Chapter 1 worked solutions – Complex numbers I

Solutions to Exercise 1A Foundation questions

1a i^2
$= i^{4(0)+2}$
$= -1$

1b i^4
$= i^{4(1)}$
$= 1$

1c i^7
$= i^{4(1)+3}$
$= -i$

1d i^{13}
$= i^{4(3)+1}$
$= i$

1e i^{29}
$= i^{4(7)+1}$
$= i$

1f i^{2010}
$= i^{4(502)+2}$
$= -1$

1g $i^3 + i^4 + i^5$
$= -i + 1 + i$
$= 1$

1h $i^7 + i^{16} + i^{21} + i^{22}$
$= -i + 1 + i - 1$
$= 0$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

2a $\overline{2i}$
$= \overline{2} \times \overline{i}$
$= 2(-i)$
$= -2i$

2b $\overline{3+i}$
$= \overline{3} + \overline{i}$
$= 3 - i$

2c $\overline{1-i}$
$= \overline{1} - \overline{i}$
$= 1 - (-i)$
$= 1 + i$

2d $\overline{5-3i}$
$= \overline{5} + \overline{-3i}$
$= \overline{5} + \overline{-3} \times \overline{i}$
$= 5 - 3\overline{i}$
$= 5 - 3(-i)$
$= 5 + 3i$

2e $\overline{-3+2i}$
$= \overline{-3} + \overline{2i}$
$= \overline{-3} + \overline{2} \times \overline{i}$
$= -3 + 2(-i)$
$= -3 - 2i$

3a $(7+3i) + (5-5i)$
$= 7 + 3i + 5 - 5i$
$= (7+5) + (3-5)i$
$= 12 - 2i$

3b $(-8+6i) + (2-4i)$
$= -8 + 6i + 2 - 4i$
$= (-8+2) + (6-4)i$
$= -6 + 2i$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

3c $(4-2i)-(3-7i)$
$=4-2i-3+7i$
$=(4-3)+(-2+7)i$
$=1+5i$

3d $(3-5i)-(-4+6i)$
$=3-5i+4-6i$
$=(3+4)+(-5-6)i$
$=7-11i$

4a $(4+5i)i$
$=4i+5i^2$
$=4i-5$
$=-5+4i$

4b $(1+2i)(3-i)$
$=3-i+6i-2i^2$
$=3-i+6i+2$
$=5+5i$

4c $(3+2i)(4-i)$
$=12-3i+8i-2i^2$
$=12-3i+8i+2$
$=14+5i$

4d $(-7+5i)(8-6i)$
$=-56+42i+40i-30i^2$
$=-56+42i+40i+30$
$=-26+82i$

4e $(5+i)^2$
$=25+10i+i^2$
$=25+10i-1$
$=24+10i$

© Cambridge University Press 2019

4f $(2-3i)^2$
$= 4-12i+9i^2$
$= 4-12i-9$
$= -5-12i$

4g $(2+i)^3$
$= 2^3+3\times 2^2 i+3\times 2i^2+i^3$
$= 8+12i-6-i$
$= 2+11i$

4h $(1-i)^4$
$= ((1-i)^2)^2$
$= (1-2i+i^2)^2$
$= (1-2i-1)^2$
$= (-2i)^2$
$= 4i^2$
$= -4$

4i $(3-i)^4$
$= ((3-i)^2)^2$
$= (9-6i+i^2)^2$
$= (9-6i-1)^2$
$= (8-6i)^2$
$= 64-96i+36i^2$
$= 64-96i-36$
$= 28-96i$

5a $(1+2i)(1-2i)$
$= 1^2-(2i)^2$
$= 1-4i^2$
$= 1+4$
$= 5$

5b $(4+i)(4-i)$
$= 4^2-i^2$
$= 16+1$
$= 17$

© Cambridge University Press 2019

5c $(5 + 2i)(5 - 2i)$
$= 5^2 - (2i)^2$
$= 25 - 4i^2$
$= 25 + 4$
$= 29$

5d $(-4 - 7i)(-4 + 7i)$
$= (-4)^2 - (7i)^2$
$= 16 - 49i^2$
$= 16 + 49$
$= 65$

6a

$$\frac{1}{i}$$
$$= \frac{i}{i \times i}$$
$$= \frac{i}{-1}$$
$$= -i$$

6b

$$\frac{2 + i}{i}$$
$$= \frac{(2 + i)i}{i \times i}$$
$$= \frac{2i + i^2}{i^2}$$
$$= \frac{2i - 1}{-1}$$
$$= -2i + 1$$
$$= 1 - 2i$$

© Cambridge University Press 2019

6c

$$\begin{aligned}
&\frac{5-i}{1-i} \\
&= \frac{(5-i)(1+i)}{(1-i)(1+i)} \\
&= \frac{5+5i-i-i^2}{1-i^2} \\
&= \frac{5+5i-i+1}{1+1} \\
&= \frac{6+4i}{2} \\
&= 3+2i
\end{aligned}$$

6d

$$\begin{aligned}
&\frac{6-7i}{4+i} \\
&= \frac{(6-7i)(4-i)}{(4+i)(4-i)} \\
&= \frac{24-6i-28i+7i^2}{4^2-i^2} \\
&= \frac{24-6i-28i-7}{16+1} \\
&= \frac{17-34i}{17} \\
&= 1-2i
\end{aligned}$$

6e

$$\begin{aligned}
&\frac{-11+13i}{5+2i} \\
&= \frac{(-11+13i)(5-2i)}{(5+2i)(5-2i)} \\
&= \frac{-55+22i+65i-26i^2}{5^2-(2i)^2} \\
&= \frac{-55+22i+65i-26i^2}{5^2-4i^2} \\
&= \frac{-55+22i+65i+26}{25+4} \\
&= \frac{-29+87i}{29} \\
&= -1+3i
\end{aligned}$$

© Cambridge University Press 2019

6f

$$\begin{aligned}
&\frac{(1+i)^2}{3-i} \\
&= \frac{1+2i+i^2}{3-i} \\
&= \frac{1+2i-1}{3-i} \\
&= \frac{2i}{3-i} \\
&= \frac{2i(3+i)}{(3-i)(3+i)} \\
&= \frac{6i+2i^2}{9-i^2} \\
&= \frac{6i-2}{9+1} \\
&= \frac{6i-2}{10} \\
&= \frac{3}{5}i - \frac{1}{5} \\
&= -\frac{1}{5} + \frac{3}{5}i
\end{aligned}$$

7a $\overline{(iz)}$

$$\begin{aligned}
&= \overline{\big(i(1+2i)\big)} \\
&= \overline{i+2i^2} \\
&= \overline{i-2} \\
&= \overline{i} - \overline{2} \\
&= -i - 2 \\
&= -2 - i
\end{aligned}$$

7b $w + \overline{z}$

$$\begin{aligned}
&= (3-i) + \overline{(1+2i)} \\
&= 3 - i + (1-2i) \\
&= 4 - 3i
\end{aligned}$$

© Cambridge University Press 2019

7c $2z + iw$
$= 2(1 + 2i) + i(3 - i)$
$= 2 + 4i + 3i - i^2$
$= 2 + 4i + 3i + 1$
$= 3 + 7i$

7d $\text{Im}(5i - z)$
$= \text{Im}\big(5i - (1 + 2i)\big)$
$= \text{Im}(5i - 1 - 2i)$
$= \text{Im}(3i - 1)$
$= 3$

7e z^2
$= (1 + 2i)^2$
$= 1 + 4i + (2i)^2$
$= 1 + 4i + 4i^2$
$= 1 + 4i - 4$
$= -3 + 4i$

8a $\overline{z} - w$
$= \overline{8 + i} - (2 - 3i)$
$= 8 - i - 2 + 3i$
$= 6 + 2i$

8b $\text{Im}(3iz + 2w)$
$= \text{Im}\big(3i(8 + i) + 2(2 - 3i)\big)$
$= \text{Im}\big(24i + 3i^2 + 2(2 - 3i)\big)$
$= \text{Im}(24i - 3 + 4 - 6i)$
$= \text{Im}(18i + 1)$
$= 18$

8c zw
$= (8 + i)(2 - 3i)$
$= 16 - 24i + 2i - 3i^2$
$= 16 - 24i + 2i + 3$
$= 19 - 22i$

© Cambridge University Press 2019

8d $65 \div z$

$$= \frac{65}{z}$$
$$= \frac{65}{8+i}$$
$$= \frac{65(8-i)}{(8+i)(8-i)}$$
$$= \frac{65(8-i)}{8^2-i^2}$$
$$= \frac{65(8-i)}{64+1}$$
$$= \frac{65(8-i)}{65}$$
$$= 8-i$$

8e

$$\frac{z}{w}$$
$$= \frac{8+i}{2-3i}$$
$$= \frac{(8+i)(2+3i)}{(2-3i)(2+3i)}$$
$$= \frac{16+24i+2i+3i^2}{2^2-(3i)^2}$$
$$= \frac{16+24i+2i-3}{2^2-9i^2}$$
$$= \frac{16+24i+2i-3}{4+9}$$
$$= \frac{16+26i-3}{4+9}$$
$$= \frac{13+26i}{13}$$
$$= 1+2i$$

9a $-zw$

$$= -(2-i)(-5-12i)$$
$$= (2-i)(5+12i)$$
$$= 10+24i-5i-12i^2$$
$$= 10+24i-5i+12$$
$$= 22+19i$$

© Cambridge University Press 2019

9b $(1+i)\overline{z}-w$

$=(1+i)\overline{(2-i)}-(-5-12i)$

$=(1+i)(2+i)-(-5-12i)$

$=2+2i+i+i^2+5+12i$

$=2+2i+i-1+5+12i$

$=6+15i$

9c

$\frac{10}{\overline{z}}$

$=\frac{10}{\overline{\overline{2-i}}}$

$=\frac{10}{2+i}$

$=\frac{10(2-i)}{(2-i)(2+i)}$

$=\frac{20-10i}{4-i^2}$

$=\frac{20-10i}{4+1}$

$=\frac{20-10i}{5}$

$=4-2i$

9d

$\frac{w}{2-3i}$

$=\frac{-5-12i}{2-3i}$

$=\frac{(-5-12i)(2+3i)}{(2-3i)(2+3i)}$

$=\frac{-10-15i-24i-36i^2}{2^2-(3i)^2}$

$=\frac{-10-15i-24i+36}{4+9}$

$=\frac{26-39i}{13}$

$=2-3i$

© Cambridge University Press 2019

9e $\text{Re}\big((1+4i)z\big)$

$= \text{Re}\big((1+4i)(2-i)\big)$

$= \text{Re}(2-i+8i-4i^2)$

$= \text{Re}(2-i+8i+4)$

$= \text{Re}(6+7i)$

$= 6$

Solutions to Exercise 1B Foundation questions

1a $z^2 + 9 = 0$

$z^2 - 9i^2 = 0$

$z^2 - (3i)^2 = 0$

$(z - 3i)(z + 3i) = 0$

$z = \pm 3i$

1b $(z - 2)^2 + 16 = 0$

$(z - 2)^2 - 16i^2 = 0$

$(z - 2)^2 - (4i)^2 = 0$

$(z - 2 - 4i)(z - 2 + 4i) = 0$

$z = 2 \pm 4i$

1c $z^2 + 2z + 5 = 0$

$z^2 + 2z + 1 + 4 = 0$

$(z + 1)^2 + 4 = 0$

$(z + 1)^2 - 4i^2 = 0$

$(z + 1)^2 - (2i)^2 = 0$

$(z + 1 - 2i)(z + 1 + 2i) = 0$

$z = -1 \pm 2i$

1d $z^2 - 6z + 10 = 0$

$z^2 - 6z + 9 + 1 = 0$

$(z - 3)^2 + 1 = 0$

$(z - 3)^2 - i^2 = 0$

$(z - 3 - i)(z - 3 + i) = 0$

$z = 3 \pm i$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

1e $16z^2 - 16z + 5 = 0$

$16z^2 - 16z + 4 + 1 = 0$

$(4z - 2)^2 + 1 = 0$

$(4z - 2)^2 - i^2 = 0$

$(4z - 2 - i)(4z - 2 + i) = 0$

$4z = 2 \pm i$

$z = \frac{1}{2} \pm \frac{1}{4}i$

1f $4z^2 + 12z + 25 = 0$

$4z^2 + 12z + 9 + 16 = 0$

$(2z + 3)^2 + 16 = 0$

$(2z + 3)^2 - 16i^2 = 0$

$(2z + 3 + 4i)(2z + 3 - 4i) = 0$

$2z = -3 \pm 4i$

$z = -\frac{3}{2} \pm 2i$

2a $z^2 + 36$
$= z^2 - 36i^2$
$= (z - 6i)(z + 6i)$

2b $z^2 - 8i^2$
$= z^2 - \left(2\sqrt{2}i\right)^2$
$= \left(z - 2\sqrt{2}i\right)\left(z + 2\sqrt{2}i\right)$

2c $z^2 - 2z + 10$
$= z^2 - 2z + 1 + 9$
$= (z - 1)^2 - 9i^2$
$= (z - 1 - 3i)(z - 1 + 3i)$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

2d $z^2 + 4z + 5$
$= z^2 + 4z + 4 + 1$
$= (z + 2)^2 + i^2$
$= (z + 2 + i)(z + 2 - i)$

2e $z^2 - 6z + 14$
$= z^2 - 6z + 9 + 5$
$= (z - 3)^2 + 5$
$= (z - 3)^2 - \left(\sqrt{5}i\right)^2$
$= (z - 3 - \sqrt{5}i)(z - 3 + \sqrt{5}i)$

2f $z^2 + z + 1$
$= z^2 + z + \frac{1}{4} + \frac{3}{4}$
$= \left(z + \frac{1}{2}\right)^2 + \frac{3}{4}$
$= \left(z + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}i}{2}\right)^2$
$= \left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

3a $\left(z - i\sqrt{2}\right)\left(z + i\sqrt{2}\right) = 0$
$z^2 - \left(i\sqrt{2}\right)^2 = 0$
$z^2 + 2 = 0$

3b $\left(z - (1 - i)\right)\left(z + (1 + i)\right) = 0$
$z^2 - (1 - i + 1 + i)z + (1 - i)(1 + i) = 0$
$z^2 - 2z + (1^2 - i^2) = 0$
$z^2 - 2z + 2 = 0$

© Cambridge University Press 2019

3c $(z+1-2i)(z+1+2i)=0$

$z^2+(1-2i+1+2i)z+(1-2i)(1+2i)=0$

$z^2+2z+(1^2-4i^2)=0$

$z^2+2z+(1+4)=0$

$z^2+2z+5=0$

3d $\left(z-\left(2-i\sqrt{3}\right)\right)\left(z-\left(2+i\sqrt{3}\right)\right)=0$

$z^2-\left(2+i\sqrt{3}+2-i\sqrt{3}\right)z+\left(2^2-\left(i\sqrt{3}\right)^2\right)=0$

$z^2-4z+(2^2+3)=0$

$z^2-4z+7=0$

4a Let $(x+iy)^2=2i$ where x and y are real.

$x^2+2ixy-y^2=2i$

$x^2-y^2+2ixy=2i$

Equating real and imaginary parts gives:

$x^2-y^2=0$

$2xy=2$

By inspection, $x=1$ and $y=1$ or $x=-1$ and $y=-1$.

So the roots are $\pm(1+i)$.

4b Let $(x+iy)^2=3+4i$ where x and y are real.

$x^2+2ixy-y^2=3+4i$

$x^2-y^2+2ixy=3+4i$

Equating real and imaginary parts gives:

$x^2-y^2=3$ and

$2xy=4$

By inspection, $x=2$ and $y=1$ or $x=-2$ and $y=-1$.

So the roots are $\pm(2+i)$.

© Cambridge University Press 2019

4c Let $(x+iy)^2 = -8-6i$ where x and y are real.

$x^2 + 2ixy - y^2 = -8-6i$

$x^2 - y^2 + 2ixy = -8-6i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -8$ and

$2xy = -6$

By inspection, $x = -1$ and $y = 3$ or $x = 1$ and $y = -3$.

So the roots are $\pm(-1+3i)$.

4d Let $(x+iy)^2 = 35+12i$ where x and y are real.

$x^2 + 2ixy - y^2 = 35+12i$

$x^2 - y^2 + 2ixy = 35+12i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 35$ and

$2xy = 12$

By inspection, $x = 6$ and $y = 1$ or $x = -6$ and $y = -1$.

So the roots are $\pm(6+i)$.

4e Let $(x+iy)^2 = -5+12i$ where x and y are real.

$x^2 + 2ixy - y^2 = -5+12i$

$x^2 - y^2 + 2ixy = -5+12i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -5$ and

$2xy = 12$

By inspection, $x = 2$ and $y = 3$ or $x = -2$ and $y = -3$.

So the roots are $\pm(2+3i)$.

© Cambridge University Press 2019

4f Let $(x+iy)^2 = 24 - 10i$ where x and y are real.

$x^2 + 2ixy - y^2 = 24 - 10i$

$x^2 - y^2 + 2ixy = 24 - 10i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 24$ and

$2xy = -10$

By inspection, $x = 5$ and $y = -1$ or $x = -5$ and $y = +1$.

So the roots are $\pm(5 - i)$.

4g Let $(x+iy)^2 = -15 - 8i$ where x and y are real.

$x^2 + 2ixy - y^2 = -15 - 8i$

$x^2 - y^2 + 2ixy = -15 - 8i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -15$ and

$2xy = -8$

By inspection, $x = 1$ and $y = -4$ or $x = -1$ and $y = 4$.

So the roots are $\pm(1 - 4i)$.

4h Let $(x+iy)^2 = 9 - 40i$ where x and y are real

$x^2 + 2ixy - y^2 = 9 - 40i$

$x^2 - y^2 + 2ixy = 9 - 40i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 9$ and

$2xy = -40$

By inspection, $x = 5$ and $y = -4$ or $x = -5$ and $y = 4$.

So the roots are $\pm(5 - 4i)$.

© Cambridge University Press 2019

Solutions to Exercise 1C Foundation questions

1a $2 = 2 + 0i$

In Cartesian form, this is $(2, 0)$.

1b $i = 0 + i$

In Cartesian form, this is $(0, 1)$.

1c $-3 + 5i$

In Cartesian form, this is $(-3, 5)$.

1d $\overline{2 + 2i} = 2 - 2i$

In Cartesian form, this is $(2, -2)$.

1e $-5(1 + i) = -5 - 5i$

In Cartesian form, this is $(-5, -5)$.

1f $(2 + i)i = 2i + i^2 = 2i - 1 = -1 + 2i$

In Cartesian form, this is $(-1, 2)$.

2a Complex number that represents $(-3, 0)$ is

$-3 + 0i = -3$

2b Complex number that represents $(0, 3)$ is

$0 + 3i = 3i$

2c Complex number that represents $(7, -5)$ is

$7 - 5i$

© Cambridge University Press 2019

2d Complex number that represents (a, b) is

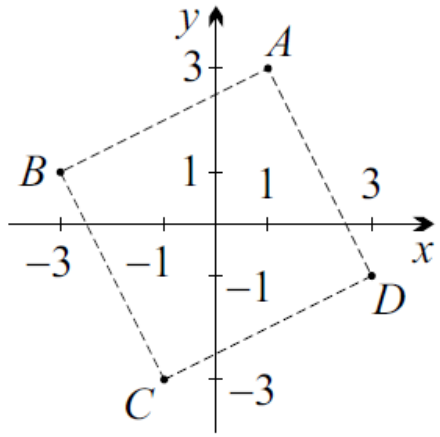
$a + bi$

3a Point A: $1 + 3i$

Point B: $i(1 + 3i) = i + 3i^2 = i - 3 = -3 + i$

Point C: $i^2(1 + 3i) = -1(1 + 3i) = -1 - 3i$

Point D: $i^3(1 + 3i) = -i(1 + 3i) = -i - 3i^2 = -i + 3 = 3 - i$



3b A square

3c An anticlockwise rotation of 90° about the origin.

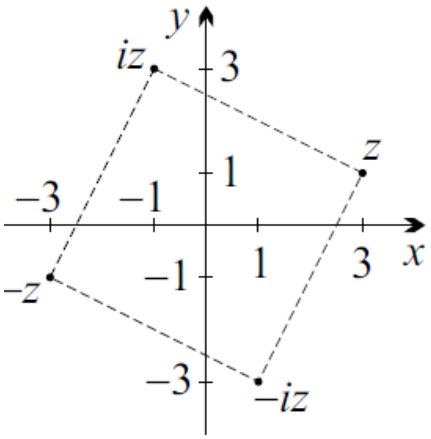
4a $z = 3 + i$

$iz = i(3 + i) = 3i + i^2 = 3i - 1 = -1 + 3i$

$-z = -(3 + i) = -3 - i$

$-iz = -i(3 + i) = -3i - i^2 = -3i + 1 = 1 - 3i$

© Cambridge University Press 2019



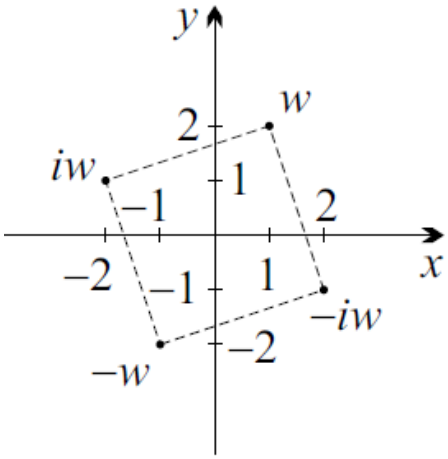
The points form a square.

4b $w = 1 + 2i$

$iw = i(1 + 2i) = i + 2i^2 = i - 2 = -2 + i$

$-w = -(1 + 2i) = -1 - 2i$

$-iw = -i(1 + 2i) = -i - 2i^2 = -i + 2 = 2 - i$



The points form a square.

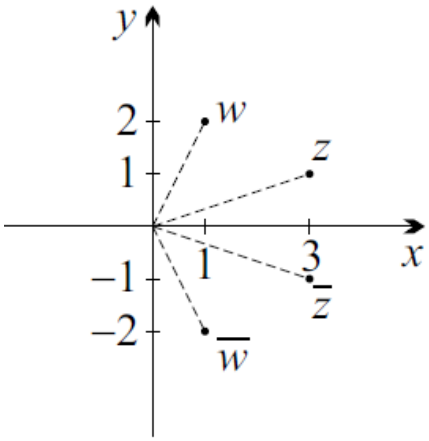
4c $z = 3 + i$

$\bar{z} = \overline{3 + i} = 3 - i$

$w = 1 + 2i$

$\overline{w} = \overline{1 + 2i} = 1 - 2i$

© Cambridge University Press 2019

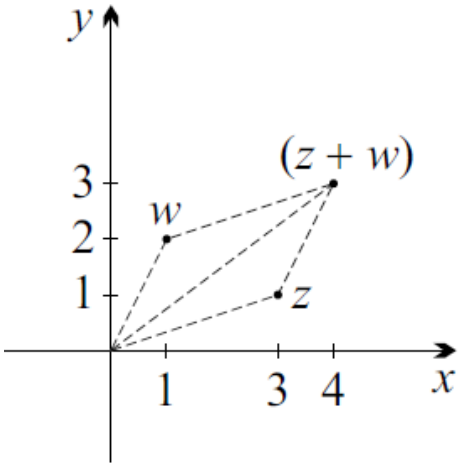


Conjugate pairs are reflections in the real axis.

4d $z = 3 + i$

$w = 1 + 2i$

$z + w = 3 + i + 1 + 2i = 4 + 3i$

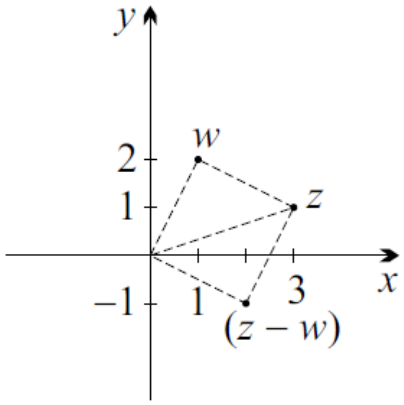


With O at the origin, the points form a parallelogram.

4e $z = 3 + i$

$w = 1 + 2i$

$z - w = 3 + i - (1 + 2i) = 2 - i$

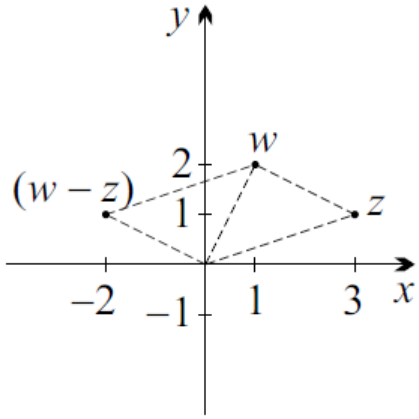


With O at the origin, the points form a parallelogram. (For this particular choice of z and w, that parallelogram happens to be a square; for other values of z and w, the points will still form a parallelogram, but not necessarily a square.)

4f $z = 3 + i$

$w = 1 + 2i$

$w - z = (1 + 2i) - (3 + i) = -2 + i$



With O at the origin, the points form a parallelogram.

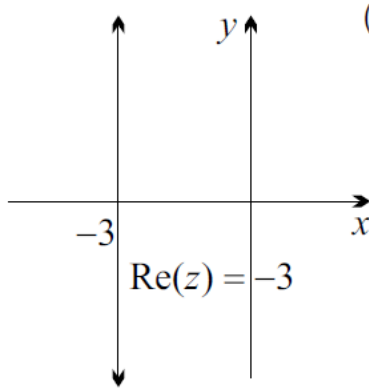
5 For the following section, let $z = x + iy$ where x and y are real.

5a $\text{Re}(z) = -3$

$\text{Re}(x + iy) = -3$

$x = -3$

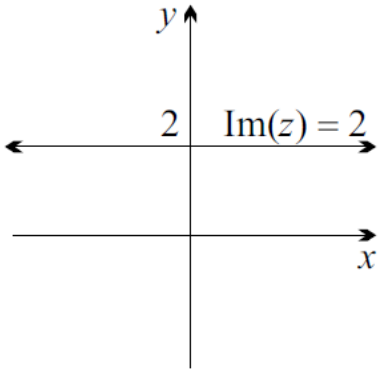
© Cambridge University Press 2019



5b $\text{Im}(z) = 2$

$\text{Im}(x + iy) = 2$

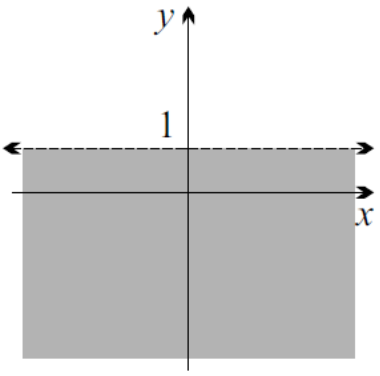
$y = 2$



5c $\text{Im}(z) < 1$

$\text{Im}(x + iy) < 1$

$y < 1$

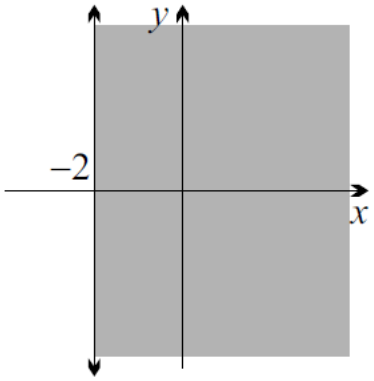


© Cambridge University Press 2019

5d $\text{Re}(z) \geq -2$

$\text{Re}(x + iy) \geq -2$

$x \geq -2$

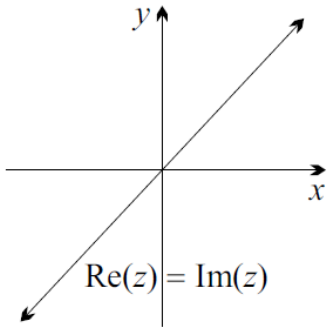


5e $\text{Re}(z) = \text{Im}(z)$

$\text{Re}(x + iy) = \text{Im}(x + iy)$

$x = y$

$y = x$



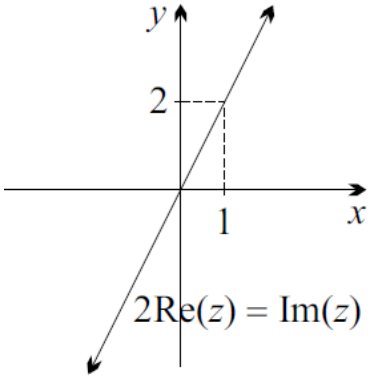
© Cambridge University Press 2019

5f $2\text{Re}(z) = \text{Im}(z)$

$2\text{Re}(x + iy) = \text{Im}(x + iy)$

$2x = y$

$y = 2x$

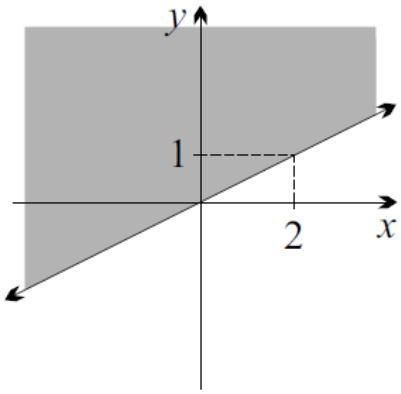


5g $\text{Re}(z) \leq 2\text{Im}(z)$

$\text{Re}(x + iy) \leq 2\text{Im}(x + iy)$

$x \leq 2y$

$y \geq \frac{1}{2}x$



© Cambridge University Press 2019

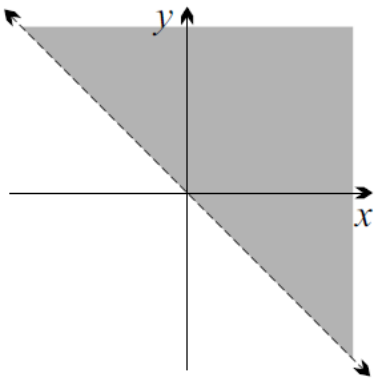
5h $\text{Re}(z) > -\text{Im}(z)$

$\text{Re}(x + iy) > -\text{Im}(x + iy)$

$x > -y$

$-y < x$

$y > -x$



© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

Solutions to Exercise 1D Foundation questions

1a $z = 3 = 3 + 0i$

$|z| = \sqrt{3^2 + 0^2}$

$= 3$

1b $z = -5i = 0 - 5i$

$|z| = \sqrt{0^2 + (-5)^2}$

$= 5$

1c $z = 1 - i$

$|z| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$

1d $z = -\sqrt{3} - i$

$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$

$= \sqrt{4}$

$= 2$

1e $z = -3 + 4i$

$|z| = \sqrt{(-3)^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

1f $z = 15 + 8i$

$|z| = \sqrt{15^2 + 8^2}$

$= \sqrt{225 + 64}$

$= \sqrt{289}$

$= 17$

2 Note that $-\pi < \text{Arg}(z) \leq \pi$.

2a $z = -2$ lies along the negative x-axis, hence $\text{Arg}(z) = \pi$

2b $z = 4i$ lies along the positive y-axis, hence $\text{Arg}(z) = \frac{\pi}{2}$

2c $z = 2 - 2i$ (fourth quadrant)

$\text{Arg}(z)$

$= \tan^{-1}\left(\frac{-2}{2}\right)$

$= \tan^{-1}(-1)$

$= -\frac{\pi}{4}$

2d $z = 1 + \sqrt{3}i$ (first quadrant)

$\text{Arg}(z)$

$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

© Cambridge University Press 2019

2e $z = -3 + 3i$ (second quadrant)

Arg(z)

$= \tan^{-1}\left(\frac{3}{-3}\right)$

$= \tan^{-1}(-1)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

2f $z = -\sqrt{3} - i$ (third quadrant)

Arg(z)

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= -\pi + \frac{\pi}{6}$

$= -\frac{5\pi}{6}$

3a For $2i = 0 + 2i$:

$r = \sqrt{0^2 + 2^2} = 2$

This lies along the positive y-axis, hence $\theta = \frac{\pi}{2}$.

The complex number is $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ or $2\text{ cis }\frac{\pi}{2}$.

3b For $-4 = -4 + 0i$:

$r = \sqrt{(-4)^2 + 0^2} = 4$

This lies along the negative x-axis, hence $\theta = \pi$.

The complex number is $4(\cos\pi + i\sin\pi)$ or $4\text{ cis }\pi$.

© Cambridge University Press 2019

3c For $1 + i$ (in first quadrant):

$$r = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$
$$= \tan^{-1} 1$$
$$= \frac{\pi}{4}$$

The complex number is $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ or $\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$.

3d For $\sqrt{3} - i$ (in fourth quadrant):

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$
$$= \sqrt{3 + 1}$$
$$= \sqrt{4}$$
$$= 2$$
$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$
$$= -\frac{\pi}{6}$$

The complex number is $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ or $2\,\text{cis}\,\left(-\frac{\pi}{6}\right)$.

3e For $-1 + \sqrt{3}i$ (in second quadrant):

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$
$$= \sqrt{1 + 3}$$
$$= \sqrt{4}$$
$$= 2$$
$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

© Cambridge University Press 2019

$= \tan^{-1}\left(-\sqrt{3}\right)$

$= \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

The complex number is $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ or $2\text{ cis }\frac{2\pi}{3}$.

3f For $-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ (in third quadrant):

$r = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}$

$= \sqrt{\frac{1}{2} + \frac{1}{2}}$

$= \sqrt{1}$

$= 1$

$\theta = \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)$

$= -\pi + \tan^{-1} 1$

$= -\pi + \frac{\pi}{4}$

$= -\frac{3\pi}{4}$

The complex number is $\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$ or $\text{cis}\left(-\frac{3\pi}{4}\right)$.

4a For $3 + 4i$ (in first quadrant):

$r = \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$

© Cambridge University Press 2019

$\theta = \tan^{-1}\frac{4}{3}$

$\doteqdot 0.93$

The complex number is $5(\cos(0.93) + i\sin(0.93))$ or 5 cis (0.93).

4b For $12 - 5i$ (in fourth quadrant):

$r = \sqrt{12^2 + (-5)^2}$

$= \sqrt{169}$

$= 13$

$\theta = \tan^{-1}\left(-\frac{5}{12}\right)$

$= -\tan^{-1}\left(\frac{5}{12}\right)$

$\doteqdot -0.39$

The complex number is $13(\cos(-0.39) + i\sin(-0.39))$ or 13 cis (-0.39).

4c For $-2 + i$ (in second quadrant):

$r = \sqrt{2^2 + (-1)^2}$

$= \sqrt{5}$

$\theta = \tan^{-1}\left(-\frac{1}{2}\right)$

$= \pi - \tan^{-1}\left(\frac{1}{2}\right)$

$\doteqdot 2.68$

The complex number is $\sqrt{5}(\cos(2.68) + i\sin(2.68))$ or $\sqrt{5}$ cis (2.68).

4d For $-1 - 3i$ (in third quadrant):

$r = \sqrt{(-1)^2 + (-3)^2}$

$= \sqrt{1 + 9}$

$= \sqrt{10}$

© Cambridge University Press 2019

$\theta = \tan^{-1}\left(\frac{-3}{-1}\right)$

$= -\pi + \tan^{-1} 3$

$\div -1.89$

The complex number is $\sqrt{10}(\cos(-1.89) + i\sin(-1.89))$ or $\sqrt{10}\text{ cis }(-1.89)$.

5a $3\text{ cis }0$

$= 3(\cos 0 + i\sin 0)$

$= 3(1 + 0i)$

$= 3$

5b $5\text{ cis }\left(-\frac{\pi}{2}\right)$

$= 5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$= 5(0 + (-1)i)$

$= -5i$

5c $4\text{ cis }\frac{\pi}{4}$

$= 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 4\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}}i$

$= 2\sqrt{2} + 2\sqrt{2}i$

© Cambridge University Press 2019

5d $6 \operatorname{cis}\left(-\frac{\pi}{6}\right)$

$= 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

$= 6\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= 3\sqrt{3} - 3i$

5e $2 \operatorname{cis} \frac{3\pi}{4}$

$= 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 2\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= -\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}i$

$= -\sqrt{2} + \sqrt{2}i$

5f $2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

$= 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$

$= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -1 - \sqrt{3}i$

6a $z = 1 - i$ (in fourth quadrant):

$r = \sqrt{(1)^2 + (-1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$

$= \tan^{-1}(-1)$

© Cambridge University Press 2019

$= -\frac{\pi}{4}$

$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

6b $\bar{z} = 1 + i$ (in first quadrant):

$r = \sqrt{(1)^2 + (1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

$\bar{z} = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

Alternatively:

$\bar{z} = \overline{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}$

$= \sqrt{2}\,\text{cis}\left(-\left(-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

6c $-z = -1 + i$ (in second quadrant):

$r = \sqrt{(-1)^2 + (1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{-1}\right)$

$= \pi - \frac{\pi}{4}$

© Cambridge University Press 2019

$= \frac{3\pi}{4}$

$-z = \sqrt{2}\ \text{cis}\ \frac{3\pi}{4}$

6d $iz = i(1-i) = 1+i$ (in first quadrant)

$r = \sqrt{(1)^2 + (1)^2}$

$= \sqrt{1+1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \frac{\pi}{4}$

$iz = \sqrt{2}\ \text{cis}\ \frac{\pi}{4}$

6e $z^2 = (1-i)^2$

$= 1 - 2i + i^2$

$= -2i$

$r = \sqrt{(0)^2 + (-2)^2}$

$= \sqrt{4}$

$= 2$

$\theta = -\frac{\pi}{2}$ since $-2i$ lies on the negative y-axis

$z^2 = 2\ \text{cis}\left(-\frac{\pi}{2}\right)$

6f $(\bar{z})^{-1} = (1+i)^{-1}$

$= \frac{1}{1+i}$

$= \frac{1}{1+i} \times \frac{1-i}{1-i}$

© Cambridge University Press 2019

$$= \frac{1-i}{1-i^2}$$

$$= \frac{1-i}{2} \qquad \text{(in fourth quadrant)}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right)$$

$$\theta = \tan^{-1}(-1)$$

$$= -\frac{\pi}{4}$$

$$(\bar{z})^{-1} = \frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

7a

$$5 \operatorname{cis} \frac{\pi}{12} \times 2 \operatorname{cis} \frac{\pi}{4}$$

$$= (5 \times 2) \operatorname{cis}\left(\frac{\pi}{12} + \frac{\pi}{4}\right)$$

$$= 10 \operatorname{cis} \frac{\pi}{3}$$

7b $3 \operatorname{cis} \theta \times 3 \operatorname{cis} 2\theta$

$$= (3 \times 3) \operatorname{cis}(\theta + 2\theta)$$

$$= 9 \operatorname{cis} 3\theta$$

© Cambridge University Press 2019

7c

$$6 \text{ cis } \frac{\pi}{2} \div 3 \text{ cis } \frac{\pi}{6}$$

$$= (6 \div 3) \text{ cis } \left(\frac{\pi}{2} - \frac{\pi}{6}\right)$$

$$= 2 \text{ cis } \frac{\pi}{3}$$

7d

$$\frac{3 \text{ cis } 5\alpha}{2 \text{ cis } 4\alpha}$$

$$= \frac{3}{2} \text{ cis}(5\alpha - 4\alpha)$$

$$= \frac{3}{2} \text{cis } \alpha$$

7e

$$\left(4 \text{ cis } \frac{\pi}{5}\right)^2$$

$$= 4^2 \text{cis } \left(\frac{\pi}{5} \times 2\right)$$

$$= 16 \text{ cis } \frac{2\pi}{5}$$

7f

$$\left(2 \text{ cis } \frac{2\pi}{7}\right)^3$$

$$= 2^3 \text{ cis } \left(3 \times \frac{2\pi}{7}\right)$$

$$= 8 \text{ cis } \frac{6\pi}{7}$$

© Cambridge University Press 2019

8a $|z - w|$

$= |1 + 3i - (-1 + i)|$

$= |2 + 2i|$

$= \sqrt{2^2 + 2^2}$

$= \sqrt{4 + 4}$

$= 2\sqrt{2}$

8b $|z - w|$

$= |(1 - i) - (4 + 2i)|$

$= |-3 - 3i|$

$= \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{9 + 9}$

$= 3\sqrt{2}$

8c $|z - w|$

$= \left|\left(4 - 2i\sqrt{3}\right) - \left(1 + i\sqrt{3}\right)\right|$

$= \left|3 - 3i\sqrt{3}\right|$

$= \sqrt{3^2 + \left(3\sqrt{3}\right)^2}$

$= \sqrt{36}$

$= 6$

8d $|z - w|$

$= \left|\left(3 + 3i\sqrt{3}\right) - \left(-3 + i\sqrt{3}\right)\right|$

$= \left|6 + 2i\sqrt{3}\right|$

$= \sqrt{6^2 + \left(2\sqrt{3}\right)^2}$

$= \sqrt{48}$

© Cambridge University Press 2019

$= 4\sqrt{3}$

8e $|z - w|$

$= |(2 + i) - (-1 - 3i)|$

$= |3 + 4i|$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$

8f $|z - w|$

$= |(-2 - i) - (-1 + i)|$

$= |-1 - 2i|$

$= \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

9a $\text{Arg}(z - w)$

$= \text{Arg}(2 + 2i)$ (in first quadrant)

$= \tan^{-1}\left(\frac{2}{2}\right)$

$= \frac{\pi}{4}$

9b $\text{Arg}(z - w)$

$= \text{Arg}(-3 - 3i)$ (in third quadrant)

$= \tan^{-1}\left(\frac{-3}{-3}\right)$

$= -\pi + \tan^{-1} 1$

$= -\pi + \frac{\pi}{4}$

$= -\frac{3\pi}{4}$

© Cambridge University Press 2019

9c $\text{Arg}(z-w)$

$= \text{Arg}\left(3-3i\sqrt{3}\right)$ (in fourth quadrant)

$= \tan^{-1}\left(\frac{-3\sqrt{3}}{3}\right)$

$= \tan^{-1}\left(-\sqrt{3}\right)$

$= -\frac{\pi}{3}$

9d $\text{Arg}(z-w)$

$= \text{Arg}\left(6+2i\sqrt{3}\right)$ (in first quadrant)

$= \tan^{-1}\left(\frac{2\sqrt{3}}{6}\right)$

$= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$= \frac{\pi}{6}$

9e $\text{Arg}(z-w)$

$= \text{Arg}(3+4i)$ (in first quadrant)

$= \tan^{-1}\left(\frac{4}{3}\right)$

$\div 0.93$

9f $\text{Arg}(z-w)$

$= \text{Arg}(-1-2i)$ (in third quadrant)

$= \tan^{-1}\left(\frac{-2}{-1}\right)$

$= -\pi + \tan^{-1} 2$

$\div -2.03$

© Cambridge University Press 2019

10 Multiplying a complex number by w produces a rotation of θ radians about the origin, so $w = \text{cis}\,\theta$.

10a

$$\begin{aligned} w &= \text{cis}\,\frac{\pi}{2} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i \times 1 \\ &= i \end{aligned}$$

10b

$$\begin{aligned} w &= \text{cis}\,\pi \\ &= \cos\pi + i\sin\pi \\ &= -1 + i \times 0 \\ &= -1 \end{aligned}$$

10c

$$\begin{aligned} w &= \text{cis}\,\frac{\pi}{3} \\ &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= \frac{1}{2} + i \times \frac{\sqrt{3}}{2} \\ &= \frac{1}{2}\left(1 + i\sqrt{3}\right) \end{aligned}$$

10c

$$\begin{aligned} w &= \text{cis}\,\frac{3\pi}{4} \\ &= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \\ &= -\frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}} \end{aligned}$$

© Cambridge University Press 2019

$$= \frac{1}{\sqrt{2}}(-1 + i)$$

10e

$$w = \text{cis}\ \frac{5\pi}{6}$$
$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$
$$= -\frac{\sqrt{3}}{2} + i \times \frac{1}{2}$$
$$= \frac{1}{2}(-\sqrt{3} + i)$$

10f

$$w = \text{cis}\left(-\frac{\pi}{2}\right)$$
$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$
$$= 0 + i \times -1$$
$$= -i$$

10g

$$w = \text{cis}\left(-\frac{\pi}{4}\right)$$
$$= \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$$
$$= \frac{1}{\sqrt{2}} + i \times -\frac{1}{\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}}(1 - i)$$

© Cambridge University Press 2019

10h

$$w = \text{cis}\left(-\frac{2\pi}{3}\right)$$

$$= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + i \times -\frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2}(1 + i\sqrt{3})$$

© Cambridge University Press 2019

Solutions to Exercise 1E Foundation questions

1a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= (5 + i) + (2 + 3i)$

$= 7 + 4i$

1b $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\overrightarrow{OA} + \overrightarrow{OC}$

$= -(5 + i) + (2 + 3i)$

$= -3 + 2i$

1c $\overrightarrow{CA} = -\overrightarrow{AC}$

$= -(-3 + 2i)$

$= 3 - 2i$

2a $\overrightarrow{OR} = \overrightarrow{OP} \times \text{cis}\,\frac{\pi}{2}$

$= (4 + 3i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= (4 + 3i)(0 + i \times 1)$

$= (4 + 3i)i$

$= 4i + 3i^2$

$= 4i - 3$

$= -3 + 4i$

2b $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= \overrightarrow{OP} + \overrightarrow{OR}$

$= (4 + 3i) + (-3 + 4i)$

$= 1 + 7i$

© Cambridge University Press 2019

Chapter 1 worked solutions – Complex numbers I

2c $\overrightarrow{QR} = \overrightarrow{PO}$

$= -\overrightarrow{OP}$

$= -(4 + 3i)$

$= -4 - 3i$

2d $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$= -\overrightarrow{OP} + \overrightarrow{OR}$

$= -(4 + 3i) + (-3 + 4i)$

$= -7 + i$

3 $\overrightarrow{OP} = \overrightarrow{AB}$

$= \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(3 + 5i) + (9 + 8i)$

$= 6 + 3i$

$\overrightarrow{OQ} = \overrightarrow{OP} \times \text{cis}\,\frac{\pi}{2}$

$= (6 + 3i) \times i$ (using $\text{cis}\,\frac{\pi}{2} = i$ from question 2a)

$= 6i + 3i^2$

$= 6i - 3$

$= -3 + 6i$

4a $C = \overrightarrow{OC}$

$= \overrightarrow{OA} \times \text{cis}\,\frac{\pi}{2}$

$= (2 + i)i$

$= 2i + i^2$

$= 2i - 1$

$= -1 + 2i$

© Cambridge University Press 2019

$$\begin{aligned} B &= \overrightarrow{OB} \\ &= \overrightarrow{OA} + \overrightarrow{AB} \\ &= \overrightarrow{OA} + \overrightarrow{OC} \\ &= (2+i) + (-1+2i) \\ &= 1 + 3i \end{aligned}$$

4b $B' = B \times \text{cis } 45°$

$$\begin{aligned} &= (1+3i) \times \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{3}{\sqrt{2}}i + \frac{3}{\sqrt{2}}i^2 \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{3}{\sqrt{2}}i - \frac{3}{\sqrt{2}} \\ &= \left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right)i \\ &= \frac{-2}{\sqrt{2}} + \frac{4}{\sqrt{2}}i \\ &= -\sqrt{2} + 2\sqrt{2}i \end{aligned}$$

5a $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$$\begin{aligned} &= -\overrightarrow{OB} + \overrightarrow{OC} \\ &= -(5+3i) + (9+6i) \\ &= 4 + 3i \end{aligned}$$

5b $\overrightarrow{BA} = \overrightarrow{BC} \times \text{cis } \frac{\pi}{2}$

$$\begin{aligned} &= (4+3i) \times i \\ &= 4i + 3i^2 \\ &= 4i - 3 \\ &= -3 + 4i \end{aligned}$$

© Cambridge University Press 2019

5c $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$

$= (5 + 3i) + (-3 + 4i)$

$= 2 + 7i$

6a $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(9 + i) + (4 + 13i)$

$= -5 + 12i$

6b $\overrightarrow{AD} = \overrightarrow{AB} \times \text{cis}\, \frac{\pi}{2}$

$= (-5 + 12i) \times i$

$= -5i + 12i^2$

$= -5i - 12$

$= -12 - 5i$

$D = \overrightarrow{OD}$

$= \overrightarrow{OA} + \overrightarrow{AD}$

$= (9 + i) + (-12 - 5i)$

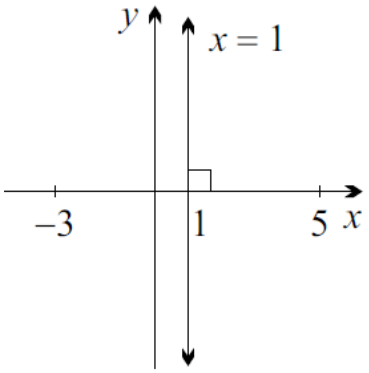
$= -3 - 4i$

© Cambridge University Press 2019

Solutions to Exercise 1F Foundation questions

1a For $|z+3| = |z-5|$, the midpoint of -3 and 5 is 1. These three points lie on the horizontal x-axis. Hence the perpendicular bisector must be vertical.

Hence the equation of the perpendicular bisector is $x = 1$.



1b For $|z-i| = |z+1|$, the midpoint of i and -1 is $\frac{-1+i}{2}$ which in Cartesian form is $(-\frac{1}{2}, \frac{1}{2})$.

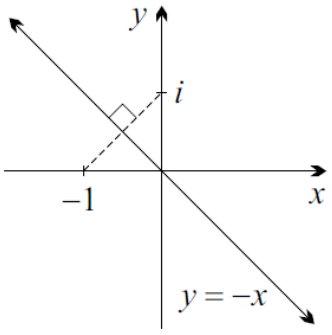
The gradient of the line between the two points is $\frac{0-1}{-1-0} = 1$ and hence the gradient of the perpendicular line is $m = -\frac{1}{1} = -1$.

For a straight line, the equation is $y = mx + b$.

Hence $y = -x + b$.

Substituting $(-\frac{1}{2}, \frac{1}{2})$ gives $\frac{1}{2} = -\left(-\frac{1}{2}\right) + b$ and so $b = 0$.

Thus the equation of the perpendicular bisector is $y = -x$.



© Cambridge University Press 2019

1c For $|z+2-2i| = |z|$, the midpoint of $-2+2i$ and 0 is $\frac{-2+2i+0}{2} = -1+i$ which in Cartesian form is $(-1, 1)$.

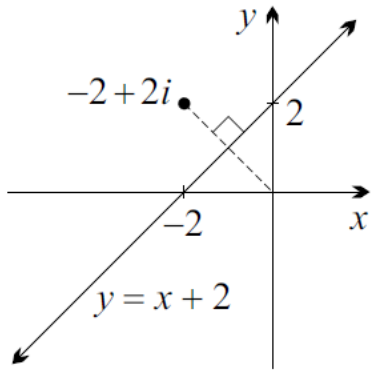
The gradient of the line between the two points is $\frac{2-0}{-2-0} = \frac{2}{-2} = -1$ and hence the gradient of the perpendicular line is $m = -\frac{1}{-1} = 1$.

For a straight line, the equation is $y = mx + b$.

Hence $y = x + b$.

Substituting in $(-1, 1)$ gives $1 = -1 + b$ and hence $b = 2$.

Thus the equation of the perpendicular bisector is $y = x + 2$.



1d For $|z-i| = |z-4+i|$, the midpoint of i and $4-i$ is 2 which in Cartesian form is $(2, 0)$.

The gradient of the line between i and $4-i$ is $\frac{-1-1}{4-0} = -\frac{2}{4} = -\frac{1}{2}$ and hence the gradient of the perpendicular line is $m = -\frac{1}{\left(-\frac{1}{2}\right)} = 2$.

For a straight line, the equation is $y = mx + b$.

Hence $y = 2x + b$.

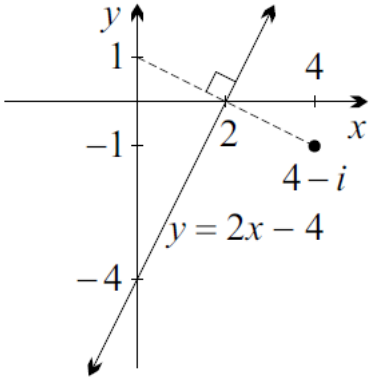
Substituting in $(2, 0)$ gives

$0 = 2(2) + b$

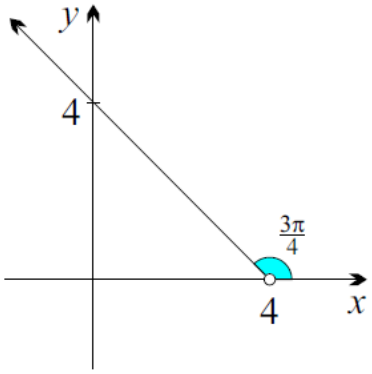
$b = -4$

Thus the equation of the perpendicular bisector is $y = 2x - 4$.

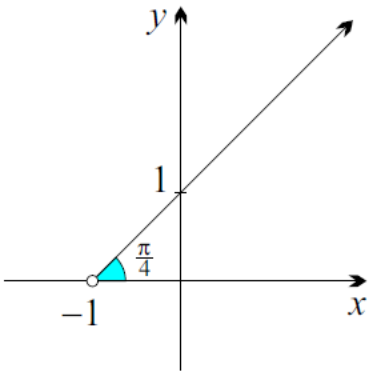
© Cambridge University Press 2019



2a From Box 29, $\arg(z-4)=\frac{3\pi}{4}$ is a line originating at the point 4, which in Cartesian form is $(4,0)$, where the angle made with the horizontal is $\frac{3\pi}{4}$.

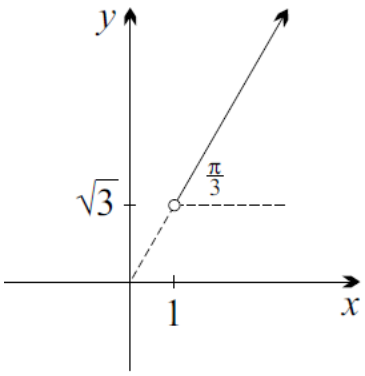


2b From Box 29, $\arg(z+1)=\frac{\pi}{4}$ is a line originating at the point -1, which in Cartesian form is $(-1,0)$, where the angle made with the horizontal is $\frac{\pi}{4}$.

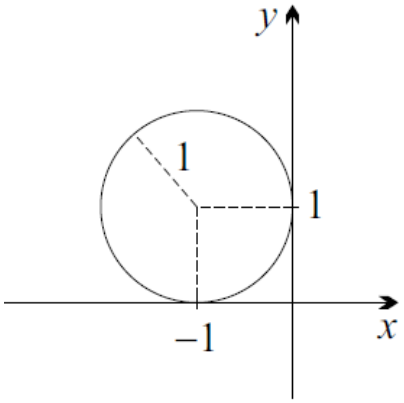


© Cambridge University Press 2019

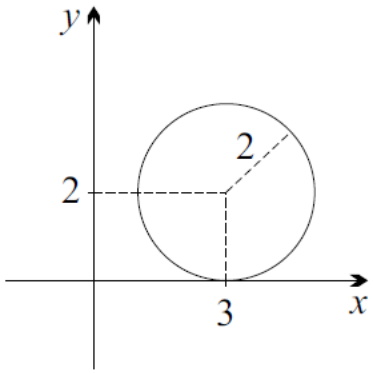
2c From Box 29, $\arg(z - 1 - i\sqrt{3}) = \frac{\pi}{3}$ is a line originating at the point $1 + i\sqrt{3}$, which in Cartesian form is $(1, \sqrt{3})$, where the angle made with the horizontal is $\frac{\pi}{3}$.



3a From Box 31, $|z + 1 - i| = 1$ is a circle with radius 1 unit and centre $(-1 + i)$, which is $(-1, 1)$ in Cartesian form.

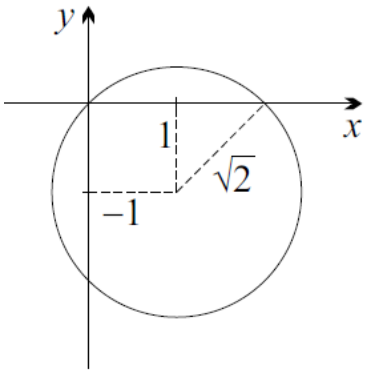


3b From Box 31, $|z - 3 - 2i| = 2$ is a circle with radius 2 units and centre $(3 + 2i)$, which is $(3, 2)$ in Cartesian form.



© Cambridge University Press 2019

3c From Box 31, $|z-1+i|=\sqrt{2}$ is a circle with radius $\sqrt{2}$ units and centre $(1-i)$, which is $(1,-1)$ in Cartesian form.



4a For the boundary $|z-8i|=|z-4|$, the midpoint of 4 and $8i$ is $\frac{4+8i}{2}=2+4i$ which in Cartesian form is $(2,4)$.

The gradient of the line between the two points is $\frac{8-0}{0-4}=-2$ and hence the gradient of the perpendicular line is $m=-\frac{1}{-2}=\frac{1}{2}$.

For a straight line, the equation is $y=mx+b$.

Hence $y=\frac{1}{2}x+b$.

Substituting $(2,4)$ gives $4=\frac{1}{2}\times 2+b$ and so $b=3$.

Thus the equation of the boundary is $y=\frac{1}{2}x+3$.

At $(0,0)$, $|-8i|=8$ and $|-4|=4$ so $|-8i|>|-4|$.

Therefore the region $|z-8i|\geq|z-4|$ includes $(0,0)$ so shade to the right of the boundary. The boundary is included.

Alternatively:

$|z-8i|\geq|z-4|$

$|x+iy-8i|\geq|x+iy-4|$

$|x+(y-8)i|\geq|(x-4)+iy|$

$\sqrt{x^2+(y-8)^2}\geq\sqrt{(x-4)^2+y^2}$

$x^2+(y-8)^2\geq(x-4)^2+y^2$

$x^2+y^2-16y+64\geq x^2-8x+16+y^2$

$-16y+64\geq-8x+16$

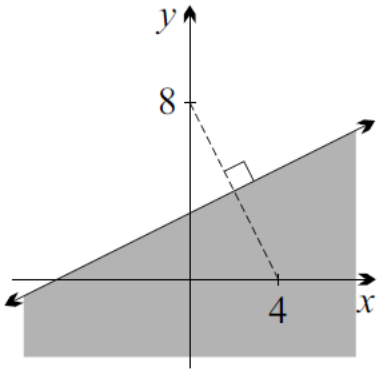
© Cambridge University Press 2019

$-16y + 8x + 48 \geq 0$

$-2y + x + 6 \geq 0$

$y \leq \frac{1}{2}x + 3$

At $(0, 0)$, $0 < 3$ so the region contains the point $(0, 0)$.



4b For the boundary $|z - 2 + i| = |z - 4 + i|$, the midpoint of $2 - i$ and $4 - i$ is $\frac{6-2i}{2} = 3 - i$ which in Cartesian form is $(3, -1)$.

The gradient of the line between the two points is $\frac{-1-(-1)}{4-2} = 0$ and hence the perpendicular line is vertical, passing through $(3, -1)$.

Thus the equation of the boundary is $x = 3$.

At $(0, 0)$, $|-2 + i| = \sqrt{5}$ and $|-4 + i| = \sqrt{17}$ so $|-2 + i| < |-4 + i|$.

Therefore the region $|z - 2 + i| \leq |z - 4 + i|$ includes $(0, 0)$ so shade to the left of the boundary. The boundary is included.

Alternatively:

$|z - 2 + i| \leq |z - 4 + i|$

$|x + iy - 2 + i| \leq |x + iy - 4 + i|$

$|(x - 2) + (y + 1)i| \leq |(x - 4) + (y + 1)i|$

$\sqrt{(x-2)^2 + (y+1)^2} \leq \sqrt{(x-4)^2 + (y+1)^2}$

$(x - 2)^2 + (y + 1)^2 \leq (x - 4)^2 + (y + 1)^2$

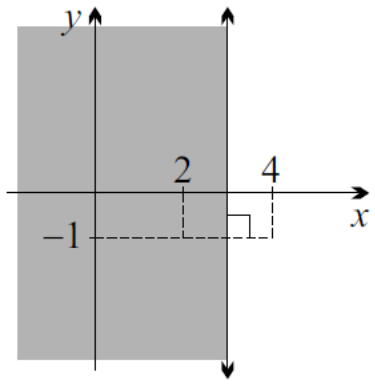
$x^2 - 4x + 4 + y^2 + 2y + 1 \leq x^2 - 8x + 16 + y^2 + 2y + 1$

$-4x + 4 + 2y + 1 \leq -8x + 16 + 2y + 1$

$4x \leq 12$

© Cambridge University Press 2019

$x \leq 3$



4c For the boundary $|z + 1 - i| = |z - 3 + i|$, the midpoint of $-1 + i$ and $3 - i$ is $\frac{2+0i}{2} = 1$ which in Cartesian form is $(1, 0)$.

The gradient of the line between the two points is $\frac{-1-1}{3-(-1)} = -\frac{1}{2}$ and hence the gradient of the perpendicular line is $m = -\frac{1}{-\frac{1}{2}} = 2$.

For a straight line, the equation is $y = mx + b$.

Hence $y = 2x + b$.

Substituting $(1, 0)$ gives $0 = 2 \times 1 + b$ and so $b = -2$.

Thus the equation of the boundary is $y = 2x - 2$.

At $(0, 0)$, $|1 - i| = \sqrt{2}$ and $|-3 + i| = \sqrt{10}$ so $|1 - i| < |-3 + i|$.

Therefore the region $|z + 1 - i| \geq |z - 3 + i|$ does not include $(0, 0)$ so shade to the right of the boundary. The boundary is included.

Alternatively:

$|z + 1 - i| \geq |z - 3 + i|$

$|x + iy + 1 - i| \geq |x + iy - 3 + i|$

$|(x + 1) + (y - 1)i| \geq |(x - 3) + (y + 1)i|$

$\sqrt{(x + 1)^2 + (y - 1)^2} \geq \sqrt{(x - 3)^2 + (y + 1)^2}$

$(x + 1)^2 + (y - 1)^2 \geq (x - 3)^2 + (y + 1)^2$

$x^2 + 2x + 1 + y^2 - 2y + 1 \geq x^2 - 6x + 9 + y^2 + 2y + 1$

$2x + 1 - 2y + 1 \geq -6x + 9 + 2y + 1$

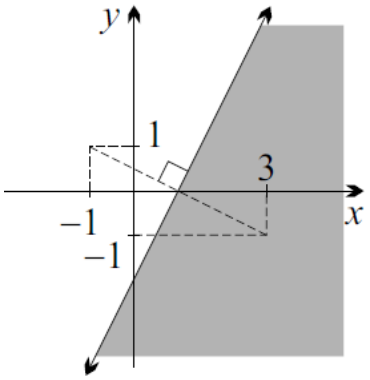
$8x - 4y \geq 8$

© Cambridge University Press 2019

$2x - y \geq 2$

$y \leq 2x - 2$

At $(0, 0)$, $0 > -2$ so the region does not contains the point $(0, 0)$.

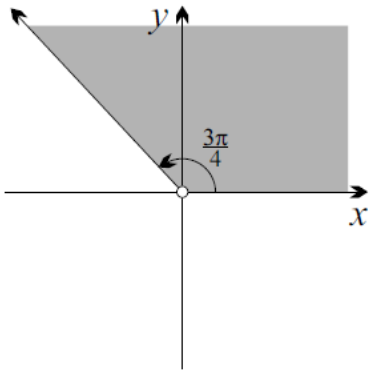


4d The boundaries are $\arg(z) = 0$ and $\arg(z) = \frac{3\pi}{4}$

$\arg(z) = 0$ is a line originating at the origin, where the angle made with the horizontal is 0.

$\arg(z) = \frac{3\pi}{4}$ is a line originating at the origin, where the angle made with the horizontal is $\frac{3\pi}{4}$.

For $0 \leq \arg(z) \leq \frac{3\pi}{4}$, shade the region between the two boundaries. The boundaries are included, except for $z = 0$ where the argument is undefined.



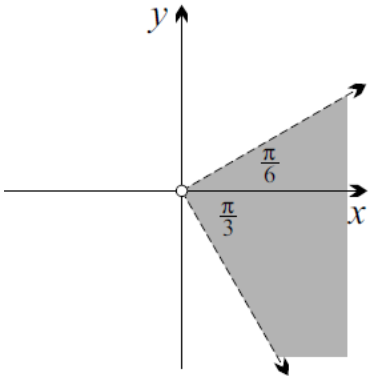
© Cambridge University Press 2019

4e The boundaries are $\arg(z) = -\frac{\pi}{3}$ and $\arg(z) = \frac{\pi}{6}$

$\arg(z) = -\frac{\pi}{3}$ is a line originating at the origin, where the angle made with the horizontal is $-\frac{\pi}{3}$. That is, $\frac{\pi}{3}$ in a clockwise direction from the positive x-axis.

$\arg(z) = \frac{\pi}{6}$ is a line originating at the origin, where the angle made with the horizontal is $\frac{\pi}{6}$.

For $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{6}$, shade the region between the two boundaries. The boundaries are not included.



4f The boundaries are $\arg(z + 2 + i) = -\frac{\pi}{4}$ and $\arg(z + 2 + i) = \frac{\pi}{4}$

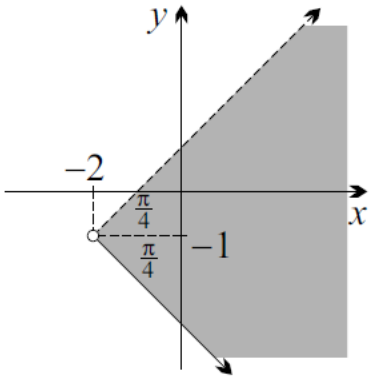
$\arg(z + 2 + i) = -\frac{\pi}{4}$ is a line originating at the point $-2 - i$, which in Cartesian form is $(-2, -1)$, where the angle made with the horizontal is $-\frac{\pi}{4}$. That is, $\frac{\pi}{4}$ in a clockwise direction from the horizontal.

$\arg(z + 2 + i) = \frac{\pi}{4}$ is a line originating at the point $-2 - i$, which in Cartesian form is $(-2, -1)$, where the angle made with the horizontal is $\frac{\pi}{4}$. That is, $\frac{\pi}{4}$ in an anticlockwise direction from the horizontal.

For $-\frac{\pi}{4} \leq \arg(z + 2 + i) < \frac{\pi}{4}$, shade the region between the two boundaries.

The boundary of $\arg(z + 2 + i) = -\frac{\pi}{4}$ is included but the boundary $\arg(z + 2 + i) = \frac{\pi}{4}$ is not.

© Cambridge University Press 2019



4g The boundary is $|z| = 2$ which is a circle with radius 2 and centre at $(0, 0)$.

At $(0, 0)$, $|0| = 0$ and $0 < 2$.

Therefore the region $|z| > 2$ does not include $(0, 0)$ so shade outside the circular boundary. The boundary is not included.

Alternatively:

$|z| > 2$

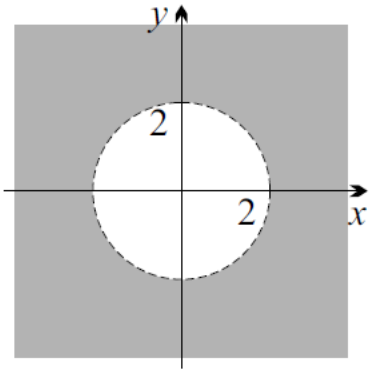
$|x + iy| > 2$

$\sqrt{x^2 + y^2} > 2$

$x^2 + y^2 > 4$

Boundary is $x^2 + y^2 = 4$, which is a circle with centre at $(0, 0)$ and radius of 2 units.

At $(0, 0)$, $x^2 + y^2 = 0 + 0 = 0$ and $0 < 4$ so the region does not contain the point $(0, 0)$. The boundary is not included.



© Cambridge University Press 2019

4h The boundary $|z + 2i| = 1$ is a circle with radius 1 and centre at $-2i$, which is $(0, -2)$ in Cartesian form.

At $(0, 0)$, $|2i| = 2$ and $2 > 1$.

Therefore the region $|z + 2i| \leq 1$ does not include $(0, 0)$ so shade inside the circular boundary. The boundary is included.

Alternatively:

$|z + 2i| \leq 1$

$|x + iy + 2i| \leq 1$

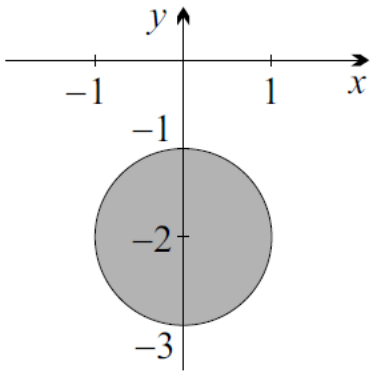
$|x + (y + 2)i| \leq 1$

$\sqrt{x^2 + (y + 2)^2} \leq 1$

$x^2 + (y + 2)^2 \leq 1$

Boundary is $x^2 + (y + 2)^2 = 1$, which is a circle with centre at $(0, -2)$ and radius of 1 unit.

At $(0, 0)$, $x^2 + (y + 2)^2 = 0 + 2^2 = 4$ and $4 > 1$ so the region does not contain the point $(0, 0)$. The boundary is included.



4i The boundary $|z - 2 + i| = 1$ is a circle with radius 1 unit and centre $(2 - i)$, which is $(2, -1)$ in Cartesian form.

The boundary $|z - 2 + i| = 2$ is a circle with radius 2 units and centre $(2 - i)$, which is $(2, -1)$ in Cartesian form.

For the region $1 < |z - 2 + i| \leq 2$, shade between the two boundaries. The boundary of $|z - 2 + i| = 2$ is included but the boundary $|z - 2 + i| = 1$ is not.

Alternatively:

$1 < |z - 2 + i| \leq 2$

© Cambridge University Press 2019

$1 < |x + iy - 2 + i| \leq 2$

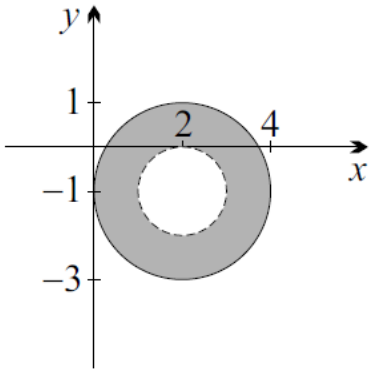
$1 < |(x - 2) + (y + 1)i| \leq 2$

$1 < \sqrt{(x - 2)^2 + (y + 1)^2} \leq 2$

$1 < (x - 2)^2 + (y + 1)^2 \leq 4$

One boundary is $(x - 2)^2 + (y + 1)^2 = 1$, which is a circle with centre at $(2, -1)$ and radius of 1 unit. This boundary is not included in the region.

The other boundary is $(x - 2)^2 + (y + 1)^2 = 4$, which is a circle with centre at $(2, -1)$ and radius of 2 units. This boundary is included in the region.



© Cambridge University Press 2019

Solutions to Exercise 1G Foundation questions

1a $P(x) = x^3 - 6x + 4$

The factors of 4 are $\pm 1, \pm 2, \pm 4$.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(2) = 0$ and hence $x - 2$ is a factor of $P(x) = x^3 - 6x + 4$.

Dividing $x^3 - 6x + 4$ by $(x - 2)$ gives $x^2 + 2x - 2$.

Thus

$$\begin{aligned} P(x) &= (x-2)(x^2+2x-2) \\ &= (x-2)((x^2+2x+1)-3) \\ &= (x-2)((x+1)^2-3) \qquad \text{(completing the square)} \\ &= (x-2)\left(x+1-\sqrt{3}\right)\left(x+1+\sqrt{3}\right) \qquad \text{(difference of two squares)} \end{aligned}$$

1b $P(x) = x^3 + 3x^2 - 2x - 2$

The factors of 2 are $\pm 1, \pm 2$.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(1) = 0$ and hence $x - 1$ is a factor of $P(x) = x^3 + 3x^2 - 2x - 2$.

Dividing $x^3 + 3x^2 - 2x - 2$ by $(x - 1)$ gives $x^2 + 4x + 2$.

Thus

$$\begin{aligned} P(x) &= (x-1)(x^2+4x+2) \\ &= (x-1)((x^2+4x+4)-2) \\ &= (x-1)((x+2)^2-2) \\ &= (x-1)\left(x+2-\sqrt{2}\right)\left(x+2+\sqrt{2}\right) \end{aligned}$$

1c $P(x) = x^3 - 3x^2 - 2x + 4$

The factors of 4 are $\pm 1, \pm 2, \pm 4$.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(1) = 0$ and hence $x - 1$ is a factor of $P(x) = x^3 - 3x^2 - 2x + 4$.

Dividing $x^3 - 3x^2 - 2x + 4$ by $(x - 1)$ gives $x^2 - 2x - 4$.

Thus

© Cambridge University Press 2019

$$P(x) = (x-1)(x^2-2x-4)$$
$$= (x-1)((x^2-2x+1)-5)$$
$$= (x-1)((x-1)^2-5)$$
$$= (x-1)\left(x-1-\sqrt{5}\right)\left(x-1+\sqrt{5}\right)$$

2a Since the coefficients of the polynomial are real, all complex roots must also have their conjugate as a root. Complex zeroes occur in conjugate pairs.

2b $1+i$ and $1-i$ are zeroes of $P(x)$.

Let the third zero of the polynomial be a, then by the sum of the roots:

$$a+1+i+1-i = -\frac{-8}{1}$$

$$a+2 = 8$$

$$a = 6$$

Hence the other root is 6.

3a Since the coefficients of the equation are real, the complex conjugate of $1-2i$ which is $1+2i$ must also be a root.

3b Since $1+2i$ and $1-2i$ are both roots, it follows that $\left(x-(1+2i)\right)$ and $\left(x-(1-2i)\right)$ must be factors of $P(x)$.

$$\left(x-(1+2i)\right)\left(x-(1-2i)\right)$$
$$= x^2-(1+2i+1-2i)x+(1+2i)(1-2i)$$
$$= x^2-2x+1-4i^2$$
$$= x^2-2x+5$$

Hence x^2-2x+5 must also be a factor of $P(x)$.

© Cambridge University Press 2019

3c Let the third root be a. Using the sum of the roots:

$a + 1 + 2i + 1 - 2i = -\frac{0}{1}$

$a + 2 = 0$

$a = -2$

So $(x + 2)$ is a factor of $P(x)$.

Thus

$P(x) = (x + 2)(x^2 - 2x + 5)$

4a Since the coefficients of the equation are real, the complex conjugate of $-3i$ which is $3i$ must also be a root.

4b Since $3i$ and $-3i$ are both roots, it follows that $(z - 3i)$ and $(z + 3i)$ must be factors of $P(z)$.

$(z - 3i)(z + 3i)$

$= z^2 - 9i^2$

$= z^2 + 9$

Hence $z^2 + 9$ is a quadratic factor of $P(z)$.

4c Let the third root be a. Using the sum of the roots:

$a + 3i + (-3i) = -\frac{3}{2}$

$a = -\frac{3}{2}$

So $\left(z + \frac{3}{2}\right)$ is a factor of $P(z)$ or $(2z + 3)$ is a factor of $P(z)$.

Hence $P(z)$ as a product of factors with real coefficients can be written as:

$P(z) = (2z + 3)(z^2 + 9)$

© Cambridge University Press 2019

5a $P(3+i)$

$= 2(3+i)^3 - 13(3+i)^2 + 26(3+i) - 10$

$= 2(9+6i+i^2)(3+i) - 13(9+6i+i^2) + 78 + 26i - 10$

$= 2(9+6i-1)(3+i) - 13(9+6i-1) + 78 + 26i - 10$

$= 2(8+6i)(3+i) - 13(8+6i) + 78 + 26i - 10$

$= 2(24+8i+18i+6i^2) - 13(8+6i) + 78 + 26i - 10$

$= 2(24+26i-6) - 13(8+6i) + 78 + 26i - 10$

$= 2(18+26i) - 13(8+6i) + 78 + 26i - 10$

$= 36 + 52i - 104 - 78i + 78 + 26i - 10$

$= 0 + 0i$

$= 0$

5b Since $3+i$ is a root of $P(z)$, and since $P(z)$ has real coefficients, the complex conjugate of $3+i$ which is $3-i$ must also be a root of $P(z)$.

Hence $P(3-i) = 0$.

5c i Let the third root be a. Using the sum of the roots of $P(z)$:

$$a + 3 + i + 3 - i = -\frac{-13}{2}$$

$$a + 6 = \frac{13}{2}$$

$$a = \frac{1}{2}$$

So $\left(z - \frac{1}{2}\right)$ is a factor of $P(z)$ or $(2z-1)$ is a factor of $P(z)$.

Hence $P(z)$ as a product of linear factors can be written as:

$P(z) = (2z-1)\big(z-(3+i)\big)\big(z-(3-i)\big)$

$= (2z-1)(z-3-i)(z-3+i)$

© Cambridge University Press 2019

5c ii $P(z) = (2z - 1)(z - 3 - i)(z - 3 + i)$ (from question 5c i)

$= (2z - 1)(z^2 - 3z + zi - 3z + 9 - 3i - zi + 3i - i^2)$

$= (2z - 1)(z^2 - 6z + 10)$

So $P(z)$ as a product of a linear factor and a quadratic factor, with real coefficients, can be written as:

$P(z) = (2z - 1)(z^2 - 6z + 10)$

© Cambridge University Press 2019