Ext 2

Ex 1A

Q17 Let $z = x + iy$ and $\omega = a + ib$

$$z + \omega = (a + x) + i(b + y)$$
$$= a + x \quad \text{(since } z + \omega \text{ is real)}$$

thus $b + y = 0$. (1)

$$z\omega = (ax - by) + i(ay + bx)$$
$$= ax - by \text{ (since } z\omega \text{ is real)}$$

thus $ay + bx = 0$. (2)

from (1), $y = -b$ so in (2)

$$-ab + bx = 0$$
$$\text{or } b(x - a) = 0 \quad (3)$$

In (3) either $b \neq 0$ so that $x = a$

and so $z = x + iy$, $\omega = x - iy$

ie $z = \bar{\omega}$

or in (3) $b = 0$ so $z = x$ and $\omega = a$ (from (1) $y = 0$)

ie $\text{Im}(z) = 0 = \text{Im}(\omega)$ #.

Ext 2

Ex 1A

Q18

$$\frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})}$$

$$= \frac{1-\bar{z}}{1-(z+\bar{z})+z\bar{z}}$$

$$= \frac{1-2\cos\theta+2i\sin\theta}{1-4\cos\theta+4\cos^2\theta+4\sin^2\theta} \qquad \left[\begin{array}{l} z+\bar{z} = 2\,\mathrm{Re}(z) \\ z\bar{z} = x^2+y^2 \end{array}\right]$$

$$= \frac{(1-2\cos\theta)+2i\sin\theta}{5-4\cos\theta} \qquad \left[\cos^2\theta+\sin^2\theta = 1\right]$$

so $\mathrm{Re}\left(\frac{1}{1-z}\right) = \frac{1-2\cos\theta}{5-4\cos\theta}$

Ext 2
Ex 1A
Q19

$$\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \frac{(1+\sin\theta+i\cos\theta)^2}{(1+\sin\theta)^2+(\cos\theta)^2}$$

$$= \frac{1+2\sin\theta+\sin^2\theta-\cos^2\theta+i\,2(1+\sin\theta)\cos\theta}{1+2\sin\theta+\sin^2\theta+\cos^2\theta}$$

$$= \frac{2\sin\theta+2\sin^2\theta+2i(1+\sin\theta)\cos\theta}{2+2\sin\theta} \quad \text{(by Pyth)}$$

$$= \frac{2(1+\sin\theta)(\sin\theta+i\cos\theta)}{2(1+\sin\theta)}$$

$$= \sin\theta+i\cos\theta \quad \not\equiv$$

Ext 2

Ex 1A

Q20

$$\frac{2}{1+z} = \frac{2(1+\bar{z})}{1+(z+\bar{z})+z\bar{z}}$$

$$= \frac{2(1+\cos\theta - i\sin\theta)}{1+2\cos\theta+(\cos^2\theta+\sin^2\theta)} \qquad \left[\begin{array}{l} z+\bar{z} = 2\,\mathrm{Re}(z) \\ z\bar{z} = x^2+y^2 \end{array}\right]$$

$$= \frac{2(1+\cos\theta - i\sin\theta)}{2(1+\cos\theta)} \qquad [\cos^2\theta+\sin^2\theta = 1]$$

$$= 1 - \frac{i\sin\theta}{1+\cos\theta}$$

Now $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\left(\frac{2t}{1+t^2}\right)}{1+\left(\frac{1-t^2}{1+t^2}\right)}$ where $t = \tan\frac{\theta}{2}$.

$$= \frac{2t}{1+t^2+1-t^2}$$

$$= \frac{2t}{2}$$

$$= t$$

So $\dfrac{2}{1+z} = 1 - it$

Ext 2

1B

Q17 a) since α is a root $\alpha^3 - 1 = 0$

taking conjugates $\overline{(\alpha^3)} - 1 = 0$

so $(\bar{\alpha})^3 - 1 = 0$

That is $\bar{\alpha}$ is also a root

since there are only two complex roots, $\beta = \bar{\alpha}$.

b) $(\alpha\bar{\alpha})^3 = \alpha^3(\bar{\alpha})^2$

$= 1$

and $\alpha\bar{\alpha}$ is real

hence $\alpha\bar{\alpha} = 1$

$= \alpha^3$

thus $\bar{\alpha} = \alpha^2$ $(\alpha \neq 0)$

also $\alpha\bar{\alpha} = (\bar{\alpha})^3$

$\alpha = (\bar{\alpha})^2$ $(\bar{\alpha} \neq 0)$

c) $\alpha^3 - 1 = 0$

so $(\alpha - 1)(\alpha^2 + \alpha + 1) = 0$

thus $\alpha^2 + \alpha + 1 = 0$ $(\alpha \neq 1)$

d) when $n = 3m$ the sum is

$(1 + \alpha + \alpha^2) + \alpha^3(1 + \alpha + \alpha^2) + \alpha^6(1 + \alpha + \alpha^2) + \cdots + \alpha^{3(m-1)}(1 + \alpha + \alpha^2)$

$= 0 + \alpha^3 \times 0 + \alpha^6 \times 0 + \cdots + \alpha^{3(m-1)} \times 0$

$= 0$ (*)

when $n = 3m + 1$ the sum is

$(1 + \alpha + \alpha^2) + \alpha^3(1 + \alpha + \alpha^2) + \cdots + \alpha^{3(m-1)}(1 + \alpha + \alpha^2) + \alpha^{3m}$

$= 0 + \alpha^{3m}$ by (*)

$= (\alpha^3)^m$ $(\alpha^3 = 1)$

$= 1$

when $n = 3m + 2$ the sum is

$(1 + \alpha + \alpha^2) + \alpha^3(1 + \alpha + \alpha^2) + \cdots + \alpha^{3(m-1)}(1 + \alpha + \alpha^2) + \alpha^{3m} + \alpha^{3m+1}$

$= 0 + \alpha^{3m}(1 + \alpha)$ by (*)

$= 1 + \alpha$ $(\alpha^3 = 1)$

$= -\alpha^2$ $(1 + \alpha + \alpha^2 = 0)$

Ex 2

Ex 1B

Q18 a) from sums & products of roots

$\alpha + \beta = -\frac{b}{2a}$ which is real

hence $\operatorname{Im}(\alpha+\beta) = 0$ (i)

$\alpha\beta = \frac{c}{a}$ which is real

so $\operatorname{Im}(\alpha\beta) = 0$ (ii)

b) from a)(i) $u+v = 0$

and $y = -v$

from a)(ii) $uy + xv = 0$

so $uy - xy = 0$

$y(u-x) = 0$

and $y \neq 0$ because $b^2 - 4ac < 0$ (not zero)

hence $u = x$

thus $\alpha = x + iy$

$\beta = u + iv$

$= x - iy$

$= \bar{\alpha}$.

Ext 2

Ex 13

Q 19 by Box 13 & 14

$$x^2 + y^2 = \sqrt{a^2 + b^2} \quad \text{(Box 14)}$$
$$x^2 - y^2 = a \quad \text{(Box 13)}$$
$$2xy = b \quad \text{(Box 13)}$$

adding and subtracting

$$2x^2 = a + \sqrt{a^2+b^2} \qquad 2y^2 = \sqrt{a^2+b^2} - a$$

$$x^2 = \frac{1}{2}\left(\sqrt{a^2+b^2} + a\right) \qquad y^2 = \frac{1}{2}\left(\sqrt{a^2+b^2} - a\right)$$

take the +ve sq. root (it does not matter which here) for x

$$x = \sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2} + a\right)}$$

the sign of y is then the sign of b, which is $\frac{b}{|b|}$ for $b \neq 0$

$$y = \frac{b}{|b|}\sqrt{\frac{1}{2}\left(\left(\sqrt{a^2+b^2}\right) - a\right)}$$

so the two sq. roots are $x+iy$ and $-(x+iy)$, hence

$$x + iy = \sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2} + a\right)} + i\frac{b}{|b|}\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2} - a\right)}$$

Note that if $b = 0$, then $x = \pm\sqrt{a}$ $(y = 0)$

Ext 2

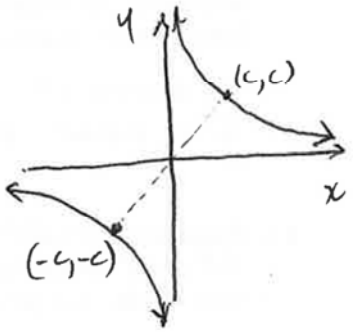
Ex 1C

Q15 a) $\text{Im}(z^2) = 2xy$

so $2xy = 2c^2$

$xy = c^2$

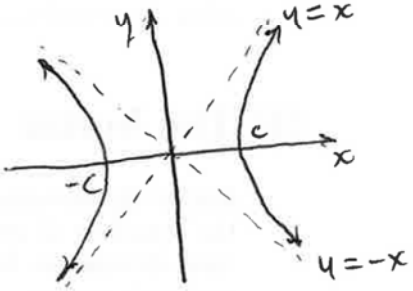
which is a rectangular hyperbola through (c, c) and $(-c, -c)$



b) $\text{Re}(z^2) = x^2 - y^2$

so $x^2 - y^2 = c^2$

which is also a rectangular hyperbola with asymptotes $y = \pm x$ and x-intercepts at $\pm c$

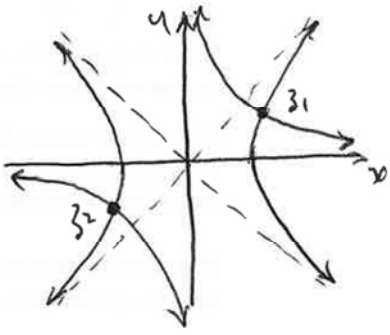


Note that if $z = x + iy$ and $z^2 = c^2 + 2ic^2$ then equating real & imaginary parts gives

$x^2 - y^2 = c^2$ (real part)

and $xy = c^2$ (imag. part).

Here the solution is the intersection of these two curves. Thus there are always two, in opposite quadrants, labelled z_1 and z_2 in the graph.

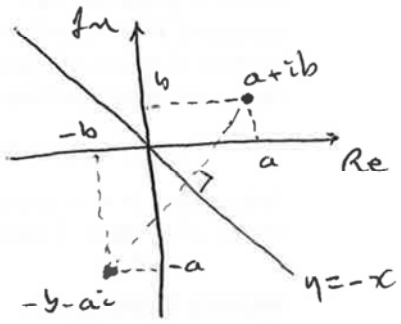


Ext 2

Ex 1C

Q16 Let $z = a + ib$ then

$$-i\bar{z} = -i(a - ib)$$
$$= -ai + i^2 b$$
$$= -b - ai$$



which is the result of swapping a with $-b$ and b with $-a$.

This is a reflection in $y = -x$

Ext 2

Ex 1C

Q17 Let $z = x + iy$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}}$$

$$= \frac{x - iy}{x^2 + y^2}$$

the sign of the imaginary part has changed, so there is a reflection in the x-axis.

Both real & imaginary parts have been multiplied by $\frac{1}{x^2+y^2}$ so there is an enlargement, centre the origin with factor $\frac{1}{x^2+y^2}$

Ex 6 2

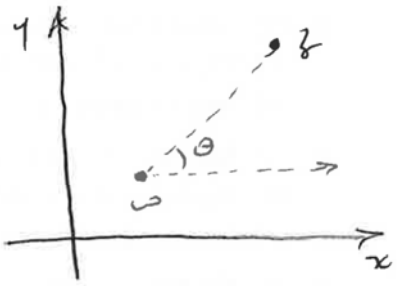
Ex 1D

Q23

a) $|z-w|^2 = (z-w)\overline{(z-w)}$

$= \left((x_2-x_1)+i(y_2-y_1)\right)$

$\times \left((x_2-x_1)-i(y_2-y_1)\right)$



$= (x_2-x_1)^2 + (y_2-y_1)^2$

$= (WZ)^2$ from coordinate geometry.

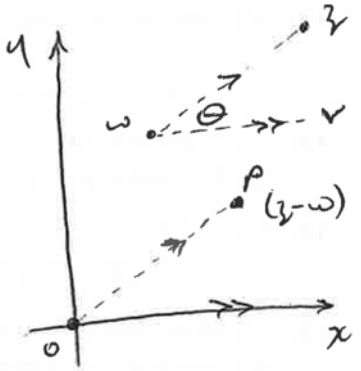
so $|z-w| = |WZ|$. ∎

b) By a) $|WZ| = |OP|$

and by shifting $WZ \parallel OP$

and $WV \parallel OX$

so $\angle ZWV = \angle POX$



ie $\theta = \arg(z-w)$.

Ext 2

Ex 1D

Q24 $z = \cis\theta$

$\omega = \cis\phi$

$z+\omega = (\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi)$

$= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + i\,2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$

$= 2\cos\left(\frac{\theta-\phi}{2}\right)\left(\cos\frac{\theta+\phi}{2} + i\sin\frac{\theta+\phi}{2}\right)$

so $\arg(z+\omega) = \frac{\theta+\phi}{2}$

$= \frac{1}{2}(\theta+\phi)$

$= \frac{1}{2}(\arg z + \arg\omega)$

also notice that $|z+\omega| = 2\cos\left(\frac{\theta-\phi}{2}\right)$

Ext 2

Ex 1D

Q25 a) $|z| = \sqrt{x^2+y^2}$ where $z = x+iy$

$\geqslant \sqrt{x^2}$

$= |x|$

$\geqslant x$

$= Re(z)$

Hence $|z| \geqslant Re(z)$ (with equality when $y=0$)

b) $|z+w|^2 = (z+w)\overline{(z+w)}$

$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$

$= |z|^2 + 2Re(z\bar{w}) + |w|^2$

$\leqslant |z|^2 + 2|z\bar{w}| + |w|^2$

$= |z|^2 + 2|z||\bar{w}| + |w|^2$

$= |z|^2 + 2|z||w| + |w|^2$

$= (|z| + |w|)^2$

Hence $|z+w|^2 \leqslant (|z| + |w|)^2$

and since all quantities are positive or zero

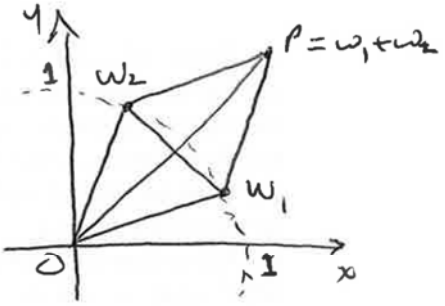
$|z+w| \leqslant |z| + |w|$

Ext 2

Fx1 Ex

Q 22

From the given information ω_1 & ω_2 lie on the unit circle in the 1st quadrant. Let P be the point which corresponds with $\omega_1+\omega_2$



Clearly OW_1PW_2 is a parallelogram and since $|\omega_1|=|\omega_2|$ it is also a rhombus.

Hence OP bisects $\angle W_1OW_2$,

and so $\arg(\omega_1+\omega_2) = \frac{1}{2}(\arg\omega_1 + \arg\omega_2)$

$= \frac{1}{2}(\alpha_1+\alpha_2)$

Now $\omega_1-\omega_2$ is represented by $\overrightarrow{\omega_2\omega_1}$

which is $\frac{\pi}{2}$ clockwise from $\overrightarrow{OP}$

(diagonals of a rhombus are perpendicular)

Thus $\arg(\omega_1-\omega_2) = \frac{1}{2}(\alpha_1+\alpha_2) - \frac{\pi}{2}$

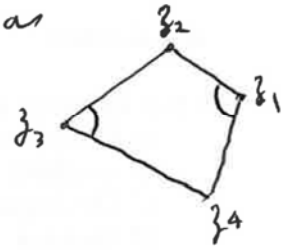
$= \frac{1}{2}(\alpha_1+\alpha_2-\pi)$ #

Ext 2

Ex 1E

Q 23

For simplicity let z_1, z_2, z_3 and z_4 be arranged in anticlockwise order, as in the diagram



Now $\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right)$ represents the angle between vectors $\overrightarrow{z_1 z_4}$ and $\overrightarrow{z_1 z_2}$ ie $\angle z_2 z_1 z_4$.

Likewise $\arg\left(\frac{z_2 - z_3}{z_4 - z_3}\right) = \angle z_4 z_3 z_2$

thus $\angle z_2 z_1 z_4 + \angle z_4 z_3 z_2 = \pi$ (as marked above)

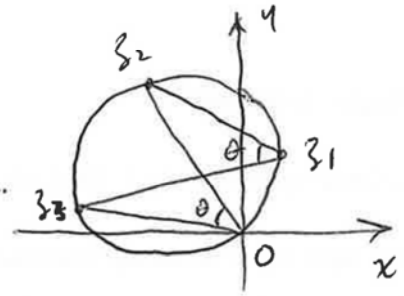
ie the opposite angles of a quadrilateral are supplementary.

Hence by the converse of opposite angles in a cyclic quadrilateral, z_1, z_2, z_3 and z_4 are concyclic.

Ext 2

Ex 1E

Q24



$$\frac{z_1^{-1} - z_2^{-1}}{z_1^{-1} - z_3^{-1}}$$

$$= \frac{z_1^{-1} - z_2^{-1}}{z_1^{-1} - z_3^{-1}} \times \frac{z_1 z_2 z_3}{z_1 z_2 z_3}$$

$$= \frac{z_3 (z_2 - z_1)}{z_2 (z_3 - z_1)}$$

$$= \frac{(z_3 - 0)}{(z_2 - 0)} \cdot \left(\frac{z_2 - z_1}{z_3 - z_1} \right)$$

$$\text{Thus } \arg\left(\frac{z_1^{-1} - z_2^{-1}}{z_1^{-1} - z_3^{-1}} \right) = \arg\left(\frac{z_3 - 0}{z_2 - 0} \right) + \arg\left(\frac{z_2 - z_1}{z_3 - z_1} \right)$$

$$= \arg\left(\frac{z_3 - 0}{z_2 - 0} \right) - \arg\left(\frac{z_3 - z_1}{z_2 - z_1} \right)$$

$$= \theta - \theta \quad \text{(angles in the same segment, see diagram above)}$$

$$= 0$$

Hence $\frac{z_1^{-1} - z_2^{-1}}{z_1^{-1} - z_3^{-1}}$ is real

so $z_1^{-1} - z_2^{-1} = k(z_1^{-1} - z_3^{-1})$ for some real $k > 0$

thus the vector from $\frac{1}{z_2}$ to $\frac{1}{z_1}$ is parallel with the vector from $\frac{1}{z_3}$ to $\frac{1}{z_1}$ and has common point $\frac{1}{z_1}$

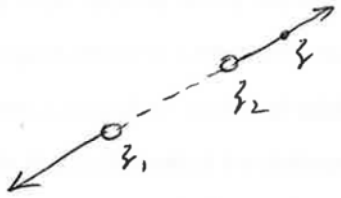
Hence $\frac{1}{z_1}$, $\frac{1}{z_2}$ and $\frac{1}{z_3}$ are collinear.

fzxt 2

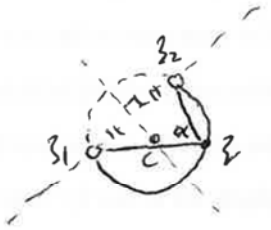
fzx1F

Q19 a) When $\alpha = 0$ vector $\overrightarrow{z_1 z}$ is parallel with $\overrightarrow{z_2 z}$ with common point z, hence z_1, z_2 and z are collinear with z on the same side of z_1 & z_2 (not between) so z is on either of the opposite rays with end points z_1 & z_2.

$\arg\left(\frac{z-z_1}{z-z_2}\right)$ is undefined at z_1 and z_2 so these points are excluded. The diagram shows one possible location of z on one of these rays.



b) the angle between vector $\overrightarrow{z_1 z}$ and $\overrightarrow{z_2 z}$ is constant, so by the converse of the angle in the same segment, z lies on an arc of a circle through z_1 and z_2, excluding the endpoints & taken anticlockwise. Since the angle at z is acute, it is the angle in a major segment. The centre will be somewhere on the perpendicular bisector of $z_1 z_2$ & on the same side as the arc.



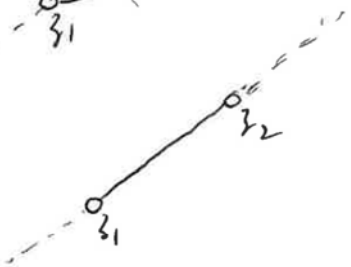
c) By similar reasoning to (b), this is the angle in a semi-circle. The centre is $\frac{z_1+z_2}{2}$



d) Likewise this is the angle in a minor segment



e) When $\alpha = \pi$, the vectors $z_1 z$ and $z_2 z$ are in opposite directions, hence z is in the line segment between z_1 and z_2



Ext 2
Ex 1F
Q20

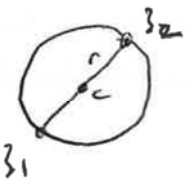
$|z-z_1|^2 + |z-z_2|^2 = (z_1-z_2)^2$

The three distances satisfy Pythagoras theorem, hence $\Delta z_1 z z_2$ is right angled at z.

Hence z lies on the circle with diameter $z_1 z_2$, including the end points.

The centre is $\frac{z_1+z_2}{2}$

The radius is $r = \frac{|z_1-z_2|}{2}$



Ext 2

Ex 1F

Q21 a) (i) squaring $k^2|z-z_1|^2 = l^2|z-z_2|^2$

putting $z = x+iy$, $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

and completing squares will give the desired result after a lot of algebra. Alternatively:

Note that $|z-w|^2 = (z-w)(\bar{z}-\bar{w})$

$= |z|^2 - z\bar{w} - \bar{z}w + |w|^2$

so

$k^2(|z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2) = l^2(|z|^2 - z\bar{z}_2 - \bar{z}z_2 + |z_2|^2)$

Rearranging

$(k^2-l^2)|z|^2 - z(k^2\bar{z}_1 - l^2\bar{z}_2) - \bar{z}(k^2z_1 - l^2z_2) = l^2|z_2|^2 - k^2|z_1|^2$

Now complete the square on the LHS

$|z|^2 - z\,\frac{k^2\bar{z}_1 - l^2\bar{z}_2}{k^2-l^2} - \bar{z}\,\frac{k^2z_1 - l^2z_2}{k^2-l^2} = \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2}$

$|z|^2 - z\left(\overline{\frac{k^2z_1 - l^2z_2}{k^2-l^2}}\right) - \bar{z}\left(\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right) + \left|\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2$

$= \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2} + \left|\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2$

so $LHS = \left|z - \frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2$

and $RHS = \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2} + \frac{k^4|z_1|^2 - k^2l^2z_1\bar{z}_2 - k^2l^2\bar{z}_1z_2 + l^4|z_2|^2}{(k^2-l^2)^2}$

$= \left[k^2l^2|z_2|^2 - l^4|z_2|^2 - k^4|z_1|^2 + k^2l^2|z_1|^2 + k^4|z_1|^2 - k^2l^2z_1\bar{z}_2 - k^2l^2\bar{z}_1z_2 + l^4|z_2|^2\right] \Big/ (k^2-l^2)^2$

$= \frac{k^2l^2(|z_1|^2 - z_1\bar{z}_2 - \bar{z}_1z_2 + |z_2|^2)}{(k^2-l^2)^2}$

so $RHS = \frac{k^2l^2|z_1-z_2|^2}{(k^2-l^2)^2}$

Ext 2

Ex 1F

Q21 (ctd) a) (i) Thus $\left|z - \frac{k^2z_1 - l^2z_2}{k^2 - l^2}\right|^2 = \frac{k^2l^2|z_1 - z_2|^2}{(k^2 - l^2)^2}$

or, taking sq. roots

$$\left|z - \frac{k^2z_1 - l^2z_2}{k^2 - l^2}\right| = \frac{kl|z_1 - z_2|}{|k^2 - l^2|}$$

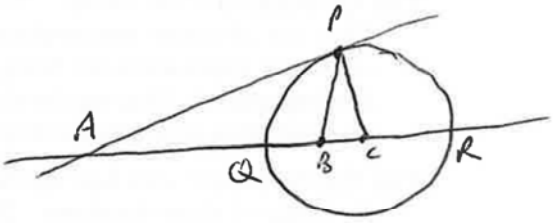
But this is the equation of a circle $|z - c| = r$

where centre $c = \frac{k^2z_1 - l^2z_2}{k^2 - l^2}$

and radius $r = \frac{kl|z_1 - z_2|}{|k^2 - l^2|}$

as required.

a) (ii) In the diagram



let $A = z_1$, $B = z_2$.

Let Q divide AB internally in the ratio $k:l$ and let R divide AB externally in ratio $k:l$.

Let C be the midpoint of QR and let P lie on the circle with diameter QR.

The task is to show that $AP : PB = k : l$

Since ratios are involved, similar Δs will be used.

First note that $Q = \frac{kz_2 + lz_1}{k + l}$ $\quad R = \frac{kz_2 - lz_1}{k - l}$

Then $C = \frac{1}{2}(Q + R)$

$$= \frac{1}{2}\left(\frac{kz_2 + lz_1}{k + l} + \frac{kz_2 - lz_1}{k - l}\right)$$

$$= \frac{1}{2}\left(\frac{k^2z_2 - klz_2 + klz_1 - l^2z_1 + k^2z_2 + klz_2 - klz_1 - l^2z_1}{k^2 - l^2}\right)$$

$$= \frac{1}{2}\,\frac{2k^2z_2 - 2l^2z_1}{k^2 - l^2} = \frac{k^2z_2 - l^2z_1}{k^2 - l^2}$$

Ext 2
Ex 1F
Q21
(ctd)
a)(ii)

and the radius of the circle is

$$QC = \left| \frac{k^2 z_2 - l^2 z_1}{k^2 - l^2} - \frac{k z_2 + l z_1}{k + l} \right|$$

$$= \left| \frac{k^2 z_2 - l^2 z_1 - k^2(k-l) z_2 - l(k-l) z_1}{k^2 - l^2} \right|$$

$$= \left| \frac{k l z_2 - k l z_1}{k^2 - l^2} \right|$$

$$= \frac{kl\,|z_2 - z_1|}{k^2 - l^2}$$

Next conside ΔAPC and ΔPBC

clearly $\angle C$ is common

Next $$\frac{AC}{PC} = \frac{\frac{1}{2}(AR + AQ)}{\frac{1}{2}(AR - AQ)}$$

$$= \frac{1 + (AQ/AR)}{1 - (AQ/AR)}$$

But, by the ratio division $AQ/AR = \frac{k-l}{k+l}$ so

$$\frac{AC}{PC} = \frac{(k+l) + (k-l)}{(k+l) - (k-l)}$$

$$= \frac{2k}{2l}$$

$$= \frac{k}{l}.$$

Last $$\frac{PC}{BC} = \frac{\frac{1}{2}(AR - AQ)}{AC - AB}$$

$$= \frac{(AR - AQ)}{(AR + AQ) - AB}$$

$$= \frac{1 - (AQ/AR)}{1 + AQ/AR - 2AB/AR}$$

Ext 2
Ex 1F
Q21 (ctd)
a)(ii)

but $\frac{AB}{AR} = \frac{k-l}{k}$ by the ratio division, so

$$\frac{PC}{BC} = \frac{\left(1 - \frac{k-l}{k+l}\right)}{\left(1 + \frac{k-l}{k+l} - \frac{2(k-l)}{k}\right)} \times \frac{(k+l)k}{(k+l)k}$$

$$= \frac{((k+l)-(k-l))k}{(k+l)k + k(k-l) - 2(k^2-l^2)}$$

$$= \frac{2kl}{2l^2}$$

$$= \frac{k}{l}.$$

Hence $\Delta APC \,|||\, \Delta PBC$ (SAS)

with ratio $k:l$.

Thus $AP:PB = k:l$ (ratio of matching sides in similar Δs)

That is P satisfies the given equation, and this is true for any point on the circle.

That is, the circle is the graph of the equation.

b) return to the original equation

$$k|z-z_1| = l|z-z_2|$$

in the limit as $k \to l$ this gives

$$k|z-z_1| = k|z-z_2|$$

that is $$|z-z_1| = |z-z_2|$$

which is the perpendicular bisector of z_1 and z_2 (AB)

Ext 2

Ex 1G

Q19 Let $P(z) = a_n z^n + \dots + a_2 z^2 + a_1 z + a_0$

where the coefficients are real,

and let $P(\alpha) = \beta$ where α and β are complex (possibly real). Then

$$P(\bar{\alpha}) = a_n(\bar{\alpha})^n + \dots + a_2(\bar{\alpha})^2 + a_1(\bar{\alpha}) + a_0$$

$$= a_n\overline{(\alpha^n)} + \dots + a_2\overline{(\alpha^2)} + a_1\overline{(\alpha)} + a_0$$

since $(\bar{z})^n = \overline{(z^n)}$

$$= \overline{(a_n\alpha^n)} + \dots + \overline{(a_2\alpha^2)} + \overline{(a_1\alpha)} + \overline{a_0}$$

since $a\bar{z} = \overline{(az)}$ when a real

$$= \overline{(a_n\alpha^n + \dots + a_2\alpha^2 + a_1\alpha + a_0)}$$

since $\bar{\omega} + \bar{z} = \overline{(\omega + z)}$

$$= \overline{P(\alpha)}$$

$$= \bar{\beta} \qquad \#.$$

Ext 2
Ex 1G
Q20

$P\left(\frac{p}{q}\right) = 0$ so

$$a_n \frac{p^n}{q^n} + \dots + a_2 \frac{p^2}{q^2} + a_1 \frac{p}{q} + a_0 = 0$$

so $a_n p^n + \dots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = 0$

thus $a_{n-1} p^{n-1} q + \dots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = -a_n p^n$

or $q\left(a_{n-1} p^{n-1} + \dots + a_2 p^2 q^{n-3} + a_1 p q^{n-2} + a_0 q^{n-1}\right) = -a_n p^n$

Hence $a_n p^n$ is a multiple of q and q is not a factor of p^n. Hence q is a factor of a_n.

Likewise

$$p\left(a_1 q^{n-1} + a_2 q^{n-2} p + \dots + a_{n-1} p^{n-1}\right) = -a_0 q^n$$

Hence $a_0 q^n$ is a multiple of p and p is not a factor of q^n. Hence p is a factor of a_0.

Ext 2

Ex1G

Q21 By way of contradiction, suppose there exists an odd polynomial with real coefficients that does not have a real zero.

By the Fundamental Theorem of Algebra, it has at least one complex zero & by BOX 3B, the number of zeroes counted by multiplicity is odd.

But the coefficients are real & so the zeroes come in conjugate pairs, by which there is an even number of zeroes.

The number of zeroes cannot be both odd & even, & so there is a contradiction.

Hence every odd polynomial with real coefficients has at least one real zero.

Ext 2

Ex 1G

Q22 Since $P(z)$ has double zero $z=\alpha$, it has factor $(z-\alpha)^2$.

a) Since $P(z)$ has real coefficients, it also has double zero $z=\bar{\alpha}$ and factor $(z-\bar{\alpha})^2$

b) Thus it is possible to write

$$P(z) = (z-\alpha)^2(z-\bar{\alpha})^2 Q(z)$$

$$= ((z-\alpha)(z-\bar{\alpha}))^2 Q(z)$$

$$= (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)^2 Q(z)$$

c) $$P'(z) = 2(z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)\cdot(2z - 2\mathrm{Re}(\alpha))\cdot Q(z) + (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)^2 Q'(z)$$

$$= (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)\left[4(z - \mathrm{Re}(\alpha))Q(z) + (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)Q'(z)\right]$$

$$= (z-\alpha)(z-\bar{\alpha})\left[4(z - \mathrm{Re}(\alpha))Q(z) + (z-\alpha)(z-\bar{\alpha})Q'(z)\right]$$

That is $(z-\alpha)$ is a factor of $P'(z)$
& hence $P'(\alpha) = 0$.
[It should also be clear that $P(\bar{\alpha}) = 0$]

d) This is essentially the result of Box 36 applied to complex zeroes. As noted there, induction may be used. The induction step will essentially be parts a), b), c) above.

Ext 2

Ex 14

Q23 a) Let $u = a + b\sqrt{2}$ and $v = p + q\sqrt{2}$ then

(i) LHS $= u^* + v^*$

$= a - b\sqrt{2} + p - q\sqrt{2}$

$= (a+p) - (b+q)\sqrt{2}$

RHS $= (u+v)^*$

$= (a + b\sqrt{2} + p + q\sqrt{2})^*$

$= ((a+p) + (b+q)\sqrt{2})^*$

$= (a+p) - (b+q)\sqrt{2}$

$=$ LHS #.

(ii) LHS $= \lambda u^*$

$= \lambda(a - b\sqrt{2})$

$= \lambda a - \lambda b\sqrt{2}$

RHS $= (\lambda u)^*$

$= (\lambda a + \lambda b\sqrt{2})^*$

$= \lambda a - \lambda b\sqrt{2}$

$=$ LHS #.

(iii) A $(u')^* = u^*$

$= (u^*)^1$

So the result is true for $n = 1$

B Assume the result is true for $n = k$, a positive integer

That is, assume that $(u^k)^* = (u^*)^k$ (1)

Let $u^k = p + q\sqrt{2}$

Now prove the result is true for $n = k+1$,

that is, prove that $(u^{k+1})^* = (u^*)^{k+1}$

Now LHS $= (u^k \cdot u)^*$

$= ((p + q\sqrt{2})(a + b\sqrt{2}))^*$

Ext 2
Ex 1G
Q23
(ctd)
a)(iii)

$$= (ap + bqc + (aq+bp)\sqrt{c})^*$$

$$= ap + bqc - (aq+bp)\sqrt{c}$$

$$RHS = (u^*)^k \cdot u^*$$

$$= (u^k)^* \cdot u^* \quad \text{by } (\dagger), \text{ the induction hypothesis.}$$

$$\text{so RHS} = (p - q\sqrt{c})(a - b\sqrt{c})$$

$$= (ap + bqc) - (aq+bp)\sqrt{c}$$

$$= \text{LHS} \ \#.$$

$\subseteq$ It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

b) Let $P(x) = a_0 + a_1 x + \dots + a_n x^n$ have rational coefficients and let $P(u) = 0$

then $P(u^*) = a_0 + a_1 u^* + a_2 (u^*)^2 + \dots + a_n (u^*)^n$

$$= a_0 + a_1 (u)^* + a_2 (u^2)^* + \dots + a_n (u^n)^*$$

by a)(iii)

$$= a_0^* + (a_1 u)^* + (a_2 u^2)^* + \dots + (a_n u^n)^*$$

by a)(ii)

$$= (a_0 + a_1 u + a_2 u^2 + \dots + a_n u^n)^*$$

by a)(i)

$$= (P(u))^*$$

$$= 0^*$$

$$= 0$$

Hence u^* is also a zero of $P(x)$. $\#$