n-1)$ players with whom Ben can swap tops. This leaves $(n-2)$ players for whom there are D_{n-2} derangements, so the number of such derangements is $(n-1) \cdot D_{n-2}$.

(b) Besides Ben, there are $(n-1)$ players for whom there are D_{n-1} derangements. Then Ben must swap tops with one of the other $(n-1)$ players, who, at this stage, does NOT have his own top.

So the number of such derangements is $D_{n-1} \cdot (n-1)$.

Now, either Ben does a direct swap with another player or he doesn't.

So $D_n = (n-1) \cdot D_{n-1} + (n-1) \cdot D_{n-2}$, for $n > 2$.

(c) $D_n = (n-1)\left(D_{n-1} + D_{n-2}\right)$

$= n \cdot D_{n-1} - D_{n-1} + (n-1) \cdot D_{n-2}$

so $D_n - n \cdot D_{n-1} = -\left(D_{n-1} - (n-1) \cdot D_{n-2}\right)$.

(d) $D_1 = 0$ and $D_2 = 1$ (if there are 2 players, the only derangement is if they swap tops)

when $n=2$, $D_n - n \cdot D_{n-1} = D_2 - 2 \cdot D_1$

$= 1 - 2(0)$

$= 1$

$= (-1)^2$.

So the result is true for $n=2$.

(25)(d)(continued)

Assume that the result is true for $n=k$, where $k \geqslant 2$.

i.e. assume that $D_k - k.D_{k-1} = (-1)^k$. (*)

Prove that the result is true for $n=k+1$.

i.e. prove that $D_{k+1} - (k+1).D_k = (-1)^{k+1}$.

$$\begin{aligned} LHS &= D_{k+1} - (k+1).D_k \\ &= -(D_k - k.D_{k-1}) \quad \text{(using part (c))} \\ &= (-1) \times (-1)^k \quad \text{(using (*))} \\ &= (-1)^{k+1} \\ &= RHS \end{aligned}$$

So, by induction, the result is true for all integers $n \geqslant 1$.

(e) When $n=1$, $D_n = D_1$

$$\begin{aligned} &= 1! \times \sum_{r=0}^{1} \frac{(-1)^r}{r!} \\ &= \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!} \\ &= 1 + (-1) \\ &= 0. \end{aligned}$$

So the result is true for $n=1$.

Assume that the result is true for $n=k$.

i.e. assume that $D_k = k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$. (*)

Prove that the result is true for $n=k+1$.

i.e. prove that $D_{k+1} = (k+1)! \times \sum_{r=0}^{k+1} \frac{(-1)^r}{r!}$.

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(25)(e) (continued)

$$D_{k+1} - (k+1).D_k = (-1)^{k+1} \quad \text{(from (d))}$$

so $D_{k+1} = (k+1).D_k + (-1)^{k+1}$

$$= (k+1) \times k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!} + (-1)^{k+1} \quad \text{(using } \circledast\text{)}$$

$$= (k+1)! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!} + (k+1)! \times \frac{(-1)^{k+1}}{(k+1)!}$$

$$= (k+1)! \times \left(\sum_{r=0}^{k} \frac{(-1)^r}{r!} + \frac{(-1)^{k+1}}{(k+1)!} \right)$$

$$= (k+1)! \times \sum_{r=0}^{k+1} \frac{(-1)^r}{r!}, \text{ as required.}$$

So, by induction, the result is true for all $n \in \mathbb{Z}^+$.