Solutions to Exercise 6G Foundation questions

1a The components of y of the displacement is given by

$$y = v_0 \sin\theta\, t - \frac{1}{2}gt^2$$

As $v_0 t = H$, $g = 10$ and the angle is equal to 90°:

$$y = H - 5t^2$$

1b When $t = 0, y = 2$.

$$2 = H - 5(0)^2$$

$$H = 2$$

When $t = \frac{1}{5}, y = 0$.

$$0 = H - 5\left(\frac{1}{5}\right)^2$$

$$H = \frac{1}{5}$$

$$H = 2 + 2 - \frac{1}{5}$$

$$= 3.8$$

2a $v_y = v_0 \sin\theta - gt = 0$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{16 \sin 90°}{10}$$

$$= 1\frac{3}{5} \text{ s}$$

Maximum height:

$$y\left(1\frac{3}{5}\right) = 16 \sin 90° \left(1\frac{3}{5}\right) - \frac{1}{2}(10)\left(1\frac{3}{5}\right)^2$$

$$= 12\frac{4}{5} \text{ metres}$$

Uncorrected proofs

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2b The equation for $y_1 = -5t^2 + vt$

As y_2 is fired one second later, $t_2 = t - 1$.

Thus, $y_2 = -5(t-1)^2 + v(t-1)$

2c At $t = 1\frac{3}{5}, y_2 = 12\frac{4}{5}$.

$12.8 = -5(0.6)^2 + v(0.6)$

$v = 24\frac{1}{3}$ m/s

3a $B - Mg = -Md$

$B - (M-m)g = (M-m)a$

3b $Ma - ma = B - Mg + mg$

$-ma - mg = B - Mg - Ma$

$-ma - mg = -Md + Mg - Mg - Ma \quad (B = -Md + Mg)$

$ma + mg = Md + Ma$

$m(a+g) = M(a+d)$

$m = M\dfrac{a+d}{a+g}$

4a $v^2 = u^2 + 2as$

Thus,

$v^2 = V_0^2 + \dfrac{2gR^2}{r} - 2gR$

4b $V_0 = \sqrt{gR}$

$\doteqdot 1680$ m/s

Uncorrected proofs

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5a $T = mg\cos\theta$

$$\frac{d^2\theta}{dt^2} = -\frac{g\sin\theta}{L}$$

5b

$a = \frac{dv}{dt} = \frac{dx}{dt} \times \frac{dv}{dx}$ by the chain rule

$= v\frac{dv}{dx}$ since $d = \frac{dx}{dt}$

Now, $v\frac{dv}{dx} = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$

By the chain rule, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Thus substituting ϖ as v, $\frac{d^2\theta}{dt^2} = \frac{d}{d\theta}(\frac{1}{2}\ \varpi^2)$

5c

$$\frac{d}{d\theta}\left(\frac{1}{2}\varpi^2\right) = a$$

$$\frac{1}{2}\varpi^2 = \frac{T - T_0}{mL}$$

$$\varpi^2 = \frac{2}{mL}(T - T_0)$$

6 Max $= 0.4$ and Min $= -1$. Therefore, $A = 0.7$. Shift down 0.3

Cycle repeats every 12 hours and 25 mins.

Period $= \frac{2\pi}{\frac{149}{12}}$

$$x = -0.7\cos\left(\frac{2\pi}{\frac{149}{12}}t\right) - 0.3$$

Using this equation to graph the harmonic motion,

The first time that $y = 0$ is when $t = 3.9794$ and returns to that position when

$t = 8.4372$ hours past 1:10 a.m.

Thus, the times are 5:09 a.m. to 9:36 a.m.

Uncorrected proofs

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Solutions to Exercise 6G Development questions

7a

$$\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\alpha}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\alpha}{2}} \qquad \text{using } \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}$$

$$=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$$

$$=\frac{1+\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

$$=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

$$=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\times\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

$$=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}$$

$$=\frac{\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos\alpha} \qquad \text{using } \cos 2x=\cos^2 x-\sin^2 x$$

$$=\frac{1+\sin\alpha}{\cos\alpha} \qquad \text{using } \cos^2 x+\sin^2 x=1 \text{ and } \sin 2x=2\sin x\cos x$$

$$=\frac{\sin\alpha+1}{\cos\alpha}$$

7b

$$y=-\frac{gx^2}{2V^2}\sec^2\beta+x\tan\beta$$

Using $\sec^2 x=1+\tan^2 x$:

Uncorrected proofs

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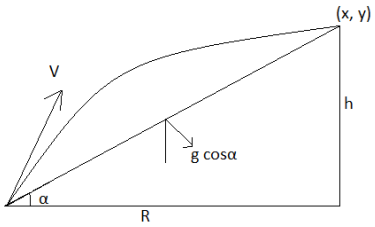
$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\beta) + x\tan\beta$$

$$y = -\frac{gx^2}{2V^2} - \frac{gx^2}{2V^2}\tan^2\beta + x\tan\beta$$

$$\frac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

7c i This case is the same as the case of projectile motion up an inclined plane with an angle of elevation α.

The equations of motion with respect to the inclined plane are:



$$\frac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$gx^2\tan^2\beta - 2V^2x\tan\beta + (2V^2y + gx^2) = 0$$

As stated in the question, since there is only one solution for $\tan\beta$,

$$\text{sum of the roots} = 2\tan\beta = -\frac{b}{a} = \frac{2V^2x}{gx^2}$$

$$\tan\beta = \frac{V^2}{gx}$$

Substituting in the equation:

$$gx^2\left(\frac{V^2}{gx}\right)^2 - 2V^2x\left(\frac{V^2}{gx}\right) + (2V^2y + gx^2) = 0$$

$$\frac{V^4}{g} - \frac{2V^4}{g} + 2V^2y + gx^2 = 0$$

$$-\frac{V^4}{g} + 2V^2y + gx^2 = 0$$

$$V^4 - 2gyV^2 - g^2x^2 = 0 \qquad \text{(multiplying by } -g\text{)}$$

Uncorrected proofs

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7c ii $y = h$ and $x = h\cot\alpha$

From part c i, we get,

$V^4 - 2gyV^2 - g^2x^2 = 0$

$V^2 = \dfrac{2gy \pm \sqrt{(2gy)^2 - 4(1)(-g^2x^2)}}{2}$ (using quadratic formula)

$V^2 = \dfrac{2gy \pm \sqrt{4g^2h^2 + 4g^2x^2}}{2}$

$V^2 = gh \pm g\sqrt{h^2 + x^2}$

$V^2 = gh \pm g\sqrt{h^2 + (h\cot\alpha)^2}$

$V^2 = gh \pm g\sqrt{h^2(1 + \operatorname{cosec}^2\alpha)}$

$V^2 = gh \pm gh\sqrt{(1 + \operatorname{cosec}^2\alpha)}$

$V^2 = gh(1 \pm \operatorname{cosec}\alpha)$

$V^2 = gh(1 + \operatorname{cosec}\alpha)$ (as we consider net above the ground)

7c iii

$\dfrac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \dfrac{gx^2}{2V^2}\right) = 0$

From part a, c i and c ii, we have:

$V^2 = gh(1 + \operatorname{cosec}\alpha)$; $\tan\beta = \dfrac{V^2}{gx}$; $x = h\cot\alpha$; $\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right) = \dfrac{1 + \sin\alpha}{\cos\alpha}$

$\tan\beta = \dfrac{V^2}{gx}$

$= \dfrac{V^2}{gh\cot\alpha}$

$= \dfrac{gh(1 + \operatorname{cosec}\alpha)}{gh\cot\alpha}$

$= \dfrac{\sin\alpha}{\cos\alpha}\left(1 + \dfrac{1}{\sin\alpha}\right)$

$= \dfrac{\sin\alpha + 1}{\cos\alpha}$

Uncorrected proofs

$$= \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$$

Hence,

$$\beta = \frac{\pi}{4} + \frac{\alpha}{2}$$

8a

$$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

Let the solutions for this quadratic in $\tan\theta$ be t_1 and t_2 and the coefficients of the quadratic be a, b and c.

$$t_1 + t_2 = -\frac{b}{a}$$

$$= \frac{x}{\frac{gx^2}{2V^2}}$$

$$= \frac{2V^2}{gx}$$

$$t_1.t_2 = \frac{c}{a}$$

$$= \frac{\left(y + \frac{gx^2}{2V^2}\right)}{\frac{gx^2}{2V^2}}$$

$$= \frac{2V^2 y}{gx^2} + 1$$

Now, since the equation is quadratic in $\tan\theta$, we get:

$t_1 = \tan\theta_1$ and $t_2 = \tan\theta_2$

$\tan(\theta_1 + \theta_2)$

$$= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}$$

$$= \frac{t_1 + t_2}{1 - t_1.t_2}$$

Uncorrected proofs

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$$= \frac{\frac{2V^2}{gx}}{1 - \left(\frac{2V^2 y}{gx^2} + 1\right)}$$

$$= \frac{2V^2}{gx} \times -\frac{gx^2}{2V^2 y}$$

$$= -\frac{x}{y}$$

8b $\theta_1 = \theta_2$ for some point $P\ (a, b)$

At the point P, the height is b and the range is a.

So, the angle of elevation for point P from the point of firing is:

$\tan \alpha = \frac{b}{a}$

From question 7c i and 7c ii, we have:

$V^2 = gy(1 + \operatorname{cosec} \alpha) = gb(1 + \operatorname{cosec} \alpha)$

From part a, using $\tan \theta_1 + \tan \theta_2 = \frac{2V^2}{gx}$:

$2 \tan \theta = \frac{2V^2}{gx}$

So $\tan \theta = \frac{V^2}{gx} = \frac{V^2}{ga}$

$\tan \theta = \frac{\sin \alpha + 1}{\cos \alpha} = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ (from question 7a)

Hence,

$\theta = \frac{\pi}{4} + \frac{\alpha}{2}$, where $\tan \alpha = \frac{b}{a}$ and the equation has only one solution.

8c

$$\frac{gx^2}{2V^2} \tan^2 \theta - x \tan \theta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

The discriminant of the above equation is zero (for the equation of parabola) as only one solution for θ.

Uncorrected proofs

$$x^2 - 4\left(\frac{gx^2}{2V^2}\right)\left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$x^2 - \frac{2gx^2y}{V^2} - \frac{g^2x^4}{V^4} = 0$$

$$x^2\left(1 - \frac{2gy}{V^2} - \frac{g^2x^2}{V^4}\right) = 0$$

$$\frac{g^2x^2}{V^4} = -\frac{2gy}{V^2} + 1 \qquad (\text{as } x \neq 0)$$

$$x^2 = \frac{V^4}{g^2}\left(-\frac{2gy}{V^2} + 1\right)$$

$$x^2 = -\frac{2V^2y}{g} + \frac{V^4}{g^2}$$

$$x^2 = -\frac{2V^2}{g}\left(y - \frac{V^2}{2g}\right)$$

or $y = \frac{V^2}{2g} - \frac{gx^2}{2V^2}$

For the vertex of the parabola, $\frac{dy}{dx} = 0$.

$$-\frac{gx}{V^2} = 0$$

$$x = 0$$

so $y = \frac{V^2}{2g}$

Therefore, coordinates of vertex are $\left(0, \frac{V^2}{2g}\right)$.

The focal length of the parabola is:

$$\frac{\frac{2V^2}{g}}{4} = \frac{V^2}{2g}$$

Uncorrected proofs

9 The equations for the positions of jth ball bearing at time t are:

$x = Vt\cos\alpha_j$ (1)

and $y = Vt\sin\alpha_j - \frac{1}{2}gt^2$ (2)

$x^2 = \left(Vt\cos\alpha_j\right)^2$ (3)

and $y^2 = \left(Vt\sin\alpha_j\right)^2 - Vgt^3\sin\alpha_j + \frac{g^2t^4}{4}$ (4)

From (2): $\sin\alpha_j = \dfrac{y + \frac{1}{2}gt^2}{Vt}$

Substituting into (4):

$$y^2 = \left(Vt\sin\alpha_j\right)^2 - gt^2\left(y + \frac{gt^2}{2}\right) + \frac{g^2t^4}{4}$$

$$= \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{2} + \frac{g^2t^4}{4}$$

$$= \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{4}$$

Therefore

$$x^2 + y^2 = \left(Vt\cos\alpha_j\right)^2 + \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{4}$$

$$x^2 + y^2 = (Vt)^2\left(\sin^2\alpha_j + \cos^2\alpha_j\right) - gt^2y - \frac{g^2t^4}{4}$$

$$x^2 + y^2 + gt^2y + \frac{g^2t^4}{4} = (Vt)^2\left(\sin^2\alpha_j + \cos^2\alpha_j\right)$$

$$x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = (Vt)^2$$

Hence, the particles are on a circle with radius Vt and centre $\left(0, -\frac{1}{2}gt^2\right)$ at any given time t.

Uncorrected proofs

10a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g \qquad 0 \le x \le l$$

$$\frac{1}{2}v^2 = \int_0^l g\,dx$$

$$\frac{v^2}{2} = [gx]_0^l$$

$$v^2 = 2g(l-0)$$

$$v^2 = 2gl$$

10b i

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x-l) \quad x > l$$

$$\frac{1}{2}v^2 = \int \big(g - gk(x-l)\big)dx$$

$$v^2 = 2g\left(\int dx - k\int (x-l)\,dx\right)$$

$$v^2 = 2gx - gk(x-l)^2 + C$$

At $x = l$, $v^2 = 2gl$.

$$2gl = 2gl - 0 + C$$

$$C = 0$$

Therefore

$v^2 = 2gx - gk(x-l)^2$ for $x > l$

10b ii $v^2 = 2gx - gk(x-l)^2$

When the jumper is at the maximum distance, $v^2 = 0$.

$$2gx_{max} - gk(x_{max} - l)^2 = 0$$

$$2x_{max} - k(x_{max}^2 - 2lx_{max} + l^2) = 0$$

$$2x_{max} - kx_{max}^2 + 2klx_{max} - kl^2 = 0$$

$$kx_{max}^2 - 2(1+kl)x_{max} + kl^2 = 0$$

Uncorrected proofs

$$x_{max} = \frac{2(1+kl) \pm \sqrt{\left(2(1+kl)\right)^2 - 4k(kl^2)}}{2k} \quad \text{using the quadratic formula}$$

$$= \frac{1}{k}\left((1+kl) \pm \sqrt{4k^2l^2 + 1 + 2kl - 4k^2l^2}\right)$$

$$= \frac{1}{k}\left(1 + kl + \sqrt{1+2kl}\right) \quad \text{since } k > 0, l > 0$$

10b iii

$$x_{max} = \frac{1}{k}\left(1 + kl + \sqrt{1+2kl}\right)$$

$$kl = 4$$

$$k = \frac{4}{l}$$

Therefore

$$x_{max} = \frac{l}{4}\left(1 + 4 + \sqrt{1+2(4)}\right)$$

$$= 2l$$

11a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{10R^2}{(R+x)^2}$$

The negative sign indicates that the acceleration due to gravity acts in the downwards direction. So, when the object is moving upwards, it will feel resistance and the speed will reduce to zero (deceleration). When an object is falling down, the speed will increase (acceleration).

11b

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{10R^2}{(R+x)^2}$$

$$\frac{v^2}{2} = -10R^2 \int \frac{dx}{(R+x)^2}$$

Uncorrected proofs

$$= 10R^2\left(\frac{1}{R+x}\right) + C$$

At $x = 0, v = V_0$.

$$C = \frac{V_0^2}{2} - 10R$$

Therefore

$$\frac{v^2}{2} = 10R^2\left(\frac{1}{R+x}\right) + \frac{V_0^2}{2} - 10R$$

For maximum height, $v = 0$ and $x = h$.

$$0 = 10R^2\left(\frac{1}{R+h}\right) + \frac{V_0^2}{2} - 10R$$

$$\frac{1}{R+h} = \frac{10R - \frac{V_0^2}{2}}{10R^2}$$

$$\frac{1}{R+h} = \frac{20R - V_0^2}{20R^2}$$

$$R + h = \frac{20R^2}{20R - V_0^2}$$

$$h = \frac{20R^2}{20R - V_0^2} - R$$

$$h = \frac{20R^2 - R(20R - V_0^2)}{20R - V_0^2}$$

$$h = \frac{RV_0^2}{20R - V_0^2}$$

11c

$$h = \frac{RV_0^2}{20R - V_0^2}$$

Given $R = 6.4 \times 10^6$ m, $V_0 = 500$ m/s:

$$h = \frac{6.4 \times 10^6 \times 500^2}{20 \times 6.4 \times 10^6 - 500^2}$$

$$= 12\,524.461\ldots$$

$$\div 12\,524 \text{ m}$$

Uncorrected proofs

12a

$$\frac{dB}{dt} = -k(B - W) \quad \text{and} \quad \frac{dW}{dt} = \frac{3}{4}k(B - W)$$

Differentiating $\frac{3}{4}B + W$:

$$\frac{d}{dt}\left(\frac{3}{4}B + W\right)$$

$$= \frac{3}{4}\frac{dB}{dt} + \frac{dW}{dt}$$

$$= \frac{3}{4}\left(-k(B - W)\right) + \frac{3}{4}k(B - W)$$

$$= \frac{3}{4}k(-B + W + B - W)$$

$$= 0$$

As $\frac{d}{dt}\left(\frac{3}{4}B + W\right) = 0$,

$\frac{3}{4}B + W$ is a constant.

Therefore

$\frac{3}{4}B + W = C$, where C *is* a constant

12b Substituting $B = 120°\text{C}$, $W = 22°\text{C}$:

$$C = \frac{3}{4}B + W = 112°\text{C}$$

$$\frac{3}{4}B + W = 112$$

$$W = 112 - \frac{3}{4}B$$

Now,

$$\frac{dB}{dt} = -k(B - W)$$

$$= -k\left(B - \left(112 - \frac{3}{4}B\right)\right)$$

Uncorrected proofs

$$= -k\left(B - 112 + \frac{3}{4}B\right)$$

$$= -k\left(\frac{7}{4}B - 112\right)$$

12c

$$\frac{dB}{dt} = -k\left(\frac{7}{4}B - 112\right)$$

$$\frac{dt}{dB} = \frac{1}{-k\left(\frac{7}{4}B - 112\right)}$$

$$dt = \frac{dB}{-k\left(\frac{7}{4}B - 112\right)}$$

$$\int dt = -\frac{1}{k}\int \frac{dB}{\frac{7}{4}B - 112}$$

$$\int dt = -\frac{4}{7k}\int \frac{\frac{7}{4}}{\frac{7}{4}B - 112}dB$$

$$t = -\frac{4}{7k}\ln\left|\frac{7}{4}B - 112\right| + C$$

At $t = 0,\ B = 120$

$$C = \frac{4}{7k}\ln 98$$

Therefore

$$t = -\frac{4}{7k}\ln\left|\frac{7}{4}B - 112\right| + \frac{4}{7k}\ln 98$$

$$-\frac{7kt}{4} = \ln\left|\frac{7}{4}B - 112\right| - \ln 98$$

$$-\frac{7kt}{4} = \ln\frac{\frac{7}{4}B - 112}{98}$$

$$e^{-\frac{7kt}{4}} = \frac{\frac{7}{4}B - 112}{98}$$

$$\frac{7}{4}B - 112 = 98e^{-\frac{7kt}{4}}$$

Uncorrected proofs

$$\frac{7}{4}B = 112 + 98e^{-\frac{7kt}{4}}$$

$$B = \frac{4}{7}\left(112 + 98e^{-\frac{7kt}{4}}\right)$$

$$B = 4\left(16 + 14e^{-\frac{7kt}{4}}\right)$$

12d When $t = 10$, $B = 92°C$.

$$B = 4\left(16 + 14e^{-\frac{7kt}{4}}\right)$$

$$92 = 4\left(16 + 14e^{-\frac{7}{4}k(10)}\right)$$

$$23 = 16 + 14e^{-\frac{35k}{2}}$$

$$14e^{-\frac{35k}{2}} = 7$$

$$e^{-\frac{35k}{2}} = \frac{1}{2}$$

$$e^{\frac{35k}{2}} = 2$$

$$\frac{35k}{2} = \ln 2$$

$$k = \frac{2}{35}\ln 2$$

When $t = 20$,

$$B = 4\left(16 + 14e^{-\frac{7}{4}\times\frac{2}{35}\ln 2\times 20}\right)$$

$$= 4\left(16 + 14e^{-2\ln 2}\right)$$

$$= 78°C$$

12e The eventual temperature of water bath occurs when the heat energy exchange results in equilibrium condition. $B = W = T$

$$\frac{dB}{dt} = 0$$

$$-k\left(\frac{7}{4}T - 112\right) = 0$$

Uncorrected proofs

$$\frac{7}{4}T - 112 = 0$$

$$T = 112 \times \frac{4}{7}$$

$$T = 64°\text{C}$$

13a

$$P = \frac{21\,000}{7 + 3e^{-\frac{t}{3}}}$$

Let $u = 7 + 3e^{-\frac{t}{3}}$

and $\frac{du}{dt} = -e^{-\frac{t}{3}}$

Therefore

$$P = 21\,000\left(\frac{1}{u}\right) = 21\,000u^{-1}$$

$$\frac{dP}{dt} = \frac{dP}{du} \times \frac{du}{dt}$$

$$\frac{dP}{dt} = -21\,000u^{-2} \times \left(-e^{-\frac{t}{3}}\right)$$

$$= -\frac{21\,000}{\left(7 + 3e^{-\frac{t}{3}}\right)^2} \times \left(-e^{-\frac{t}{3}}\right)$$

$$= \frac{21\,000}{7 + 3e^{-\frac{t}{3}}} \times \left(-\frac{1}{7 + 3e^{-\frac{t}{3}}}\right) \times \left(-e^{-\frac{t}{3}}\right)$$

$$= P\left(\frac{e^{-\frac{t}{3}}}{7 + 3e^{-\frac{t}{3}}}\right)$$

$$= \frac{P}{3}\left(\frac{\left(7 + 3e^{-\frac{t}{3}}\right) - 7}{7 + 3e^{-\frac{t}{3}}}\right)$$

$$= \frac{P}{3}\left(1 - \frac{7}{7 + 3e^{-\frac{t}{3}}}\right)$$

Uncorrected proofs

$$= \frac{P}{3}\left(1 - \frac{21\,000}{3000\left(7 + 3e^{-\frac{t}{3}}\right)}\right)$$

$$= \frac{P}{3}\left(1 - \frac{P}{3000}\right)$$

13b At the end of 2008, $t = 0$.

When $t = 0$,

$$P = \frac{21000}{7 + 3}$$

$$= 2100$$

13c Eventually, population growth stops when:

$$\frac{dP}{dt} = 0$$

$$\frac{P}{3}\left(1 - \frac{P}{3000}\right) = 0$$

$$P = 3000$$

13d Rate of growth at the end of 2008:

$$r = \frac{dP}{dt}$$

$$= \frac{P}{3}\left(1 - \frac{P}{3000}\right)$$

$$= \frac{2100}{3}\left(1 - \frac{2100}{3000}\right)$$

$$= 700\left(1 - \frac{7}{10}\right)$$

$$= 210$$

Annual % rate of growth

$$= \frac{r}{P_{2008}} \times 100$$

Uncorrected proofs

$$= \frac{210}{2100} \times 100\%$$

$$= 10\%$$

14a $y = bx - ax^2$

$$y = -a\left(x^2 - \frac{b}{a}x\right)$$

$$y = -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a} - \frac{b^2}{4a}\right)$$

$$y - \frac{b^2}{4} = -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a}\right)$$

$$y - \frac{b^2}{4} = -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a^2}\right)$$

$$y - \frac{b^2}{4} = -a\left(x - \frac{b}{2a}\right)^2$$

$$-\frac{1}{a}\left(y - \frac{b^2}{4}\right) = \left(x - \frac{b}{2a}\right)^2$$

or $\left(x - \frac{b}{2a}\right)^2 = -\frac{1}{a}\left(y - \frac{b^2}{4}\right)$

Comparing with the standard equation for a parabola: $(x-h)^2 = 4A(y-k)$,

$$4A = -\frac{1}{a}$$

$$A = -\frac{1}{4a}$$

The focal length is positive so the focal length for this parabola is $\frac{1}{4a}$.

Uncorrected proofs

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14b

$$y = x\tan\alpha - x^2\frac{g\sec^2\alpha}{2V^2}$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - x\tan\alpha \times \frac{2V^2}{g\sec^2\alpha}\right)$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - x \times \frac{\sin\alpha}{\cos\alpha} \times \frac{2V^2\cos^2\alpha}{g}\right)$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - \frac{2V^2\sin\alpha\cos\alpha}{g}x\right)$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - \frac{V^2\sin 2\alpha}{g}x + \left(\frac{V^2\sin 2\alpha}{2g}\right)^2 - \left(\frac{V^2\sin 2\alpha}{2g}\right)^2\right)$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - \frac{V^2\sin 2\alpha}{g}x + \left(\frac{V^2\sin 2\alpha}{2g}\right)^2\right) - \frac{g\sec^2\alpha}{2V^2} \times -\left(\frac{V^2\sin 2\alpha}{2g}\right)^2$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x^2 - \frac{V^2\sin 2\alpha}{g}x + \left(\frac{V^2\sin 2\alpha}{2g}\right)^2\right) + \frac{g\sec^2\alpha}{2V^2} \times \left(\frac{V^2\sin\alpha\cos\alpha}{g}\right)^2$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x - \frac{V^2\sin 2\alpha}{2g}\right)^2 + \frac{V^2\sin^2\alpha\cos^2\alpha}{2g\cos^2\alpha}$$

$$y = -\frac{g\sec^2\alpha}{2V^2}\left(x - \frac{V^2\sin 2\alpha}{2g}\right)^2 + \frac{V^2\sin^2\alpha}{2g}$$

$$y - \frac{V^2\sin^2\alpha}{2g} = -\frac{g\sec^2\alpha}{2V^2}\left(x - \frac{V^2\sin 2\alpha}{2g}\right)^2$$

$$-\frac{2V^2}{g\sec^2\alpha}\left(y - \frac{V^2\sin^2\alpha}{2g}\right) = \left(x - \frac{V^2\sin 2\alpha}{2g}\right)^2$$

$$\text{or } \left(x - \frac{V^2\sin 2\alpha}{2g}\right)^2 = -\frac{2V^2}{g\sec^2\alpha}\left(y - \frac{V^2\sin^2\alpha}{2g}\right)$$

Standard equation for a parabola is $(x-h)^2 = 4A(y-k)$

where the coordinates of the vertex are (h, k),

the equation of its directrix is $y = k - A$ and

the coordinates of the focus are $(h, k + A)$.

Uncorrected proofs

14b i The coordinates of the vertex are (h, k) or

$$\left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 \sin^2 \alpha}{2g}\right)$$

14b ii

$$4A = -\frac{2V^2}{g \sec^2 \alpha}$$

$$A = -\frac{V^2}{2g \sec^2 \alpha}$$

$$= -\frac{V^2 \cos^2 \alpha}{2g}$$

The focal length of the trajectory is $\frac{V^2 \cos^2 \alpha}{2g}$.

14b iii The equation of the directrix is $y = k - A$ or

$$y = \frac{V^2 \sin^2 \alpha}{2g} - \left(-\frac{V^2 \cos^2 \alpha}{2g}\right)$$

$$y = \frac{V^2}{2g}(\sin^2 \alpha + \cos^2 \alpha)$$

$$y = \frac{V^2}{2g}$$

The coordinates of focus are $(h, k + A)$ or

$$\left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 \sin^2 \alpha}{2g} - \frac{V^2 \cos^2 \alpha}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 (\sin^2 \alpha - \cos^2 \alpha)}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, -\frac{V^2 (\cos^2 \alpha - \sin^2 \alpha)}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, -\frac{V^2 \cos 2\alpha}{2g}\right)$$

Uncorrected proofs

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15a

$$\frac{dy}{dt} = -k\sqrt{y}$$

$$dt = -\frac{1}{k}\frac{dy}{\sqrt{y}}$$

Integrating both sides

$$\int dt = -\frac{1}{k}\int \frac{dy}{\sqrt{y}}$$

$$t = -\frac{2}{k}\sqrt{y} + C$$

At time $t = 0, y = y_0$.

$$0 = -\frac{2}{k}\sqrt{y_0} + C$$

$$C = \frac{2}{k}\sqrt{y_0}$$

Therefore

$$t = -\frac{2}{k}\sqrt{y} + \frac{2}{k}\sqrt{y_0}$$

At $t = T, y = 0$

$$T = \frac{2}{k}\sqrt{y_0}$$

$$k = \frac{2}{T}\sqrt{y_0}$$

Hence

$$t = -\frac{2}{k}\sqrt{y} + \frac{2}{k}\sqrt{y_0}$$

$$t = -\frac{2}{k}\left(\sqrt{y} - \sqrt{y_0}\right)$$

$$t = -\frac{2T}{2\sqrt{y_0}}\left(\sqrt{y} - \sqrt{y_0}\right) \qquad \text{(substituting } k = \frac{2}{T}\sqrt{y_0}\text{)}$$

$$t = -\frac{T\sqrt{y}}{\sqrt{y_0}} + \frac{T\sqrt{y_0}}{\sqrt{y_0}}$$

Uncorrected proofs

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$$\frac{t}{T} = -\frac{\sqrt{y}}{\sqrt{y_0}} + 1$$

$$\frac{\sqrt{y}}{\sqrt{y_0}} = 1 - \frac{t}{T}$$

15b At $t = 15,\ y = \frac{y_0}{2}$

$$\sqrt{\frac{\frac{y_0}{2}}{y_0}} = 1 - \frac{15}{T}$$

$$\sqrt{\frac{y_0}{2y_0}} = 1 - \frac{15}{T}$$

$$\sqrt{\frac{1}{2}} = 1 - \frac{15}{T}$$

$$\frac{15}{T} = 1 - \frac{1}{\sqrt{2}}$$

$$\frac{15}{T} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$\frac{T}{15} = \frac{\sqrt{2}}{\sqrt{2} - 1}$$

$$T = \frac{15\sqrt{2}}{\sqrt{2} - 1}$$

$$= \frac{15\sqrt{2}}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= 15\left(2 + \sqrt{2}\right)$$

$$\div 51.2 \text{ seconds}$$

Uncorrected proofs

16a

$$y = \frac{3}{2}x^{\frac{2}{3}}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E$$

At $t = 0$, bead is released from rest from point $\left(1, \frac{3}{2}\right)$.

$$\frac{dx}{dt} = \dot{x} = 0 \text{ and } \frac{dy}{dt} = \dot{y} = 0 \text{ at } t = 0$$

$$\frac{1}{2}m \times 0 + \frac{1}{2}m \times 0 + mg \times \frac{3}{2} = E$$

$$E = \frac{3}{2}mg$$

16b

$$y = \frac{3}{2}x^{\frac{2}{3}}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$$\dot{y} = x^{-\frac{1}{3}} \times \dot{x}$$

$$\dot{x} = x^{\frac{1}{3}}\dot{y}$$

$$\dot{x}^2 = x^{\frac{2}{3}}\dot{y}^2$$

$$= \frac{2}{3}y\dot{y}^2$$

Using the equation for E:

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E$$

$$\frac{1}{2}m\left(\frac{2}{3}y\dot{y}^2\right) + \frac{1}{2}m\dot{y}^2 + mgy = \frac{3}{2}mg$$

$$\frac{2}{3}y\dot{y}^2 + \dot{y}^2 + 2gy = 3g$$

$$\dot{y}^2\left(\frac{2}{3}y + 1\right) + 2gy = 3g$$

Uncorrected proofs

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$$\dot{y}^2\left(\frac{2}{3}y+1\right)=g(3-2y)$$

$$\dot{y}^2(2y+3)=3g(3-2y)$$

$$\dot{y}^2=\frac{3g(3-2y)}{3+2y}$$

16c At the origin, $x=0$ and $y=0$.

$$\dot{x}=x^{\frac{1}{3}}\dot{y}$$

$$=0\times\dot{y}$$

$$=0$$

$$\dot{y}=\pm\sqrt{\frac{3g(3-2y)}{3+2y}}$$

$$=\pm\sqrt{\frac{3g(3-0)}{3+0}}$$

$$=\pm\sqrt{3g}$$

Since y is decreasing to that point on the curve,

$$\dot{y}=-\sqrt{3g}$$

16d

$$\int_0^{\alpha}\sqrt{\frac{1+u}{1-u}}\,du=\sin^{-1}\alpha+1-\sqrt{1-\alpha^2}\qquad\text{for } 0\leq\alpha\leq 1$$

As our motion is also restricted in the same interval,

$$\dot{y}=\frac{dy}{dt}=-\sqrt{\frac{3g(3-2y)}{3+2y}}$$

$$\frac{dt}{dy}=-\frac{1}{\sqrt{3g}}\sqrt{\frac{3+2y}{3-2y}}$$

Uncorrected proofs

$$dt = -\frac{1}{\sqrt{3g}}\sqrt{\frac{3+2y}{3-2y}}\,dy$$

$$\int_0^t dt = -\frac{1}{\sqrt{3g}}\int_{\frac{3}{2}}^0 \sqrt{\frac{1+\frac{2y}{3}}{1-\frac{2y}{3}}}\,dy$$

$$t = \frac{1}{\sqrt{3g}}\int_0^{\frac{3}{2}} \sqrt{\frac{1+\frac{2y}{3}}{1-\frac{2y}{3}}}\,dy$$

Let $u = \frac{2y}{3}$

$$\frac{du}{dy} = \frac{2}{3}$$

$$dy = \frac{3}{2}du$$

When $y = 0, u = 0$

When $y = \frac{3}{2}, u = 1$

Therefore

$$t = \frac{1}{\sqrt{3g}}\int_0^1 \frac{3}{2}\sqrt{\frac{1+u}{1-u}}\,du$$

$$= \frac{3}{2\sqrt{3g}}\int_0^1 \sqrt{\frac{1+u}{1-u}}\,du$$

$$= \frac{3}{2\sqrt{3g}}\left(\sin^{-1}1 + 1 - \sqrt{1-1^2}\right) \quad \text{(using the given relationship with } \alpha = 1\text{)}$$

$$= \frac{3}{2\sqrt{3g}}\left(\frac{\pi}{2} + 1 - \sqrt{0}\right)$$

$$= \frac{3}{2\sqrt{3g}}\left(\frac{\pi}{2} + 1\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{g}}\left(\frac{\pi}{2} + 1\right)$$

Uncorrected proofs

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16e

$$\int_0^{\alpha} \sqrt{\frac{1+u}{1-u}}\, du$$

$$= \int_0^{\alpha} \frac{\sqrt{1+u}}{\sqrt{1-u}} \times \frac{\sqrt{1+u}}{\sqrt{1+u}} du$$

$$= \int_0^{\alpha} \frac{1+u}{\sqrt{1-u^2}}\, du$$

$$= \int_0^{\alpha} \frac{1}{\sqrt{1-u^2}}\, du - \frac{1}{2}\int_0^{\alpha} \frac{-2u}{\sqrt{1-u^2}}\, du$$

$$= \left[\sin^{-1} u - \sqrt{1-u^2}\right]_0^{\alpha}$$

$$= \left(\sin^{-1}\alpha - \sqrt{1-\alpha^2}\right) - \left(\sin^{-1} 0 - \sqrt{1-0^2}\right)$$

$$= \sin^{-1}\alpha - \sqrt{1-\alpha^2} + 1$$

$$= \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$$

At $\alpha = 1$,

$$\sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2}$$

$$= \frac{\pi}{2} + 1 - \sqrt{1-1}$$

$$= \frac{\pi}{2} + 1 - \sqrt{0}$$

$$= \frac{\pi}{2} + 1$$

Uncorrected proofs

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