Solutions to Exercise 6H Chapter review

1a i $v = 2x - 1$

$$\frac{dx}{dt} = 2x - 1$$

$$\frac{dt}{dx} = \frac{1}{2x-1}$$

$$\int dt = \int \frac{1}{2x-1}\, dx$$

$$t = \frac{1}{2}\int \frac{2}{2x-1}\, dx$$

$$t = \frac{1}{2}\ln|2x-1| + C$$

At $t = 0,\ x = 1.$

$$0 = \frac{1}{2}\ln|1| + C$$

$$C = 0$$

Therefore

$$t = \frac{1}{2}\ln|2x-1|$$

1a ii

$$t = \frac{1}{2}\ln|2x-1|$$

$$2t = \ln|2x-1|$$

$$e^{2t} = 2x - 1$$

$$x = \frac{1}{2}(e^{2t} + 1)$$

1b i $v = -6x^2$

$$\frac{dx}{dt} = -6x^2$$

$$\int dt = -\int \frac{dx}{6x^2}$$

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$$t = \frac{1}{6x} + C$$

At $t = 0, x = 1$.

$$C = -\frac{1}{6}$$

Therefore

$$t = \frac{1}{6x} - \frac{1}{6}$$

$$t = \frac{1}{6}(x^{-1} - 1)$$

1b ii

$$t = \frac{1}{6}(x^{-1} - 1)$$

$$6t = x^{-1} - 1$$

$$x^{-1} = 1 + 6t$$

$$x = \frac{1}{1 + 6t}$$

or $x = (1 + 6t)^{-1}$

2a $v = 2x - 1$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx}$$

$$\ddot{x} = (2x - 1)(2)$$

$$\ddot{x} = 2(2x - 1)$$

2b $v = -6x^2$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx}$$

$$\ddot{x} = -6x^2(-12x)$$

$$\ddot{x} = 72x^3$$

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3a $\ddot{x} = 6$

$\ddot{x} = \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6$

$\frac{1}{2}v^2 = \int 6\,dx$

$v^2 = \int 12\,dx$

$v^2 = 12x + C$

At $x = 0, v = 0.$

$C = 0$

Therefore

$v^2 = 12x$

3b $\ddot{x} = \sin 6x$

$\ddot{x} = \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sin 6x$

$v^2 = 2\int \sin 6x\,dx$

$v^2 = -\frac{1}{3}\cos 6x + C$

At $v = 0,\ x = 0,$ hence

$C = \frac{1}{3}$

Therefore

$v^2 = -\frac{1}{3}\cos 6x + \frac{1}{3}$

$v^2 = \frac{1}{3}(1 - \cos 6x)$

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4 $\ddot{x} = 4v$

$\ddot{x} = \frac{dv}{dt} = 4v$

$\frac{dt}{dv} = \frac{1}{4v}$

$\int dt = \int \frac{dv}{4v}$

$t = \frac{1}{4}\ln v + C$

At $t = 0, \ v = 2$, hence

$C = -\frac{1}{4}\ln 2$

Therefore

$t = \frac{1}{4}\ln v - \frac{1}{4}\ln 2$

$t = \frac{1}{4}\ln \frac{v}{2}$

5

$\ddot{x} = \frac{1}{3v}$

$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{3v}$

$\frac{dv}{dx} = \frac{1}{3v^2}$

$\frac{dx}{dv} = 3v^2$

$\int dx = \int 3v^2 dv$

$x = v^3 + C$

At $x = 0, \ v = 1$, hence

$C = -1$

Therefore

$x = v^3 - 1$

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6a $F = 18t, \quad m = 3$

$$F = m\ddot{x} = m\frac{d^2x}{dt^2} = 18t$$

$$\frac{d^2x}{dt^2} = 6t$$

$$\frac{dx}{dt} = 3t^2 + C$$

Initially particle is at rest at origin, so $\frac{dx}{dt} = 0$ when $t = 0$, hence:

$$C = 0$$

Therefore

$$\frac{dx}{dt} = 3t^2$$

$$\int dx = 3\int t^2 dt$$

$$x = t^3 + D$$

Using initial conditions, $t = 0$ and $x = 0$:

$$D = 0$$

Therefore

$$x = t^3$$

At $t = 2$,

$$x = 2^3 = 8$$

6b $F = 4x - 3, \quad m = 2$

$$F = m\ddot{x} = m\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x - 3$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2}(4x - 3)$$

$$v^2 = \int (4x - 3)\, dx$$

$$v^2 = 2x^2 - 3x + C$$

At $t = 0, x = 0, v = 0$, hence

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$C = 0$

Therefore

$v^2 = 2x^2 - 3x$

At $x = 4$,

$v^2 = 2(16) - 3(4)$

$= 20$

$v = 2\sqrt{5}$ (taking positive value of v)

6c

$F = \dfrac{1}{1+v}, \quad m = 0.5$

$F = m\ddot{x} = m\dfrac{dv}{dt} = \dfrac{1}{1+v}$

$\dfrac{dv}{dt} = \dfrac{2}{1+v}$

$\dfrac{dt}{dv} = \dfrac{1}{2}(1+v)$

$\int dt = \dfrac{1}{2}\int (1+v)\, dv$

$t = \dfrac{1}{2}\left(v + \dfrac{v^2}{2}\right) + C$

At $t = 0, \quad v = 0$, hence

$C = 0$

Therefore

$t = \dfrac{v}{2} + \dfrac{v^2}{4}$

At $v = 6$,

$t = \dfrac{6}{2} + \dfrac{36}{4}$

$t = 12$

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6d $F = 2 + v^2, \quad m = 2$

$F = m\ddot{x} = mv\dfrac{dv}{dx} = (2 + v^2)$

$\dfrac{dv}{dx} = \dfrac{2 + v^2}{2v}$

$\dfrac{dx}{dt} = \dfrac{2v}{2 + v^2}$

$\int dx = \int \dfrac{2v}{2 + v^2}\, dv$

$x = \ln(2 + v^2) + C$ (since $2 + v^2$ is positive)

At $x = 0, \quad v = 0$, hence

$C = -\ln 2$

Therefore

$x = \ln(2 + v^2) - \ln 2$

$x = \ln\left(\dfrac{2 + v^2}{2}\right)$

At $v = 2$,

$x = \ln\left(\dfrac{2 + 4}{2}\right)$

$x = \ln 3$

7 $m = 2, \quad F_1 = 40$ N in NE direction

$F_1 = 20\sqrt{2}\hat{\imath} + 20\sqrt{2}\hat{\jmath}$

$F_2 = 20\hat{\imath} + 40\hat{\jmath}$

The resultant force is:

$F_r = F_1 + F_2$

$= \left(20\sqrt{2} + 20\right)\hat{\imath} + (20\sqrt{2} + 40)\hat{\jmath}$

$= 20\left(\left(\sqrt{2} + 1\right)\hat{\imath} + \left(\sqrt{2} + 2\right)\hat{\jmath}\right)$

Now $F_r = m\ddot{x}_r$

$20\left(\left(\sqrt{2} + 1\right)\hat{\imath} + \left(\sqrt{2} + 2\right)\hat{\jmath}\right) = 2 \times \ddot{x}_r$

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$$\ddot{x}_r = 10\left((\sqrt{2}+1)\hat{\imath} + (\sqrt{2}+2)\hat{\jmath}\right)$$

$$|\ddot{x}_r| = 10\sqrt{(\sqrt{2}+1)^2 + (\sqrt{2}+2)^2}$$

$$= 10\sqrt{3 + 2\sqrt{2} + 6 + 4\sqrt{2}}$$

$$= 10\sqrt{9 + 6\sqrt{2}}$$

$$= 41.815\ldots$$

Magnitude of resultant acceleration is about 41.8 m/s^2

The direction of resultant acceleration is same as that of the resultant force.

Direction of $\ddot{x}_r$:

$$\tan\alpha = \frac{\sqrt{2}+2}{\sqrt{2}+1}$$

$$\alpha = 54.735\ldots°$$

Angle north of east is about 54.7° so direction of acceleration is 035.3°T.

8a The plane undergoes a constant deceleration until it stops completely. So, the forces acting on the plane are just the deceleration. Hence,

$$\ddot{x} = -k$$

$$v_0 = 100, \quad d = 2000$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -k$$

$$\frac{1}{2}v^2 = -k\int dx$$

$$v^2 = -2k\int dx$$

$$= -2kx + C$$

At $x = 0, \quad v = 100$, hence

$$C = 10\,000$$

Therefore

$$v^2 = -2kx + 10\,000$$

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$v^2 = 10\,000 - 2kx$

At $x = 2000, \quad v = 0.$

$0 = 10\,000 - 4000k$

$k = \frac{5}{2}$

Therefore

$v^2 = 10\,000 - 5x$

8b i $v^2 = 10\,000 - 5x$

At $x = 1$ km $= 1000$ m,

$v^2 = 10\,000 - 5000$

$v^2 = 5000$

$v = 50\sqrt{2}$ m/s (taking positive v as plane continues to move in forward direction)

8b ii $v^2 = 10\,000 - 5x$

At $v = 50,$

$2500 = 10\,000 - 5x$

$x = 1500$ m

8c $v^2 = 10\,000 - 5x$

Now, while the brakes are being applied, the plane continues to move in the forward direction due to the existing momentum. So, the velocity is positive.
Hence, $v = \sqrt{10\,000 - 5x}$

8d $v = \sqrt{10\,000 - 5x}$

$\frac{dx}{dt} = \sqrt{10\,000 - 5x}$

$\frac{dt}{dx} = \frac{1}{\sqrt{10\,000 - 5x}}$

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$$\int dt = \int \frac{dx}{\sqrt{10\,000 - 5x}}$$

$$t = -\frac{2}{5}\sqrt{10\,000 - 5x} + C$$

When $t = 0, x = 0$, hence

$$0 = -\frac{2}{5}\sqrt{10\,000} + C$$

$$C = \frac{2}{5} \times 100$$

$$C = 40$$

Therefore

$$t = -\frac{2}{5}\sqrt{10\,000 - 5x} + 40$$

$$\sqrt{10\,000 - 5x} = \frac{5}{2}(40 - t)$$

$$10\,000 - 5x = \frac{25}{4}(40 - t)^2$$

$$5x = 10\,000 - \frac{25}{4}(1600 - 80t + t^2)$$

$$x = 2000 - \frac{5}{4}(1600 - 80t + t^2)$$

$$x = 2000 - 2000 + 100t - \frac{5}{4}t^2$$

$$x = 100t - \frac{5}{4}t^2$$

The plane stops when $x = 2000$.

$$t = -\frac{2}{5}\sqrt{10\,000 - 5x} + 40$$

$$= -\frac{2}{5}\sqrt{10\,000 - 10\,000} + 40$$

$$= 40 \text{ s}$$

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9a $v_0 = \frac{1}{2}\text{km/s} = 500\text{ m/s}, \quad v_1 = 250\text{ m/s}, \quad x_1 = 50\text{ m}$

$$\ddot{x} = v\frac{dv}{dx} = -kv$$

$$\frac{dv}{dx} = -k$$

$$\int dv = -k\int dx$$

$$v = -kx + C$$

At $x = 0,\ v_0 = 500$, hence

$$C = 500$$

Therefore

$$v = 500 - kx$$

At $x = 50,\ v = 250$, hence

$$k = 5$$

Therefore

$$v = \frac{dx}{dt} = 500 - 5x$$

$$\frac{dt}{dx} = \frac{1}{500 - 5x}$$

$$\int dt = \int \frac{dx}{500 - 5x}$$

$$\int dt = -\frac{1}{5}\int \frac{-5}{500 - 5x}\,dx$$

$$t = -\frac{1}{5}\ln|500 - 5x| + D$$

At $t = 0, x = 0$, hence

$$D = \frac{1}{5}\ln 500$$

Therefore

$$t = -\frac{1}{5}\ln|500 - 5x| + \frac{1}{5}\ln 500$$

$$t = \frac{1}{5}\ln\left(\frac{500}{500 - 5x}\right) \quad \text{for } x < 100$$

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$$5t = \ln\left(\frac{100}{100 - x}\right)$$

$$\frac{100}{100 - x} = e^{5t}$$

$$100 - x = 100e^{-5t}$$

$$x = 100 - 100e^{-5t}$$

$$x = 100(1 - e^{-5t})$$

9b

$$t = \frac{1}{5}\ln\left(\frac{100}{100 - x}\right)$$

At $x = 50$,

$$t = \frac{1}{5}\ln\left(\frac{100}{100 - 50}\right)$$

$$t = \frac{1}{5}\ln 2$$

$$v = 500 - 5x$$

Eventually, the torpedo will stop after a distance of 100 m as the velocity reduces to zero.

10a

$$\ddot{x} = -\frac{1}{2}e^{-x}\ \text{m/s}^2$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-x}$$

$$v^2 = -\int e^{-x}\,dx$$

$$= e^{-x} + C$$

At $x = 0$, $v = 1$, hence

$$C = 0$$

Therefore

$$v^2 = e^{-x}$$

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10b v is always positive as v is initially positive (1 m/s), and is never zero ($e^{-x} \neq 0$).

$v^2 = e^{-x}$

$v = e^{-\frac{x}{2}}$

$\frac{dx}{dt} = e^{-\frac{x}{2}}$

$\frac{dt}{dx} = e^{\frac{x}{2}}$

$\int dt = \int e^{\frac{x}{2}}\, dx$

$t = 2e^{\frac{x}{2}} + C$

At $t = 0,\ x = 0$, hence

$C = -2$

Therefore

$t = 2e^{\frac{x}{2}} - 2$

$t = 2\left(e^{\frac{x}{2}} - 1\right)$

$e^{\frac{x}{2}} = \frac{t}{2} + 1$

$\frac{x}{2} = \ln\left(\frac{t}{2} + 1\right)$

$x = 2\ln\frac{1}{2}(t + 2)$

10c

$x = 2\ln\frac{1}{2}(t + 2)\,, v = e^{-\frac{x}{2}}$

As $t \to \infty,\ x \to \infty$ and $v \to 0$

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11a $\ddot{x} = a + bv^2 \qquad a, b > 0$

$$\ddot{x} = v\frac{dv}{dx} = a + bv^2$$

$$\frac{dv}{dx} = \frac{a + bv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{a + bv^2}$$

$$\int dx = \int \frac{v}{a + bv^2}\, dv$$

$$x = \frac{1}{2b}\int \frac{2bv}{a + bv^2}\, dv$$

$$= \frac{1}{2b}\ln(a + bv^2) + C \qquad \text{(since } a, b > 0 \text{ so } a + bv^2 \text{ is positive)}$$

At $t = 0, \quad x = 0, \quad v = 0$, hence

$$C = -\frac{1}{2b}\ln a$$

Therefore

$$x = \frac{1}{2b}\ln(a + bv^2) - \frac{1}{2b}\ln a$$

$$x = \frac{1}{2b}\ln\left(\frac{a + bv^2}{a}\right)$$

$$2bx = \ln\left(\frac{a + bv^2}{a}\right)$$

$$e^{2bx} = \frac{a + bv^2}{a}$$

$$e^{2bx} = 1 + \frac{b}{a}v^2$$

$$\frac{b}{a}v^2 = e^{2bx} - 1$$

$$v^2 = \frac{a}{b}(e^{2bx} - 1)$$

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11b

$$\ddot{x} = a + bv^2 = \frac{dv}{dt}$$

$$\frac{dt}{dv} = \frac{1}{a + bv^2}$$

$$\int dt = \int \frac{dv}{a + bv^2}$$

$$t = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\frac{\sqrt{b}\, v}{\sqrt{a}}\right) + C$$

At $t = 0,\ \ v = 0$, hence

$$C = 0$$

Therefore

$$t = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\frac{\sqrt{b}\, v}{\sqrt{a}}\right)$$

$$\sqrt{ab}\, t = \tan^{-1}\left(\frac{\sqrt{b}\, v}{\sqrt{a}}\right)$$

$$\frac{\sqrt{b}\, v}{\sqrt{a}} = \tan \sqrt{ab}\, t$$

$$v = \frac{\sqrt{a}}{\sqrt{b}} \tan \sqrt{ab}\, t$$

12a

$$\ddot{x} = -16x = v\frac{dv}{dx}$$

$$\int v\, dv = -16 \int x\, dx$$

$$\frac{1}{2}v^2 = -8x^2 + C$$

At $t = 0,\ \ x = 1,\ \ v = 4$, hence

$$C = 16$$

Therefore

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$$\frac{1}{2}v^2 = -8x^2 + 16$$

$$v^2 = 32 - 16x^2$$

$$v^2 = 16(2 - x^2)$$

12b At $x = x_{max}$, $v = 0$.

$$0 = 16(2 - (x_{max})^2)$$

$$(x_{max})^2 = 2$$

$$x_{max} = \sqrt{2}$$

12c $\ddot{x} = -16x$

Amplitude $a = x_{max} = \sqrt{2}$

Let $x = \sqrt{2}\cos(nt + \alpha)$

$\dot{x} = -\sqrt{2}n\sin(nt + \alpha)$ and

$\ddot{x} = -\sqrt{2}n^2\cos(nt + \alpha)$

Substituting for $\ddot{x}$ and x in $\ddot{x} = -16x$:

$-\sqrt{2}n^2\cos(nt + \alpha) = -16\sqrt{2}\cos(nt + \alpha)$

$n^2 = 16$

$n = 4$

At $t = 0$, $x = 1$, so $x = \sqrt{2}\cos(nt + \alpha)$ becomes:

$1 = \sqrt{2}\cos(0 + \alpha)$

$$\cos\alpha = \frac{1}{\sqrt{2}}$$

$$\alpha = \frac{\pi}{4}$$

Therefore

$$x = \sqrt{2}\cos\left(4t + \frac{\pi}{4}\right)$$

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13a $x = 5 + \sqrt{3}\sin 3t - \cos 3t$

$\dot{x} = 3\sqrt{3}\cos 3t + 3\sin 3t$

$\ddot{x} = -9\sqrt{3}\sin 3t + 9\cos 3t$

$\ddot{x} = -9\left(\sqrt{3}\sin 3t - \cos 3t\right)$

$\ddot{x} = -9\left(\left(5 + \sqrt{3}\sin 3t - \cos 3t\right) - 5\right)$

$\ddot{x} = -3^2(x - 5)$

which is of the form $\ddot{x} = -n^2(x - c)$

As the equation satisfies the conditions of restoring force of SHM, the given particle is under SHM.

13b $\sqrt{3}\sin 3t - \cos 3t$

$= 2\left(\frac{\sqrt{3}}{2}\sin 3t - \frac{1}{2}\cos 3t\right)$

$= 2\left(\cos\frac{\pi}{6}\sin 3t - \sin\frac{\pi}{6}\cos 3t\right)$

$= 2\sin\left(3t - \frac{\pi}{6}\right)$

Therefore

$R = 2, \quad \alpha = \frac{\pi}{6}$

Hence

$x = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$

13c The amplitude is: $a = R = 2$

The centre of motion is: $x = 5$

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13d

$x = 5 + 2\sin(3t - \frac{\pi}{6})$

When $v = v_{max}$, $x = 5$

$5 = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$

$\sin(3t - \frac{\pi}{6}) = 0$

$3t = \frac{\pi}{6}$

$t = \frac{\pi}{18}$

14 $v_{max} = 2, \quad \ddot{x}_{max} = 6$

For SHM: $v_{max} = an$ and $\ddot{x}_{max} = an^2$

$an = 2$ and $an^2 = 6$

$\frac{an^2}{an} = \frac{6}{2}$

$n = 3$

$a = \frac{2}{3}$

So, the amplitude is: $a = \frac{2}{3}$ and the frequency is: $n = 3$

Period of motion is:

$T = \frac{2\pi}{n} = \frac{2\pi}{3}$ s

15a $a = 5$ cm, $m = 0.5$ kg, cycles $= 60/$s, $n = 2\pi \times 60 = 120\pi$

$v_{max} = an$

$= 5 \times 120\pi$

$= 1884.95 \ldots$ cm/s

$\div 18.8$ m/s

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15b $\ddot{x}_{max} = an^2$

$a = 5 \text{ cm} = 0.05 \text{ m}$

$F_{max} = m\ddot{x}_{max}$

$= 0.5 \times an^2$

$= 0.5 \times 0.05 \times (120\pi)^2$

$= 3553.057 \ldots$

$= 3553 \text{ N}$

16a At $x = 0.01$ m, $|v| = 0.09$ m/s and $x = -0.02$ m, $|v| = 0.06$ m/s

$v^2 = -n^2((x-c)^2 - a^2)$

Here, $c = 0$.

$(0.09)^2 = -n^2((0.01)^2 - a^2)$ and $(0.06)^2 = -n^2((-0.02)^2 - a^2)$

$\left(\frac{0.09}{0.06}\right)^2 = \frac{(0.01)^2 - a^2}{(-0.02)^2 - a^2}$

$\frac{9}{4} = \frac{0.0001 - a^2}{0.0004 - a^2}$

$0.0004 - 4a^2 = 0.0036 - 9a^2$

$5a^2 = 0.0032$

$a = \sqrt{\frac{32}{50\,000}}$

$= \frac{4\sqrt{2}}{100\sqrt{5}}$

$= \frac{\sqrt{2}}{25\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{\sqrt{10}}{125}$

Hence amplitude is $\frac{\sqrt{10}}{125}$ metres.

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16b

$$a = \frac{\sqrt{10}}{125},$$

$$(0.09)^2 = -n^2((0.01)^2 - a^2)$$

$$n^2 = \frac{(0.09)^2}{\frac{10}{15\,625} - 0.0001}$$

$$= \frac{0.0081}{0.00054}$$

$$= \frac{810}{54}$$

$$= 15$$

$$n = \sqrt{15}$$

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{15}}$$

Hence period is $\frac{2\pi}{\sqrt{15}}$ seconds.

16c Let $x = a\cos nt$

$$x = \frac{\sqrt{10}}{125}\cos\frac{2\pi t}{\sqrt{15}}$$

When $x = 0.01$,

$$0.01 = \frac{\sqrt{10}}{125}\cos\frac{2\pi t_1}{\sqrt{15}}$$

$$t_1 = \frac{\sqrt{15}}{2\pi}\cos^{-1}\frac{1.25}{\sqrt{10}}$$

$$= 0.717\,75\ldots \text{ s}$$

When $x = -0.02$,

$$-0.02 = \frac{\sqrt{10}}{125}\cos\frac{2\pi t_2}{\sqrt{15}}$$

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$$t_2 = \frac{\sqrt{15}}{2\pi}\cos^{-1}\left(\frac{-2.5}{\sqrt{10}}\right)$$

$$= 1.530\ 24 \ldots \text{ s}$$

$$t_2 - t_1 = 1.530\ 24 \ldots - 0.717\ 75 \ldots$$

$$\div 0.81 \text{ s}$$

So, it takes about 0.81 seconds for the particle to go from A to B.

17 The equation of motion for the aircraft is:

$$M\ddot{x} = T - kv^2$$

$$M\ v\frac{dv}{dx} = T - kv^2$$

$$\frac{dv}{dx} = \frac{T - kv^2}{Mv}$$

$$\frac{dx}{dv} = \frac{Mv}{T - kv^2}$$

$$\int dx = \frac{M}{2k}\int \frac{2kv}{T - kv^2}\ dv$$

$$x = -\frac{M}{2k}\ln(T - kv^2) + C \qquad \text{(since } T - kv^2 \text{ is positive as aircraft is moving)}$$

At $x = 0,\ \ v = 0$, hence

$$C = \frac{M}{2k}\ln T$$

Therefore

$$x = \frac{M}{2k}\ln\left(\frac{T}{T - kv^2}\right)$$

So, when the speed of the aircraft is V, the distance travelled is:

$$x = \frac{M}{2k}\ln\left(\frac{T}{T - kV^2}\right)$$

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18a $m = 2$ kg, $v_0 = 20$ m/s

The equation of motion of the object when it is travelling upwards is:

$$m\ddot{y} = mv\frac{dv}{dy} = -mg - \frac{v^2}{10}$$

$$2v\frac{dv}{dy} = -20 - \frac{v^2}{10}$$

$$2v\frac{dv}{dy} = -\frac{200 + v^2}{10}$$

$$\frac{dv}{dy} = -\frac{200 + v^2}{20v}$$

$$\frac{dy}{dv} = -\frac{20v}{200 + v^2}$$

$$\int dy = -10\int \frac{2vdv}{200 + v^2}$$

$y = -10\ln(200 + v^2) + C$ (since $200 + v^2$ is positive)

At $y = 0, \quad v = 20$, hence

$C = 10\ln 600$

Therefore

$y = -10\ln(200 + v^2) + 10\ln 600$

$$y = 10\ln\frac{600}{200 + v^2}$$

At maximum height, $v = 0$.

$$h = 10\ln\frac{600}{200 + 0}$$

$h = 10\ln 3$

Hence maximum height is $10\ln 3$ metres.

18b For half its maximum height, $y = 5\ln 3$.

$$y = 10\ln\frac{600}{200 + v^2}$$

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$$5\ln 3 = 10\ln\frac{600}{200+v^2}$$

$$10\ln 3^{\frac{1}{2}} = 10\ln\frac{600}{200+v^2}$$

$$\sqrt{3} = \frac{600}{200+v^2}$$

$$200\sqrt{3} + \sqrt{3}v^2 = 600$$

$$\sqrt{3}v^2 = 600 - 200\sqrt{3}$$

$$3v^2 = 600\sqrt{3} - 600$$

$$v^2 = 200\sqrt{3} - 200$$

$$|v| = 12.100\ldots$$

Hence speed of object is about 12.1 m/s.

19a $m = 0.03$ kg, $H = 6$ m, $f_r = 0.6v$ N, $g = 10$ m/s^2

Taking downwards as positive, equation of motion for the balloon is:

$$m\ddot{y} = mg - f_r$$

$$0.03\ddot{y} = 0.3 - 0.6v$$

$$\ddot{y} = 10 - 20v$$

19b Terminal velocity is reached when the resultant forces/acceleration is zero.

$$\ddot{y} = 0 = 10 - 20v_t$$

$$20v_t = 10$$

$$v_t = 0.5 \text{ m/s}$$

Terminal speed is 0.5 m/s.

19c

$$\ddot{y} = \frac{dv}{dt} = 10 - 20v$$

$$\frac{dt}{dv} = \frac{1}{10-20v}$$

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$$\int dt = \int \frac{1}{10-20v}\, dv$$

$$= -\frac{1}{20}\int \frac{-20}{10-20v}\, dv$$

$t = -\frac{1}{20}\ln(10-20v) + C$ (since terminal velocity is 0.5 so $10-2v$ is positive)

At $t = 0, \quad v = 0,$ hence

$0 = -\frac{1}{20}\ln 10 + C$

$C = \frac{1}{20}\ln 10$

$t = \frac{1}{20}\ln\left(\frac{10}{10-20v}\right)$

$e^{20t} = \frac{10}{10-20v}$

$e^{-20t} = \frac{10-20v}{10}$

$e^{-20t} = 1-2v$

$v = \frac{1}{2}(1-e^{-20t})$

19d For half its terminal speed, $v = 0.25$.

$t = \frac{1}{20}\ln\left(\frac{10}{10-20v}\right)$

$= \frac{1}{20}\ln\left(\frac{10}{10-5}\right)$

$= \frac{1}{20}\ln 2$

$= 0.034\,657\ldots$

Time to reach half its terminal speed is about 0.35 seconds.

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20a Taking downwards as positive, the equation of motion of the particle is:

$m\ddot{y} = mg - mkv$

$\ddot{y} = \frac{dv}{dt} = g - kv$

At $t = 0, v = 0$ and $\frac{dv}{dt} = 0, v = V$

$g - kV = 0$

$k = \frac{g}{V}$

Therefore

$\frac{dv}{dt} = g - \frac{g}{V}v$

$\frac{dv}{dt} = \frac{g}{V}(V - v)$

20b

$\frac{dv}{dt} = g - kv$

$\frac{dt}{dv} = \frac{1}{g - kv}$

$\int dt = \int \frac{1}{g - kv}\, dv$

$\int dt = -\frac{1}{k}\int \frac{-k}{g - kv}\, dv$

$t = -\frac{1}{k}\ln(g - kv) + C$ (since particle is falling so $g - kv$ is positive)

At $t = 0,\ v = 0$, hence

$C = \frac{1}{k}\ln g$

Therefore

$t = -\frac{1}{k}\ln(g - kv) + \frac{1}{k}\ln g$

$t = \frac{1}{k}\ln\left(\frac{g}{g - kv}\right)$

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$$t = \frac{V}{g}\ln\left(\frac{g}{g - \frac{gv}{V}}\right) \qquad (\text{since } k = \frac{g}{V})$$

$$t = \frac{V}{g}\ln\left(\frac{1}{1 - \frac{v}{V}}\right)$$

$$t = \frac{V}{g}\ln\left(\frac{V}{V - v}\right)$$

At $v = \frac{1}{2}V$,

$$t = \frac{V}{g}\ln\left(\frac{V}{V - \frac{1}{2}V}\right)$$

$$t = \frac{V}{g}\ln 2$$

20c

$$\ddot{y} = v\frac{dv}{dy} = \frac{g}{V}(V - v)$$

$$\frac{dv}{dy} = \frac{g}{Vv}(V - v)$$

$$\frac{dy}{dv} = \frac{Vv}{g(V - v)}$$

$$\int dy = \frac{V}{g}\int \frac{v}{V - v}\, dv$$

$$= -\frac{V}{g}\int \frac{V - v - V}{V - v} dv$$

$$= -\frac{V}{g}\int \left(\frac{V - v}{V - v} + \frac{-V}{V - v}\right) dv$$

$$= -\frac{V}{g}\int \left(1 + \frac{-V}{V - v}\right) dv$$

$$y = -\frac{V}{g}(v + V\ln(V - v)) + C$$

At $t = 0, v = 0$ and $y = 0$, hence

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$$y = -\frac{V}{g}(V \ln V) + C$$

$$C = \frac{V^2}{g} \ln V$$

Therefore

$$y = -\frac{V}{g}(v + V \ln(V - v)) + \frac{V^2}{g} \ln V$$

$$= -\frac{V}{g}v - \frac{V^2}{g} \ln(V - v) + \frac{V^2}{g} \ln V$$

$$= \frac{V^2}{g} \ln\left(\frac{V}{V - v}\right) - \frac{V}{g}v$$

At $v = \frac{1}{2}V$,

$$y = \frac{V^2}{g} \ln\left(\frac{V}{V - \frac{1}{2}V}\right) - \frac{V}{g}\left(\frac{1}{2}V\right)$$

$$= \frac{V^2}{g} \ln 2 - \frac{V^2}{2g}$$

$$= \frac{V^2}{g}\left(\ln 2 - \frac{1}{2}\right)$$

21a Particle P_2 is dropped from point B.

Taking upwards as positive, the equation of motion is:

$$m\ddot{y} = mv\frac{dv}{dy} = -mg + mkv^2$$

$$v\frac{dv}{dy} = -g + kv^2$$

When terminal speed, V, is reached, the forces on the particle are balanced and the resultant acceleration is zero.

$$0 = -g + kV^2$$

$$V = \sqrt{\frac{g}{k}}$$

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21b The equation of motion for particle P_1 is:

$$\ddot{y} = v\frac{dv}{dy} = -(g + kv^2)$$

$$\frac{dv}{dy} = -\frac{g + kv^2}{v}$$

$$\frac{dy}{dv} = -\frac{v}{g + kv^2}$$

$$\int dy = -\int \frac{v}{g + kv^2}\, dv$$

$$= -\frac{1}{2k}\int \frac{2kv}{g + kv^2}\, dv$$

$$y = -\frac{1}{2k}\ln(g + kv^2) + C$$

At $t = 0$, $y = 0$ and $v = U$, hence

$$C = \frac{1}{2k}\ln(g + kU^2)$$

Therefore

$$y = -\frac{1}{2k}\ln(g + kv^2) + \frac{1}{2k}\ln(g + kU^2)$$

$$y = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv^2}\right)$$

When the particles collide, the velocity of P_1 is v_1.

$$y_1 = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv_1^2}\right)$$

21c H = height P_1 would reach if it did not collide with P_2

$= y_{1max}$

At y_{1max}, $v = 0$.

$$y_{1max} = H = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + k(0)}\right)$$

$$H = \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

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Since $V = \sqrt{\frac{g}{k}}$,

$V^2 = \frac{g}{k}$

Therefore

$H = \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$

21d

$y_2 = \frac{1}{2k}\ln\left|\frac{g}{g - kv_2^2}\right|$

At the point of collision:

$v_2 = \frac{1}{2}V = \frac{1}{2}\sqrt{\frac{g}{k}}$

$y_2 = \frac{1}{2k}\ln\left|\frac{g}{g - k\left(\frac{1}{4}\frac{g}{k}\right)}\right|$

$= \frac{1}{2k}\ln\left|\frac{g}{g - \frac{g}{4}}\right|$

$= \frac{1}{2k}\ln\left(\frac{1}{\frac{3}{4}}\right)$

$= \frac{1}{2k}\ln\frac{4}{3}$

21e $H = y_1 + y_2$

$\frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right) = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv_1^2}\right) + \frac{1}{2k}\ln\frac{4}{3}$

$\frac{1}{2k}\ln\frac{3}{4}\left(1 + \frac{U^2}{V^2}\right) = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv_1^2}\right)$

$\frac{3}{4}\left(1 + \frac{U^2}{V^2}\right) = \frac{g + kU^2}{g + kv_1^2}$

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$$\frac{3}{4}\left(1+\frac{U^2}{V^2}\right)=\frac{1+\frac{k}{g}U^2}{1+\frac{k}{g}v_1^2}$$

$$\frac{3}{4}\left(1+\frac{U^2}{V^2}\right)=\frac{1+\frac{U^2}{V^2}}{1+\frac{v_1^2}{V^2}} \qquad (\text{since } \frac{g}{k}=V^2)$$

$$1+\frac{v_1^2}{V^2}=\frac{4}{3}$$

$$\frac{v_1}{V}=\sqrt{\frac{4}{3}-1}$$

$$v_1=\frac{V}{\sqrt{3}}$$

22a $v = 48\underset{\sim}{\imath} + 36\underset{\sim}{\jmath}$

$v_{x0} = 48$ and $v_{y0} = 36$

The equations of motion for the particle are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

Integrating both with respect to time, we get:

$\dot{x} = C_1$ and $\dot{y} = C_2 - gt$

Given the initial velocity vector is: $v_{x0} = 48$ and $v_{y0} = 36$

$\dot{x} = 48$ and $\dot{y} = 36 - gt = 36 - 10t$

Integrating again with respect to time, we get:

$x = 48t + C_3$ and $y = 36t - 5t^2 + C_4$

At $t = 0,\ x = 0, y = 0$, hence

$C_3 = 0$ and $C_4 = 0$

Therefore

$x = 48t$ and $y = 36t - 5t^2$

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22b The maximum height of the projectile is reached when $\dot{y} = 0$.

$0 = 36 - 10t$

$t = \frac{36}{10} = 3.6$ s

Maximum height, H, found $w$hen $t = 3.6$ is substituted into $y = 36t - 5t^2$.

$H = 36(3.6) - 5(3.6)^2$

$= 64.8$ m

22c To find horizontal range, consider when $y = 0$.

$0 = 36t - 5t^2$

$0 = t(36 - 5t)$

$t = 0, t = \frac{36}{5} = 7.2$ s

Horizontal range, R, found when $t = 7.2$ is substituted into $x = 48t$

$R = 48 \times 7.2$

$= 345.6$ m

22d When $t = 1.6$ s,

$\dot{x} = 48$

$\dot{y} = 36 - 5t = 36 - 5(1.6) = 28$

Velocity as component after 1.6 s is $48\underset{\sim}{i} + 28\underset{\sim}{j}$.

23a $x = 12t$ (1)

$y = 9t - 5t^2$ (2)

From (1): $t = \frac{x}{12}$

Substituting for t in (2):

$y = 9\left(\frac{x}{12}\right) - 5\left(\frac{x}{12}\right)^2$

$y = \frac{3}{4}x - \frac{5}{144}x^2$

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23b For horizontal range, R, consider when $y = 0$.

$9t - 5t^2 = 0$

$t(9 - 5t) = 0$

$t = 0$ or $t = \frac{9}{5} = 1.8$ s

Horizontal range, R, found when $t = 1.8$ is substituted into $x = 12t$.

$R = 12(1.8)$

$= 21.6$ m

For maximum height, H,

$\frac{dy}{dt} = 0$

$9 - 10t = 0$

$t = \frac{9}{10} = 0.9$ s

Maximum height, H, found when $t = 0.9$ is substituted into $y = 9t - 5t^2$.

$H = 9(0.9) - 5(0.9)^2$

$= 4.05$ m

23c

$y = \frac{3}{4}x - \frac{5}{144}x^2$

$\frac{dy}{dx} = \frac{3}{4} - \frac{5}{72}x$

At $x = 0$,

$\frac{dy}{dx} = \frac{3}{4}$

Hence the gradient at $x = 0$ is $\frac{3}{4}$.

So, the angle of projection is $\tan^{-1}\frac{3}{4}$.

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23d $x = 12t$ and $y = 9t - 5t^2$

$\dot{x} = 12$ and $\dot{y} = 9 - 10t$

At $t = 0$,

$v_x = 12$ and $v_y = 9$

Initial velocity is $12\hat{\imath} + 9\hat{\jmath}$.

Magnitude of initial velocity is $\sqrt{12^2 + 9^2}$ or 15 m/s.

Angle of projection is $\tan^{-1}\frac{9}{12}$ or $\tan^{-1}\frac{3}{4}$.

23e At $y = 4$,

$4 = 9t - 5t^2$

$5t^2 - 9t + 4 = 0$

$(5t - 4)(t - 1) = 0$

$t = 0.8$ or $t = 1$

Particle is 4 m high when $t = 0.8$ s and $t = 1.0$ s.

So, the horizontal displacements are:

$x_{0.8} = 12(0.8) = 9.6$ m

$x_{1.0} = 12(1) = 12$ m

24a $\alpha = \tan^{-1} 2, V = 5$

so $\tan\alpha = 2$, and

$\cos\alpha = \frac{1}{\sqrt{2^2 + 1^2}} = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{2^2 + 1^2}} = \frac{2}{\sqrt{5}}$

Therefore

$\dot{x}_0 = V\cos\alpha$ and $\dot{y}_0 = V\sin\alpha$

$\dot{x}_0 = 5\left(\frac{1}{\sqrt{5}}\right)$ and $\dot{y}_0 = 5\left(\frac{2}{\sqrt{5}}\right)$

$\dot{x}_0 = \sqrt{5}$ and $\dot{y}_0 = 2\sqrt{5}$

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24b $\ddot{x} = 0$

Integrating with respect to time:

$\dot{x} = C_1$

At $t = 0$, $\dot{x}_0 = \sqrt{5}$, hence

$C_1 = \sqrt{5}$

Therefore

$\dot{x} = \sqrt{5}$

Integrating with respect to time:

$x = \sqrt{5}t + C_2$

At $t = 0$, $x = 0$, hence

$C_2 = 0$

Therefore

$x = \sqrt{5}t$ or $x = t\sqrt{5}$

$\ddot{y} = -10$

Integrating with respect to time:

$\dot{y} = -10t + C_3$

At $t = 0$, $\dot{y}_0 = 2\sqrt{5}$, hence $C_3 = 2\sqrt{5}$

Therefore

$\dot{y} = 2\sqrt{5} - 10t$

Integrating with respect to time:

$y = 2\sqrt{5}t - 5t^2 + C_4$

At $t = 0$, $y = 0$, hence $C_4 = 0$

Therefore

$y = 2\sqrt{5}t - 5t^2$

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24c For greatest height: $\dot{y} = 0$

$0 = 2\sqrt{5} - 10t$

$t = \dfrac{1}{\sqrt{5}}$ seconds

Greatest height, H, found when $t = \dfrac{1}{\sqrt{5}}$ is substituted into $y = 2\sqrt{5}t - 5t^2$.

$H = 2\sqrt{5}\left(\dfrac{1}{\sqrt{5}}\right) - 5\left(\dfrac{1}{\sqrt{5}}\right)^2$

$= 2 - 1$

$= 1$ metre

24d For time of flight for the apple, consider when $y = 0$.

$2\sqrt{5}t - 5t^2 = 0$

$t\left(2\sqrt{5} - 5t\right) = 0$

$t = 0$ or $t = \dfrac{2\sqrt{5}}{5}$

So apple was in the air for $\dfrac{2\sqrt{5}}{5}$ or $\dfrac{2}{5}\sqrt{5}$ seconds.

Horizontal distance, R, is found by substituting for t in $x = t\sqrt{5}$.

$R = \dfrac{2}{5}\sqrt{5} \times \sqrt{5}$

$= 2$ metres

24e $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5} - 10t$

Adam caught the apple when $t = \dfrac{2\sqrt{5}}{5}$ s, so

$\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5} - \dfrac{20\sqrt{5}}{5} = -2\sqrt{5}$

Final speed $= \sqrt{\left(\sqrt{5}\right)^2 + \left(-2\sqrt{5}\right)^2}$

$= \sqrt{25}$

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$= 5$ m/s

Hence the final speed is the same as the initial speed.

The angle at which the projectile was caught is:

$\tan^{-1}\left(-\frac{2\sqrt{5}}{\sqrt{5}}\right)$

$= \tan^{-1}(-2)$

$= \tan^{-1} 2$ below the horizontal

Hence, the final direction is the opposite of the initial direction.

24f $x = \sqrt{5}t$ (1)

$y = 2\sqrt{5}t - 5t^2$ (2)

From (1): $t = \frac{x}{\sqrt{5}}$

Substituting for t in (2):

$y = 2\sqrt{5}\left(\frac{x}{\sqrt{5}}\right) - 5\left(\frac{x}{\sqrt{5}}\right)^2$

$y = 2x - x^2$

25a $\dot{x}_0 = 200$ and $\dot{y}_0 = 0$

25b $\ddot{x} = 0$ and $\ddot{y} = -g$

Integrating with respect to time and, as there is assumed to be no air resistance, the constants will be the initial velocities. (Assume upwards is positive)

$\dot{x} = 200$ and $\dot{y} = -gt = -10t$

Integrating with respect to time and with origin at the window, the constants will be zero.

$x = 200t$ and $y = -5t^2$

For Cartesian equation, consider:

$x = 200t$ (1)

$y = -5t^2$ (2)

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From (1): $t = \frac{x}{200}$

Substituting for t in (2):

$y = -5\left(\frac{x}{200}\right)^2$

$y = -\frac{x^2}{8000}$

or $y = -\frac{1}{8000}x^2$

25c For the horizontal range, $y = -45$.

$-45 = -5t^2$

$t = 3$ s

Horizontal range, R, is found by substituting $t = 3$ into $x = 200t$.

$R = 200(3)$

$= 600$ metres

Horizontal distance that the bullet travels is 600 metres.

25d The bullet hits the ground when $t = 3$.

When $t = 3$,

$\dot{x} = 200$

$\dot{y} = -10(3) = -30$

The angle at which the bullet hits the ground is:

$\tan^{-1}\left(-\frac{30}{200}\right)$

$= \tan^{-1}\left(-\frac{3}{20}\right)$

$= 8.530\,76\ldots°$

$= 8°32'$ below the horizontal.

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26a Let $\alpha = 45°, \quad \theta = 30°, \quad V = 30$

$\ddot{x} = 0 \quad$ and $\quad \ddot{y} = -10$

Integrating with respect to time:

$\dot{x} = C_1 \quad$ and $\quad \dot{y} = -10t + C_2$

At $t = 0, \quad v_x = V\cos\alpha$ and $v_y = V\sin\alpha$, hence

$C_1 = 30\cos 45° \quad$ and $\quad C_2 = 30\sin 45°$

$C_1 = 15\sqrt{2} \quad$ and $\quad C_2 = 15\sqrt{2}$

Therefore

$\dot{x} = 15\sqrt{2}$ and $\dot{y} = -10t + 15\sqrt{2}$

Integrating with respect to time:

$x = 15\sqrt{2}t + C_3 \quad$ and $\quad y = -5t^2 + 15\sqrt{2}t + C_4$

At $t = 0, \ x = 0$ and $y = 0$, hence

$C_3 = 0$ and $C_4 = 0$

Therefore

$x = 15\sqrt{2}t$ and $y = 15\sqrt{2}t - 5t^2$

26b $x = 15\sqrt{2}t \qquad (1)$

$y = 15\sqrt{2}t - 5t^2 \qquad (2)$

From (1): $t = \dfrac{x}{15\sqrt{2}}$

Substituting for t in (2):

$$y = 15\sqrt{2}\left(\frac{x}{15\sqrt{2}}\right) - 5\left(\frac{x}{15\sqrt{2}}\right)^2$$

$$y = x - \frac{1}{90}x^2$$

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26c The equation of line of inclined hill is:

$$y = \tan 30° \, x$$

$$= \frac{1}{\sqrt{3}} x$$

P is the point of intersection of inclined hill and the parabolic trajectory:

$$\frac{1}{\sqrt{3}} x = x - \frac{x^2}{90}$$

$$\frac{x^2}{90} - x + \frac{1}{\sqrt{3}} x = 0$$

$$x\left(\frac{x}{90} - \left(1 - \frac{1}{\sqrt{3}}\right)\right) = 0$$

$$x_p = 0 \text{ or } x_p = \frac{90\left(\sqrt{3} - 1\right)}{\sqrt{3}} = 30\left(3 - \sqrt{3}\right)$$

$$y_p = \frac{1}{\sqrt{3}} x_p \text{ so}$$

$$y_p = 0 \text{ or } y_p = 30\left(\sqrt{3} - 1\right)$$

So, the distance OP is:

$$OP = \sqrt{\left(30\left(3 - \sqrt{3}\right) - 0\right)^2 + \left(30\left(\sqrt{3} - 1\right) - 0\right)^2}$$

$$= 30\sqrt{\left(3 - \sqrt{3}\right)^2 + \left(\sqrt{3} - 1\right)^2}$$

$$= 30\sqrt{9 - 6\sqrt{3} + 3 + 3 - 2\sqrt{3} + 1}$$

$$= 30\sqrt{16 - 8\sqrt{3}}$$

$$= 60\sqrt{4 - 2\sqrt{3}}$$

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27a $x = Vt\cos\alpha$ (1)

$y = Vt\sin\alpha - \frac{1}{2}gt^2$ (2)

From (1): $t = \frac{x}{V\cos\alpha}$

Substituting for t in (2):

$$y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{\frac{1}{2}gx^2}{V^2}\sec^2\alpha$$

$$= x\tan\alpha - \frac{1}{2}g\frac{x^2}{V^2}\tan^2\alpha - \frac{1}{2}g\frac{x^2}{V^2}$$

$$2V^2y = 2V^2x\tan\alpha - gx^2\tan^2\alpha - gx^2$$

$$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$$

27b $V = 200, \quad g = 10, \quad x = 3000, \quad y = 500$

$$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$$

$$9\times10^7\tan^2\alpha - 24\times10^7\tan\alpha + (4\times10^7 + 9\times10^7) = 0$$

$$9\tan^2\alpha - 24\tan\alpha + 13 = 0$$

$$\tan\alpha = \frac{24 \pm \sqrt{24^2 - 4(9)(13)}}{2(9)}$$

$$\tan\alpha = \frac{24 \pm \sqrt{108}}{18}$$

$$\tan\alpha = \frac{4 \pm \sqrt{3}}{3}$$

$\alpha = 62.373\,65\ldots°$ or $37.088\,66\ldots°$

$\alpha = 62°22'$ or $37°5'$

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28a $V = 14, \quad R = 10, \quad H_w = 8$

$\ddot{x} = 0$

Integrating with respect to time:

$\dot{x} = C_1$

At $t = 0$, $v_x = V\cos\theta$, hence $C_1 = V\cos\theta$.

Therefore

$\dot{x} = V\cos\theta$

Integrating with respect to time:

$x = Vt\cos\theta + C_2$

At $t = 0$, $x = 0$, hence $C_2 = 0$.

Therefore

$x = Vt\cos\theta$

$= 14t\cos\theta$

$\ddot{y} = -9.8$

Integrating with respect to time:

$\dot{y} = -9.8t + C_3$

At $t = 0$, $v_y = V\sin\theta$, hence $C_3 = V\sin\theta$.

Therefore

$\dot{y} = -9.8t + V\sin\theta$

or $\dot{y} = V\sin\theta - 9.8t$

Integrating with respect to time:

$y = Vt\sin\theta - 4.9t^2 + C_4$

At $t = 0$, $y = 0$, hence $C_4 = 0$.

Therefore

$y = Vt\sin\theta - 4.9t^2$

$= 14t\sin\theta - 4.9t^2$

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28b $x = 14t\cos\theta$ (1)

$y = 14t\sin\theta - 4.9t^2$ (2)

From (1): $t = \dfrac{x}{14\cos\theta}$

Substituting for t in (2):

$$y = 14\left(\frac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\frac{x}{14\cos\theta}\right)^2$$

$$y = x\tan\theta - \frac{4.9}{196}x^2\sec^2\theta$$

$$y = x\tan\theta - \frac{1}{40}x^2(\tan^2\theta + 1)$$

$$y = mx - \left(\frac{m^2+1}{40}\right)x^2 \qquad \text{where } m = \tan\theta$$

28c At $x = 10, \quad y = h$

$$h = 10m - \frac{100}{40}m^2 - \frac{100}{40}$$

$$h = 10m - 2.5m^2 - 2.5$$

$$2.5m^2 - 10m + (2.5 + h) = 0$$

$$m = \frac{10 \pm \sqrt{100 - 4(2.5)(2.5+h)}}{5}$$

$$m = \frac{10 \pm \sqrt{75 - 10h}}{5}$$

$$m = \frac{10 \pm 5\sqrt{3 - \frac{10}{25}h}}{5}$$

$$m = 2 \pm \sqrt{3 - 0.4h}$$

$$(m-2)^2 = 3 - 0.4h$$

$$h = \frac{3 - (m-2)^2}{0.4}$$

$$h = 7.5 - \frac{(m-2)^2}{0.4}$$

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For maximum h, $\dfrac{dh}{dm} = 0.$

$-5(m-2) = 0$

$m = 2$

When $m = 2, h = 7.5$ m

28d $y_l = 3.9, \quad y_h = 5.9, \quad x = 10$

$y = mx - \left(\dfrac{m^2+1}{40}\right)x^2$

$y = 10m - \dfrac{100}{40}m^2 - \dfrac{100}{40}$

$y = 10m - 2.5m^2 - 2.5$

When $y = y_l = 3.9,$

$3.9 = 10m_l - 2.5m_l^2 - 2.5$

$2.5m_l^2 - 10m_l + 6.4 = 0$

$m_l = \dfrac{10 \pm \sqrt{100 - 4(2.5)(6.4)}}{5}$

$m_l = \dfrac{10 \pm \sqrt{36}}{5}$

$m_l = \dfrac{4}{5}$ or $\dfrac{16}{5}$

$m_l = 0.8$ or 3.2

When $y = y_h = 5.9,$

$5.9 = 10m_h - 2.5m_h^2 - 2.5$

$2.5m_l^2 - 10m_l + 8.4 = 0$

$m_h = \dfrac{10 \pm \sqrt{100 - 4(2.5)(8.4)}}{5}$

$m_h = \dfrac{10 \pm \sqrt{16}}{5}$

$m_h = \dfrac{6}{5}$ or $\dfrac{14}{5}$

$m_h = 1.2$ or 2.8

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Hence, the paintball will pass through the hole for $0.8 \leq m \leq 1.2$ or $2.8 \leq m \leq 3.2$.

29a $\ddot{x} = -k\dot{x}$

$$\ddot{x} = \frac{dv}{dt} = -kv$$

$$\frac{dt}{dv} = -\frac{1}{kv}$$

$$\int dt = -\int \frac{1}{kv}\, dv$$

$$\int dt = -\frac{1}{k}\int \frac{k}{kv}\, dv$$

$$t = -\frac{1}{k}\ln kv + C$$

At $t = 0$, $v = u\cos\alpha$, hence

$$C = \frac{1}{k}\ln(ku\cos\alpha)$$

Therefore

$$t = -\frac{1}{k}\ln kv + \frac{1}{k}\ln(ku\cos\alpha)$$

$$t = \frac{1}{k}\ln\frac{u\cos\alpha}{v}$$

$$kt = \ln\frac{u\cos\alpha}{v}$$

$$e^{kt} = \frac{u\cos\alpha}{v}$$

$$e^{-kt} = \frac{v}{u\cos\alpha}$$

$$v = u\cos\alpha\, e^{-kt}$$

Therefore

$$\dot{x} = ue^{-kt}\cos\alpha$$

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29b

$$\dot{y} = \frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$$

Differentiating with respect to time:

$$\ddot{y} = -k\left(\frac{1}{k}(ku\sin\alpha + g)e^{-kt}\right) - 0$$

$$= -k\left(\frac{1}{k}(ku\sin\alpha + g)e^{-kt}\right)$$

$$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g + g\right)\right)$$

$$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right) + \frac{g}{k}\right)$$

$$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)\right) - g$$

$$= -k(\dot{y}) - g$$

Therefore

$$\ddot{y} = -k\dot{y} - g$$

29c When the particle reaches its maximum height, the vertical velocity is zero.

$$\dot{y} = 0$$

$$\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right) = 0$$

$$(ku\sin\alpha + g)e^{-kt} = g$$

$$e^{-kt} = \frac{g}{ku\sin\alpha + g}$$

$$-kt = \ln\left(\frac{g}{ku\sin\alpha + g}\right)$$

$$t = -\frac{1}{k}\ln\left(\frac{g}{ku\sin\alpha + g}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{ku\sin\alpha + g}{g}\right)$$

$$t = \frac{1}{k}\ln\left(\frac{ku\sin\alpha}{g} + \frac{g}{g}\right)$$

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$$t = \frac{1}{k}\ln\left(\frac{ku}{g}\sin\alpha + 1\right)$$

$$\text{or } t = \frac{1}{k}\ln\left(1 + \frac{ku}{g}\sin\alpha\right)$$

29d The limiting value of horizontal displacement is when the time is very large.

$$\dot{x} = ue^{-kt}\cos\alpha$$

$$\frac{dx}{dt} = ue^{-kt}\cos\alpha$$

$$\int dx = u\cos\alpha \int e^{-kt}dt$$

$$x = -\frac{u}{k}\cos\alpha\, e^{-kt} + C$$

When $t = 0, x = 0$, hence

$$0 = -\frac{u}{k}\cos\alpha + C$$

$$C = \frac{u}{k}\cos\alpha$$

Therefore

$$x = -\frac{u}{k}\cos\alpha\, e^{-kt} + \frac{u}{k}\cos\alpha$$

$$\lim_{t\to\infty} x = \lim_{t\to\infty}\left(-\frac{u}{k}\cos\alpha\, e^{-kt} + \frac{u}{k}\cos\alpha\right)$$

As $t \to \infty, e^{-kt} \to 0$, so

$$\lim_{t\to\infty} x = 0 + \frac{u}{k}\cos\alpha$$

$$\lim_{t\to\infty} x = \frac{u}{k}\cos\alpha$$

Hence, the limiting value of horizontal displacement is $\frac{u}{k}\cos\alpha$.

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