Solutions to Exercise 6F Foundation questions

1a $x = 20t$

$y = -5t^2 + 20\sqrt{3}t$

$\dot{x} = 20$

$\dot{y} = -10t + 20\sqrt{3}$

1b

Using $v_o \sin\theta\, t - \frac{1}{2}gt^2 = 0$

$t\left(v_o \sin\theta - \frac{1}{2}gt\right) = 0$

$t = 0$ or $t = \frac{2v_o \sin\theta}{g} = \frac{2 \times 40 \sin 60°}{10} = 4\sqrt{3}$ seconds

$R = xt$

$= v_0 \cos\theta\, t$

$= \frac{2v_0 \cos\theta\, v_o \sin\theta}{g}$

$= \frac{v_0^2 \sin 2\theta}{g}$

$= \frac{40^2 \sin 2(60°)}{10}$

$= 80\sqrt{3}$ metres

1c $v_y = v_0 \sin\theta - gt = 0$

$t = \frac{v_0 \sin\theta}{g}$

$= \frac{40 \sin 60°}{10}$

$= 2\sqrt{3}$

Maximum height:

$y(2\sqrt{3}) = 40 \sin 60° (2\sqrt{3}) - \frac{1}{2}(10)(2\sqrt{3})^2$

Uncorrected proofs

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$= 60$ metres

1d

$$R = xt$$
$$= v_0 \cos\theta \, t$$
$$= \frac{2v_0 \cos\theta \, v_o \sin\theta}{g}$$
$$= \frac{v_0^2 \sin 2\theta}{g}$$
$$= \frac{40^2 \sin 2(30°)}{10}$$
$$= 80\sqrt{3} \text{ metres}$$

It is false.

2a

$$x = 39\cos\left(\tan^{-1}\frac{12}{5}\right)t$$
$$= 15t$$
$$y = -5t^2 + 39\sin\left(\tan^{-1}\frac{12}{5}\right)t$$
$$= -5t^2 + 36t$$
$$\dot{x} = 15$$
$$\dot{y} = -10t + 36$$

2b

$$t = \frac{x}{v_0 \cos\theta}$$
$$= \frac{30}{39\cos\left(\tan^{-1}\frac{12}{5}\right)}$$
$$= 2$$
$$y = -5(2)^2 + 36(2)$$

Uncorrected proofs

$= 52$

2c $\dot{x} = 15$

$\dot{y} = -10t + 36$

At $t = 2$,

$v = \sqrt{15^2 + 16^2}$

$\div 21.9 \text{ m/s}$

2d $v_y = v_0 \sin\theta - gt = 0$

$t = \dfrac{v_0 \sin\theta}{g}$

$= \dfrac{39 \sin\left(\tan^{-1}\dfrac{12}{5}\right)}{10}$

$= 3.6$

Therefore, after.

3a $x = 20\cos(30°)\, t$

$= 10\sqrt{3}t$

$y = -5t^2 + 20\sin(30°)\, t$

$= -5t^2 + 10t$

3b $-75 = -5t^2 + 10t$

$-5t^2 + 10t + 75 = 0$

$t^2 - 2t - 15 = 0$

$(t + 3)(t - 5) = 0$

$t = -3, 5$

When $t = 5$,

$x = 20\cos(30°)\,(5)$

$= 50\sqrt{3}$

Uncorrected proofs

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3c

$$v_y = v_0 \sin\theta - gt = 0$$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{20 \sin 30°}{10}$$

$$= 1$$

Maximum height:

$$y(1) = 20 \sin 30° (1) - \frac{1}{2}(10)(1)^2$$

$$= 5 \text{ metres}$$

Thus, $75 + 5 = 80$ metres.

3d $\dot{x} = 10\sqrt{3}$

$$\dot{y} = -10t + 10$$

At $t = 5$,

$$v = \sqrt{\left(10\sqrt{3}\right)^2 + (-40)^2}$$

$$\doteqdot 44 \text{ m/s}$$

The direction is given by:

$$\tan\theta = \frac{40}{10\sqrt{3}}$$

$$\theta \doteqdot 67°$$

Uncorrected proofs

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3e $x = 10\sqrt{3}t$

$t = \dfrac{x}{10\sqrt{3}}$

$y = -5t^2 + 10t$

$= -5\left(\dfrac{x}{10\sqrt{3}}\right)^2 + 10\left(\dfrac{x}{10\sqrt{3}}\right)$

$= -\dfrac{1}{60}x^2 + \dfrac{1}{\sqrt{3}}x$

4a

$\dfrac{363.6 \times 1000}{1 \times 60 \times 60} = 101 \text{ m/s}$

4b $x = 101t$

$y = -5t^2$

4c $-600 = -5t^2$

$t^2 = 120$

$t = 2\sqrt{30}$

4d $\dot{x} = 101$

$\dot{y} = -10t$

At $t = 2\sqrt{30}$,

$v = \sqrt{101^2 + \left(-20\sqrt{30}\right)^2}$

$= 149 \text{ m/s}$

$\tan\theta = \dfrac{20\sqrt{30}}{101}$

$\theta \div 47° 19'$

Uncorrected proofs

4e At $t = 2\sqrt{30}$,

$$x = \frac{101 \times 2\sqrt{30}}{1000}$$

$$\div 1.106 \text{ km}$$

5a $x = 50\cos(45°)\, t$

$$= 25\sqrt{2}t$$

$$y = -5t^2 + 50\sin(45)\, t$$

$$= -5t^2 + 25\sqrt{2}t$$

When $x = 75$,

$$t = \frac{75}{25\sqrt{2}}$$

$$= \frac{3}{2}\sqrt{2}$$

At $t = \frac{3}{2}\sqrt{2}$,

$$y = -5\left(\frac{3}{2}\sqrt{2}\right)^2 + 25\sqrt{2}\left(\frac{3}{2}\sqrt{2}\right)$$

$$= 52.5$$

5b $\dot{x} = 25\sqrt{2}$

$$\dot{y} = -10t + 25\sqrt{2}$$

At $t = \frac{3}{2}\sqrt{2}$,

$$v = \sqrt{\left(25\sqrt{2}\right)^2 + \left(-10\left(\frac{3}{2}\sqrt{2}\right) + 25\sqrt{2}\right)^2}$$

$$= 5\sqrt{58} \text{ m/s}$$

$$\tan\theta = \frac{10\sqrt{2}}{25\sqrt{2}} = \frac{2}{5}$$

$$\theta = \tan^{-1}\frac{2}{5}$$

Uncorrected proofs

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6a The components of x and y of the displacement are given by

$x = v_0 \cos\theta\, t$

$y = v_0 \sin\theta\, t - \frac{1}{2}gt^2$

As $v_0 = V$, $g = 10$ and the angle is equal to θ:

$x = V \cos\theta\, t$

$y = V \sin\theta\, t - 5t^2$

6b When $t = 2, x = 24\sqrt{5}$.

$24\sqrt{5} = V \cos\theta \times 2$

$V \cos\theta = 12\sqrt{5}$

When $= 2, y = 28$.

$28 = V \sin\theta\,(2) - 5(2)^2$

$2V \sin\theta = 48$

$V \sin\theta = 24$

6c $V^2 \sin^2\theta = 576$

$V^2 \cos^2\theta = 720$

$V^2 \cos^2\theta + V^2 \sin^2\theta = 1296$

$V^2(\cos^2\theta + \sin^2\theta) = 1296$

$V = |\sqrt{1296}| = 36$

$36 \sin\theta = 24$

$\sin\theta = \frac{24}{36}$

$\theta = 41° 49'$

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6d $v_y = v_0 \sin\theta - gt = 0$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{36 \sin 41° 49'}{10}$$

$$= 2.4$$

Total flight time $= 2.4(2) = 4.8$

$x(4.8) = 36 \cos 41° 49' (4.8)$

$= 129$ metres

Uncorrected proofs

Solutions to Exercise 6F Development questions

7a $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

$$t = \frac{x}{V\cos\alpha}$$

Substituting for t in equation for y:

$$y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$$

$$y = \frac{x\sin\alpha}{\cos\alpha} - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$y = \frac{x}{\cos^2\alpha}\left(\sin\alpha\cos\alpha - \frac{gx}{2V^2}\right)$$

7b For horizontal range, $y = 0$.

$$0 = \frac{x}{\cos^2\alpha}\left(\sin\alpha\cos\alpha - \frac{gx}{2V^2}\right)$$

$$V^2(2\sin\alpha\cos\alpha) = gx$$

$$x = R = \frac{V^2\sin 2\alpha}{g}$$

7c i $V = 30; \quad g = 10; \quad R = 45$

$$45 = \frac{30^2\sin 2\alpha}{10}$$

$$\sin 2\alpha = \frac{1}{2}$$

$$\alpha = \frac{1}{2}\sin^{-1}\frac{1}{2}$$

$$\alpha = 15° \text{ or } 75°$$

Uncorrected proofs

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7c ii For $\alpha = 15°$,

$$y_1 = \frac{x}{\cos^2 15°}\left(\sin 15° \cos 15° - \frac{10x}{1800}\right)$$

For $\alpha = 75°$,

$$y_2 = \frac{x}{\cos^2 75°}\left(\sin 75° \cos 75° - \frac{10x}{1800}\right)$$

Evaluating y_1 and y_2 at $x = 40$:

$y_1 \div 1.1908$ m and $y_2 \div 16.5868$ m

So, the ball will reach 45 m if thrown at 75° but not for 15°.

8a $V = 20, \quad \theta = \tan^{-1}\frac{4}{3}$

$$\sin\theta = \frac{4}{5} \text{ and } \cos\theta = \frac{3}{5}$$

$$x = V\cos\theta\, t$$

$$= 20 \times \frac{3}{5} \times t$$

$$= 12t$$

$$y = V\sin\theta\ t - \frac{1}{2}gt^2$$

$$= 20 \times \frac{4}{5} \times t - 5t^2$$

$$y = 16t - 5t^2$$

8b

$$R = \frac{V^2 \sin 2\theta}{g}$$

$$= \frac{400 \times 2 \times \frac{4}{5} \times \frac{3}{5}}{10}$$

$$= 38.4 \text{ m}$$

The ball reaches maximum height when the vertical velocity component is zero.

$$\frac{dv_y}{dt} = -g$$

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$v_y = -gt + C$

At $t = 0$, $v_y = V \sin\theta = 16$

$C = 16$

Therefore

$\frac{dy}{dt} = 16 - gt$

At maximum height,

$\frac{dy}{dt} = 0$

so $t = \frac{16}{g} = 1.6$ s

$y = V \sin\theta \ t - \frac{1}{2}gt^2$

$y = 16 \times 1.6 - \frac{1}{2} \times 10 \times 1.6^2$

$y = H = 12.8$ m

8c i

$\alpha = \tan^{-1}\frac{1}{5}$

$\sin\alpha = \frac{1}{\sqrt{26}}$ and $\cos\alpha = \frac{5}{\sqrt{26}}$

Trajectory of the incline is:

$y = x\tan\alpha = \frac{x}{5}$

The horizontal position when it reaches maximum height (at $t = 1.6$ s) is:

$x = V\cos\theta\, t$

$= 12 \times 1.6$

$= 19.2$ m

The height of the incline platform at 19.2 m is:

$H_p = \frac{19.2}{5} = 3.84$ m

Uncorrected proofs

So, the difference in height between the ball and the incline is:

$H - H_p$

$= 12.8 - 3.84$

$= 8.96$ m

Hence the ball is about 9 m above the road when it reaches its greatest height.

8c ii $T = 2.72$ s

The ball hits the plane when their trajectories collide:

$$y = \frac{x}{5} = \frac{25x}{9}\left(\frac{12}{25} - \frac{x}{80}\right)$$

$$\frac{9}{125} = \frac{12}{25} - \frac{x}{80}$$

$$\frac{x}{80} = \frac{51}{125}$$

$$x = 51 \times \frac{80}{125}$$

$= 32.64$ m

But we have to calculate the distance travelled by the ball up the inclination, hence that distance will be:

$$\frac{32.64}{\cos(\tan^{-1} 0.2)} = 33.28 \text{ m}$$

9a Ball thrown at angle θ from 1 m above the ground with a velocity of $u = 20$ m/s.

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

$$\frac{dx}{dt} = C_1 \quad \text{and} \quad \frac{dy}{dt} = -gt + C_2$$

At $t = 0$, $v_x = V\cos\theta$ and $v_y = V\sin\theta$.

So $C_1 = V\cos\theta$ and $C_2 = V\sin\theta$.

Therefore

$$\frac{dx}{dt} = V\cos\theta \quad \text{and} \quad \frac{dy}{dt} = V\sin\theta - gt$$

Uncorrected proofs

$x = Vt\cos\theta + C_3$ and $y = Vt\sin\theta - \frac{gt^2}{2} + C_4$

At $t = 0$, $x = 0$ and $y = 1$.

So $C_3 = 0$ and $C_4 = 1$.

Therefore

$x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{gt^2}{2} + 1$

Since $V = 20$ m/s and $g = 10$ m/s^2,

$x = 20t\cos\theta$ and $y = 20t\sin\theta - 5t^2 + 1$

$x = 20t\cos\theta$ and $y = -5t^2 + 20t\sin\theta + 1$

9b $x = 16,\ y = 16,\ t = T$

$16 = 20T\cos\theta$ and $16 - 1 = 20T\sin\theta - 5T^2$

$4 = 5T\cos\theta$ and $3 = 4T\sin\theta - T^2$

9c $4 = 5T\cos\theta$ (1)

$3 = 4T\sin\theta - T^2$ (2)

Using (1):

$T = \frac{4}{5\cos\theta}$

Substituting for T in (2):

$3 = 4 \times \left(\frac{4}{5\cos\theta}\right)\sin\theta - \left(\frac{16}{25\cos^2\theta}\right)$

$3 = \frac{16}{5}\tan\theta - \frac{16}{25}\sec^2\theta$

$75 = 80\tan\theta - 16(\tan^2\theta + 1)$

$16\tan^2\theta - 80\tan\theta + 91 = 0$

Uncorrected proofs

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9d Let $\tan\theta = X$.

$16X^2 - 80X + 91 = 0$

$(4X - 7)(4X - 13) = 0$

$X = 1.75$ or $X = 3.25$

Alternatively,

$$X = \frac{80 \pm \sqrt{80^2 - 4 \times 16 \times 91}}{32}$$

$X = 3.25$ or $X = 1.75$

Hence

$\tan\theta = 1.75$ or 3.25

$\theta = 60.255\ldots^\circ$ or $72.897\ldots^\circ$

So, to the nearest minute, the two possible values of θ are 60°15′and 72°54′.

10a $h = 2.4$, $V = 144$ km/h $= 40$ m/s, $\alpha = 7^\circ$

The equations of motion for the ball are:

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

$$\frac{dx}{dt} = C_1 \quad \text{and} \quad \frac{dy}{dt} = -gt + C_2$$

At $t = 0$, $\frac{dx}{dt} = V\cos 7^\circ$ and $\frac{dy}{dt} = -V\sin 7^\circ$.

So $C_1 = V\cos 7^\circ$ and $C_2 = -V\sin 7^\circ$.

Therefore

$$\frac{dx}{dt} = V\cos 7^\circ \quad \text{and} \quad \frac{dy}{dt} = -gt - V\sin 7^\circ$$

$$x = Vt\cos 7^\circ + C_3 \quad \text{and } y = -\frac{gt^2}{2} - Vt\sin 7^\circ + C_4$$

At $t = 0$, $x = 0$, $y = 2.4$.

So $C_3 = 0$ and $C_4 = 2.4$.

Therefore

Uncorrected proofs

$x = Vt\cos 7°$ and $y = -\dfrac{gt^2}{2} - Vt\sin 7° + 2.4$

Since $V = 40$ m/s and $g = 10$ m/s^2,

$x = 40t\cos 7°$ and $y = 2.4 - 40t\sin 7° - 5t^2$

10b When the ball hits the pitch, $y = 0$.

$5t^2 + 40t\sin 7° - 2.4 = 0$

Using the quadratic formula:

$$t = \frac{-40\sin 7° \pm \sqrt{(40\sin 7°)^2 - 4 \times 5 \times (-2.4)}}{10}$$

$t = \dfrac{1}{10}(3.596\ldots)$ (ignoring the negative value)

$\fallingdotseq 0.36$ s

10c $x = 40t\cos 7°$ (1)

$y = 2.4 - 40t\sin 7° - 5t^2$ (2)

Using (1):

$$t = \frac{x}{40\cos 7°}$$

Substituting for t in (2):

$$y = 2.4 - x\tan 7° - \frac{x^2}{320\cos^2 7°}$$

The ball hits the pitch after 0.36 s and at $y = 0$.

$x = Vt\cos 7°$

$= 40 \times 0.36 \times \cos 7°$

$\fallingdotseq 14.29$ m

The angle at which the ball hits the pitch is the slope of the tangent to the trajectory at the point of contact.

$$\frac{dy}{dx}\Big|_{x=14.29} = -\tan 7° - \frac{2x}{320\cos^2 7°}$$

$$= -\left(\tan 7° + \frac{2 \times 14.29}{320\cos^2 7°}\right)$$

Uncorrected proofs

$\frac{dy}{dx}|_{x=14.29} = -0.213 \ldots = \tan\delta$

$\delta \div -12.05°$

The negative sign indicates that it is an angle of declination. The angle at which the ball hits the pitch is 12°.

10d The ball was pitched at 14.29m from the bowler. The length of the wicket is 19 m.

So, the length for the batsman is: $(19 - 14.29) = 4.71 \text{ m} < 5$.

Hence, it is not a short-pitched delivery.

11a As the particles are projected at the same time, each particle will have the same flight time the moment they collide. Also, when they collide, their x- and y-coordinates will be the same.

As time $= t$ for both the particles when they collide and $y_1 = y_2$,

$y_1 = -\frac{1}{2}gt^2 + tV_1 \sin\theta_1$

$y_2 = -\frac{1}{2}gt^2 + tV_2 \sin\theta_2$

$-\frac{1}{2}gt^2 + tV_1 \sin\theta_1 = -\frac{1}{2}gt^2 + tV_2 \sin\theta_2$

$V_1 \sin\theta_1 = V_2 \sin\theta_2$

11b i $AB = 200; \quad V_1 = 30; \quad \theta_1 = \sin^{-1}\frac{4}{5}; \quad \theta_2 = \sin^{-1}\frac{3}{5}; \quad g = 10$

Using the condition obtained in part a for collision of the particles:

$V_1 \sin\theta_1 = V_2 \sin\theta_2$

$30 \times \frac{4}{5} = V_2 \times \frac{3}{5}$

$V_2 = 40 \text{ m/s}$

The x-coordinate at which the particles collide is the same for both the projectiles:

$V_1 t \cos\theta_1 = 200 - V_2 t \cos\theta_2$

$18t = 200 - 32t$

$50t = 200$

Uncorrected proofs

$t = 4$

11b ii The height of both the particles is same at time of collision:

$y_1 = y_2 = V_1 t_c \sin\theta_1 - 5t_c^2$

$y_1 = y_2 = 30 \times 4 \times \frac{4}{5} - 5 \times 16$

$y_1 = y_2 = 16 \text{ m}$

11b iii The coordinates of collision of the particles are:

$x = V_1 \cos\theta_1 \, t_c = 72 \text{ m}$ and $y = 16 \text{ m}$

The trajectories of the particles are:

$y_1 = x_1 \tan\theta_1 - \frac{5x_1^2}{324}$ and $y_2 = (200 - x_2)\tan\theta_2 - \frac{5(200 - x_2)^2}{1024}$

Differentiating y_1 and y_2 with respect to x:

$D_1 = \tan\theta_1 - \frac{5x_1}{162}$ and $D_2 = -\tan\theta_2 + \frac{5(200 - x_2)}{512}$

At the point of collision, $x_1 = (200 - x_2) = 72$

$D_1 = \frac{4}{3} - \frac{20}{9}$ and $D_2 = -\frac{3}{4} + \frac{5}{4}$

$D_1 = \tan\delta_1 = -\frac{8}{9}$

$D_2 = \tan\delta_2 = \frac{1}{2}$

$Y_1 - 16 = -\frac{8}{9}(X_1 - 72)$ and $Y_2 - 16 = -\frac{1}{2}(X_2 - 72)$

The angle between these two lines is given by:

$$\tan\delta = \frac{\tan\delta_1 - \tan\delta_2}{1 + \tan\delta_1 \tan\delta_2}$$

$$= \frac{-\frac{8}{9} - \left(\frac{1}{2}\right)}{1 + \left(\frac{8}{9} \times \frac{1}{2}\right)}$$

$\delta = 68°$

So, the obtuse angle between the particles is: $180° - 68° = 112°$

Uncorrected proofs

12a $V = 30; \quad \alpha = 60^\circ; \quad \ddot{x} = -\frac{1}{6}\dot{x}; \quad \ddot{y} = -10 - \frac{1}{6}\dot{y}$

$$\ddot{x} = \frac{dv_x}{dt} = -\frac{1}{6}v_x$$

$$\int dt = -6\int \frac{dv_x}{v_x}$$

$$t = -6\ln v_x + C$$

At $t = 0; \quad v_x = 30\cos 60° = 15$

$$C = 6\ln 15$$

Therefore

$$t = 6\ln\frac{15}{v_x}$$

$$v_x = 15e^{-\frac{t}{6}} = \dot{x}$$

$$\ddot{y} = \frac{dv_y}{dt} = -10 - \frac{1}{6}v_y$$

$$\int dt = -\int \frac{dv_y}{10 + \frac{v_y}{6}}$$

$$t = -6\ln\left(10 + \frac{v_y}{6}\right) + D$$

At $t = 0, \; v_y = 30\sin 60° = 15\sqrt{3}$

$$D = 6\ln\left(10 + \frac{15\sqrt{3}}{6}\right)$$

Therefore

$$t = 6\ln\left(\frac{10 + \frac{15\sqrt{3}}{6}}{10 + \frac{v_y}{6}}\right)$$

$$10 + \frac{v_y}{6} = 5\left(2 + \frac{\sqrt{3}}{2}\right)e^{-\frac{t}{6}}$$

$$v_y = 15\left(\left(4 + \sqrt{3}\right)e^{-\frac{t}{6}} - 4\right) = \dot{y}$$

Hence

Uncorrected proofs

$$\dot{y} = 15\left((4+\sqrt{3})e^{-\frac{1}{6}t} - 4\right)$$

12b

$$v_x = 15e^{-\frac{t}{6}} = \dot{x} = \frac{dx}{dt}$$

$$\int dx = 15\int e^{-\frac{t}{6}}\,dt$$

$$x = -90e^{-\frac{t}{6}} + C$$

At $t = 0,\ x = 0,$

$$C = 90$$

Therefore

$$x = -90e^{-\frac{t}{6}} + 90$$

$$x = 90\left(1 - e^{-\frac{1}{6}t}\right)$$

12c At greatest height, $\dot{y} = 0$.

$$15\left((4+\sqrt{3})e^{-\frac{t}{6}} - 4\right) = 0$$

$$e^{-\frac{t}{6}} = \frac{4}{4+\sqrt{3}}$$

$$t = 6\ln\left(\frac{4+\sqrt{3}}{4}\right)$$

12d Using equations from part b and part c:

$$x = 90\left(1 - e^{-\frac{t}{6}}\right) \quad \text{where } t = 6\ln\left(\frac{4+\sqrt{3}}{4}\right)$$

$$x = 90\left(1 - e^{-\frac{6}{6}\ln\left(\frac{4+\sqrt{3}}{4}\right)}\right)$$

Uncorrected proofs

$$x = 90\left(1 - \frac{4}{4 + \sqrt{3}}\right)$$

$$x = 27.195 \ldots$$

$$\fallingdotseq 27 \text{ m}$$

13a $V = 20; \quad g = 10; \quad R = 20$

The equations of the motion of the ball are:

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

$$\dot{x} = V\cos\alpha \quad \text{and} \quad \dot{y} = V\sin\alpha - gt$$

$$x = V\cos\alpha \; t \quad \text{and} \quad y = Vt\sin\alpha - \frac{1}{2}gt^2$$

$$t = \frac{x}{V\cos\alpha}$$

Trajectory of the ball:

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

The wall is at 20 m from the point of hitting the ball. So, when the ball hits the wall, $x = 20$ m.

$$y = 20\tan\alpha - \frac{10 \times 20^2}{2 \times 20^2 \times \cos^2\alpha}$$

$$y = 20\tan\alpha - 5\sec^2\alpha$$

13b

$$\frac{d}{d\alpha}(\sec\alpha)$$

$$= \frac{d}{d\alpha}\left(\frac{1}{\cos\alpha}\right)$$

$$= \frac{d}{d\alpha}(\cos^{-1}\alpha)$$

$$= -\cos^{-2}\alpha \times -\sin\alpha$$

$$= \frac{\sin\alpha}{\cos^2\alpha}$$

Uncorrected proofs

$$= \frac{1}{\cos\alpha} \times \frac{\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha \tan\alpha$$

13c From part a, $h = 20\tan\alpha - 5\sec^2\alpha$

For maximum value of h, $h'(\alpha) = 0$.

$20\sec^2\alpha - 5 \times 2\sec\alpha \times \sec\alpha\tan\alpha = 0$

$20\sec^2\alpha - 10\sec^2\alpha\tan\alpha = 0$

$10\sec^2\alpha(2 - \tan\alpha) = 0$

$\sec^2\alpha \neq 0$ so $2 - \tan\alpha = 0$

$\tan\alpha = 2$

13d Maximum height:

Taking $\tan\alpha = 2$, we get

$$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}} \text{ and } \cos\alpha = \frac{1}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}$$

so $\sec\alpha = \sqrt{5}$

Using $h = 20\tan\alpha - 5\sec^2\alpha$,

$$h_{max} = 20(2) - 5\left(\sqrt{5}\right)^2$$

$$= 15 \text{ m}$$

13e From part a, the trajectory of the ball is:

$$y = x\tan\alpha - \frac{x^2\sec^2\alpha}{80}$$

The coordinates of impact point are: $(20, 15)$.

The angle of impact is:

$$\frac{dy}{dx}|_{20,15} = \tan\alpha - \frac{x\sec^2\alpha}{40}$$

$$\tan\theta = 2 - \frac{20 \times 5}{40} = -0.5$$

Uncorrected proofs

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$\theta = -26.565\ldots°$

$= 63.434\ldots°$ below the horizontal

$\doteqdot 63°26'$ below the horizontal

The time of flight of the ball:

$20 = 20\cos\alpha\, T$

$T = \dfrac{1}{\cos\alpha}$

$= \sec\alpha$

$= \sqrt{5}$

So, the speed of the ball when it hits the wall:

$v_x = 20\cos\alpha$

$= 20 \times \dfrac{1}{\sqrt{5}}$

$= 4\sqrt{5}$

$v_y = 20\sin\alpha - gt$

$= 20\sin\alpha - 10T$

$= 20 \times \dfrac{2}{\sqrt{5}} - 10 \times \sqrt{5}$

$= 8\sqrt{5} - 10\sqrt{5}$

$= -2\sqrt{5}$

The speed with which the ball hits the wall is:

$V = \sqrt{\left(4\sqrt{5}\right)^2 + \left(2\sqrt{5}\right)^2}$

$= \sqrt{80 + 20}$

$= 10$ m/s

Uncorrected proofs

14a $y_0 = 40; \quad V = 20; \quad g = 10;$

The equations of motion for the stone are:

$\ddot{x} = 0 \quad$ and $\quad \ddot{y} = -g$

At $t = 0, \quad \dot{x} = V\cos\theta \quad$ and $\quad \dot{y} = V\sin\theta$

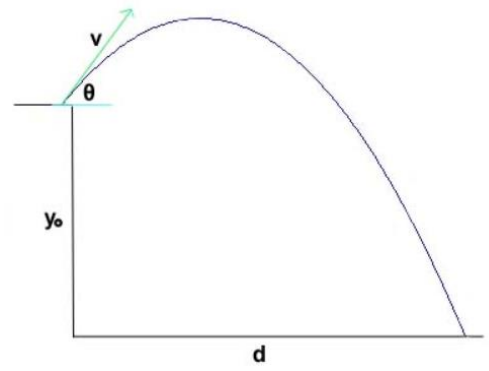
$\dot{x} = V\cos\theta \quad$ and $\quad \dot{y} = V\sin\theta - gt$

At $t = 0, \ x = 0$ and $y = 0$ (assuming cliff as level zero)

$x(t) = Vt\cos\theta \quad$ and $\quad y(t) = Vt\sin\theta - \frac{1}{2}gt^2$

14b When the stone hits the lake, the stone has gone through two sets of motion.

First it has been a projectile from the cliff and the second when it reaches the cliff level while descending and then still falls 40 m below into the lake.



$x(T) = 20T\cos\theta$

$\cos\theta = \dfrac{x(T)}{20T}$

$\sin\theta = \sqrt{1 - \left(\dfrac{x(T)}{20T}\right)^2}$

$y(T) = -40 = 20T\sin\theta - 5T^2$

$\dfrac{5T^2 - 40}{20T} = \sin\theta = \sqrt{1 - \left(\dfrac{x(T)}{20T}\right)^2}$

$(5T^2 - 40)^2 = (20T)^2 - \left(x(T)\right)^2$

$\left(x(T)\right)^2 = 400T^2 - (5T^2 - 40)^2$

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14c $\left(x(T)\right)^2 = 400T^2 - (5T^2 - 40)^2$

For maximum $\left(x(T)\right)^2$,

$\frac{d}{dt}\left(x(T)\right)^2 = 0$

$\frac{d}{dt}\left(x(T)\right)^2 = 800T - 20T(5T^2 - 40) = 0$

$100T^3 - 1600T = 0$

$100T(T^2 - 16) = 0$

$T = 0$ or $T^2 - 16 = 0$

$T \neq 0$ as the stone is being thrown at that time.

So $T = 4$.

$x(T) = \sqrt{400(16) - (5(16) - 40)^2}$

$= 40\sqrt{3}$

$\cos\theta = \frac{x(T)}{20T}$

$= \frac{40\sqrt{3}}{20 \times 4}$

$= \frac{\sqrt{3}}{2}$

$\theta = 30^\circ$

15a The equation of motion with respect to time are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta - gt$

$x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$

At maximum height, $\dot{y} = 0$.

$V\sin\theta - gt = 0$

$t = \frac{V\sin\theta}{g}$

Substituting for t in equation for y:

Uncorrected proofs

$$y_{max} = H = V\left(\frac{V\sin\theta}{g}\right)\sin\theta - \frac{1}{2}g\left(\frac{V\sin\theta}{g}\right)^2$$

$$H = \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$

$$H = \frac{V^2\sin^2\theta}{2g}$$

15b i As H is the same,

$$\frac{V^2\sin^2\theta}{2g} = \frac{9V^2\sin^2\frac{1}{2}\theta}{8g}$$

$$4\sin^2\theta = 9\sin^2\frac{\theta}{2}$$

Using $\cos 2x = 1 - 2\sin^2 x$,

$$4\sin^2\theta = 9 \times \frac{1}{2}(1 - \cos\theta)$$

$$8(1 - \cos^2\theta) = 9 - 9\cos\theta$$

$$8 - 8\cos^2\theta = 9 - 9\cos\theta$$

$$8\cos^2\theta - 9\cos\theta + 1 = 0$$

$$(8\cos\theta - 1)(\cos\theta - 1) = 0$$

$$\cos\theta = \frac{1}{8} \text{ or } \cos\theta = 1$$

Since $\theta \neq 0, \theta = \cos^{-1}\frac{1}{8}$

15b ii Time taken by P_1 to reach maximum height:

$$T_1 = \frac{V\sin\theta}{g}$$

Time taken by P_2 to reach maximum height:

$$T_2 = \frac{\frac{3V}{2}\sin\frac{\theta}{2}}{g}$$

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$$\frac{T_1}{T_2} = \frac{2\sin\theta}{3\sin\frac{\theta}{2}}$$

Now $\cos\theta = \frac{1}{8}$, so using $\sin^2\theta + \cos^2\theta = 1$,

$$\sin\theta = \frac{\sqrt{63}}{8} \quad \text{and} \quad \sin\frac{\theta}{2} = \sqrt{\frac{1}{2} - \frac{1}{16}} = \frac{\sqrt{7}}{4}$$

Therefore

$$\frac{T_1}{T_2} = \frac{2}{3} \times \frac{\sqrt{63}}{8} \times \frac{4}{\sqrt{7}} = 1$$

Hence $T_1 = T_2$.

So, the particles reach the maximum height at the same time.

Intuitionally, the gravitational force acting on the particles is equal and constant. So, as they reach the same maximum height, they must take the same time too.

16a $v_i = V$, Angle of projectile $= \alpha$

The equations of motion are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

The range of the projectile is when the projectile hits the ground for the first time. That is, when $y = 0$.

$$Vt\sin\alpha = \frac{1}{2}gt^2$$

$$t = \frac{2V\sin\alpha}{g}$$

Substituting for t in the equation for x:

$$x = V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$$x = \frac{V^2\sin 2\alpha}{g} \qquad (\text{using } \sin 2\theta = 2\sin\theta\cos\theta)$$

Uncorrected proofs

16b i

$$R = \frac{V^2 \sin 2\alpha}{g}$$

α varies between 15° and 45°

The ranges will vary as:

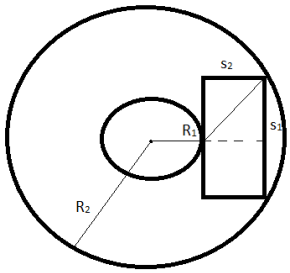
$$\frac{V^2 \sin(2 \times 15°)}{g} \leq R_s \leq \frac{V^2 \sin(2 \times 45°)}{g}$$

$$\frac{V^2 \sin\ 30°}{g} \leq R_s \leq \frac{V^2 \sin 90°}{g}$$

$$\frac{V^2}{2g} \leq R_s \leq \frac{V^2}{g}$$

16b ii Suppose one of the sides $'s_1'$ of the rectangular garden is tangent to the inner circle and the other side is $'s_2'$.

For the minimum value of R_1 and R_2, the optimal case is when s_1 is the larger side and is tangent to inner circle at its midpoint.



$$R_2 \geq \left(R_1 + s_2 + \frac{\left(\frac{s_1}{2}\right)^2}{R_1 + s_2} \right)$$

$$4R_1R_2 + 4R_2s_2 \geq 4R_1^2 + 8R_1s_2 + 4s_2^2 + s_1^2$$

Substituting values and solving for $\frac{V^2}{g}$:

$$\frac{V^2}{g} \geq 6\sqrt{2}$$

$$\frac{V^2}{2g} \geq 3\sqrt{2}$$

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17a $\ddot{x} = -k\dot{x} = -kv_x$

$\ddot{y} = -g - k\dot{y} = -g - kv_y$

$\ddot{x} = \frac{d^2x}{dt^2} = -k\left(\frac{dx}{dt}\right)$

$\frac{dv_x}{dt} = -kv_x$

Integrating once,

$\dot{x} = \frac{dx}{dt}$

$\int dt = -\int \frac{dv_x}{kv_x}$

$t = -\frac{1}{k}\ln(kv_x) + C$

At $t = 0,\ v_x = V\cos\alpha$.

$C = \frac{1}{k}\ln(kV\cos\alpha)$

Therefore

$t = -\frac{1}{k}\ln(kv_x) + \frac{1}{k}\ln(kV\cos\alpha)$

$t = \frac{1}{k}\ln\left(\frac{V\cos\alpha}{v_x}\right)$

$kt = \ln\left(\frac{V\cos\alpha}{v_x}\right)$

$e^{kt} = \frac{V\cos\alpha}{v_x}$

$v_x = \frac{V\cos\alpha}{e^{kt}}$

$v_x = V\cos\alpha\ e^{-kt}$

Hence

$\dot{x} = V\cos\alpha\ e^{-kt}$

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$$\ddot{y} = \frac{d^2y}{dt^2} = -g - kv_y$$

$$\frac{dv_y}{dt} = -g - kv_y$$

$$\int dt = -\int \frac{dv_y}{g + kv_y}$$

$$t = -\frac{1}{k}\ln(g + kv_y) + D$$

At $t = 0$, $v_y = V\sin\alpha$.

$$D = \frac{1}{k}\ln(g + kV\sin\alpha)$$

Therefore

$$t = \frac{1}{k}\ln\left(\frac{g + kV\sin\alpha}{g + kv_y}\right)$$

$$kt = \ln\left(\frac{g + kV\sin\alpha}{g + kv_y}\right)$$

$$e^{kt} = \frac{g + kV\sin\alpha}{g + kv_y}$$

$$g + kv_y = (g + kV\sin\alpha)e^{-kt}$$

$$kv_y = (g + kV\sin\alpha)e^{-kt} - g$$

$$v_y = \frac{1}{k}(g + kV\sin\alpha)e^{-kt} - \frac{g}{k}$$

$$v_y = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$$

Hence

$$\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$$

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17b

$$v_x = V\cos\alpha\ e^{-kt} = \dot{x} = \frac{dx}{dt}$$

$$\int dx = \int V\cos\alpha\ e^{-kt}dt$$

$$x = -\frac{V\cos\alpha}{k}e^{-kt} + C$$

At $t = 0,\ x = 0.$

$$C = \frac{V\cos\alpha}{k}$$

Therefore

$$x = -\frac{V\cos\alpha}{k}e^{-kt} + \frac{V\cos\alpha}{k}$$

$$x = \frac{V\cos\alpha}{k}(1 - e^{-kt})$$

$$v_y = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k} = \dot{y} = \frac{dy}{dt}$$

$$\int dy = \int \left(\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}\right)dt$$

$$y = -\frac{1}{k}\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}t + D$$

$$y = -\left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t + D$$

At $t = 0,\ y = 0.$

$$D = \frac{g}{k^2} + \frac{V\sin\alpha}{k}$$

Therefore

$$y = -\left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t + \frac{g}{k^2} + \frac{V\sin\alpha}{k}$$

$$y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right) - \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t$$

$$y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t$$

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17c At maximum height, $\dot{y} = 0$.

$$\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k} = 0$$

$$\frac{g}{k} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt}$$

$$g = (g + kV\sin\alpha)e^{-kt}$$

$$e^{-kt} = \frac{g}{g + kV\sin\alpha}$$

Substituting in equation for x:

$$x = \frac{V\cos\alpha}{k}(1 - e^{-kt})$$

$$= \frac{V\cos\alpha}{k}\left(1 - \frac{g}{g + kV\sin\alpha}\right)$$

$$= \frac{V\cos\alpha}{k}\left(\frac{g + kV\sin\alpha - g}{g + kV\sin\alpha}\right)$$

$$= \frac{V\cos\alpha}{k(g + kV\sin\alpha)}(g + kV\sin\alpha - g)$$

$$= \frac{V\cos\alpha \times kV\sin\alpha}{k(g + kV\sin\alpha)}$$

$$= \frac{V^2\cos\alpha\sin\alpha}{(g + kV\sin\alpha)}$$

$$= \frac{V^2(2\sin\alpha\cos\alpha)}{2(g + kV\sin\alpha)}$$

$$= \frac{V^2\sin 2\alpha}{2(g + kV\sin\alpha)}$$

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6F

(18) (a) The path has equation

$$y = -\frac{g}{2V^2}(1+\tan^2\alpha)x^2 + x\tan\alpha \qquad (*)$$

Now, $k = \frac{V^2}{2g}$, so $\frac{1}{k} = \frac{2g}{V^2}$, so $\frac{1}{4k} = \frac{g}{2V^2}$.

So $(*)$ becomes $y = -\frac{1}{4k}(1+\tan^2\alpha)x^2 + x\tan\alpha$

$$4ky = -x^2 - x^2\tan^2\alpha + 4kx\tan\alpha$$

$$x^2\tan^2\alpha - 4kx\tan\alpha + (4ky + x^2) = 0$$

(b) $X^2\tan^2\alpha - 4kX\tan\alpha + (4kY + X^2) = 0 \qquad (**)$

If there are two real and distinct solutions for $\tan\alpha$,

then $\Delta > 0$,

so $(-4kX)^2 - 4 \cdot X^2(4kY + X^2) > 0$

$$16k^2X^2 - 16kX^2Y - 4X^4 > 0$$

$\div 4X^2$, $X \neq 0$

$$4k^2 - 4kY - X^2 > 0$$

$$X^2 < 4k^2 - 4kY.$$

(c) The product of the roots of equation $(**)$ is

$$\tan\alpha_1 \tan\alpha_2 = \frac{4kY + X^2}{X^2}$$

$$= 1 + \frac{4kY}{X^2}$$

> 1, since $k, X, Y > 0$. (A)

Suppose that α_1 and α_2 are both less than 45°.

Then $\tan\alpha_1$ and $\tan\alpha_2$ are both less than one.

So $\tan\alpha_1 \tan\alpha_2 < 1$, which contradicts (A).

So α_1 and α_2 cannot both be less than 45°.

(6F) (19)

(a) P has coordinates $(d\cos\beta, d\sin\beta)$.

(b) $d\cos\beta = V t \cos\alpha$ ①

$d\sin\beta = V t \sin\alpha - \frac{1}{2} g t^2$ ②

From ①, $t = \dfrac{d\cos\beta}{V\cos\alpha}$.

Substituting into ②:

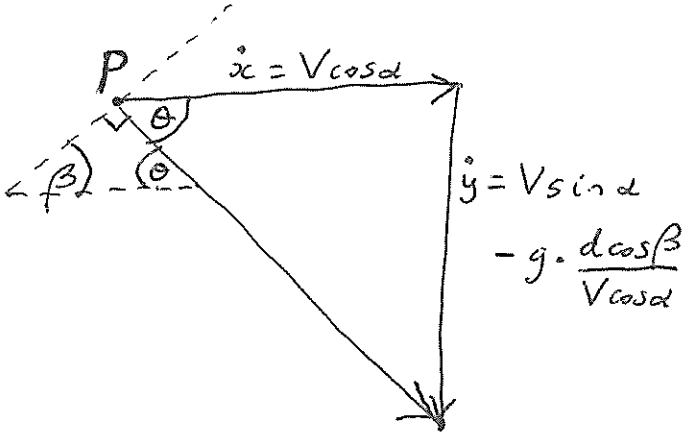
$$d\sin\beta = V\sin\alpha \cdot \frac{d\cos\beta}{V\cos\alpha} - \frac{g}{2} \cdot \frac{d^2\cos^2\beta}{V^2\cos^2\alpha}$$

$\div d, d \neq 0$

$$\sin\beta = \tan\alpha\cos\beta - \frac{g d\cos^2\beta}{2V^2\cos^2\alpha}$$

$$\text{so } d = \frac{2V^2\cos^2\alpha}{g\cos^2\beta}\left(\tan\alpha\cos\beta - \sin\beta\right).$$

(c) At P, $t = \dfrac{d\cos\beta}{V\cos\alpha}$.



$\theta + \beta = \frac{\pi}{2}$, so $\theta = \frac{\pi}{2} - \beta$.

$$\tan\theta = \frac{|\dot{y}|}{|\dot{x}|} \quad (\text{note that } \dot{y} < 0)$$

$$\text{so } \tan\left(\frac{\pi}{2} - \beta\right) = \frac{\frac{g d\cos\beta}{V\cos\alpha} - V\sin\alpha}{V\cos\alpha}$$

$$\text{so } \cot\beta = \frac{g d\cos\beta}{V^2\cos^2\alpha} - \tan\alpha.$$

(d) From (b), $\dfrac{g d\cos\beta}{V^2\cos^2\alpha} = \dfrac{2}{\cos\beta}\left(\tan\alpha\cos\beta - \sin\beta\right)$

$$= 2\tan\alpha - 2\tan\beta.$$

Substituting into (c),

$$\cot\beta = \left(2\tan\alpha - 2\tan\beta\right) - \tan\alpha$$

$$\tan\alpha = \cot\beta + 2\tan\beta.$$

6F (20)(a) $y = 2\sin(x-\theta)\cos x$

(i) $y' = 2\sin(x-\theta).(-\sin x) + 2\cos(x-\theta)\cos x$

$= 2\left(\cos(x-\theta)\cos x - \sin(x-\theta)\sin x\right)$

$= 2\cos(2x-\theta)$

(ii) Integrating (i),

$\int \frac{dy}{dx}dx = \int 2\cos(2x-\theta)\,dx$

$y = \sin(2x-\theta) + C$

When $x=0$, $y = 2\sin(-\theta)$

$= -2\sin\theta$

$\rightarrow -2\sin\theta = \sin(-\theta) + C$

$C = -\sin\theta$

so $2\sin(x-\theta)\cos x = \sin(2x-\theta) - \sin\theta$.

(b)(i) The Cartesian equation of the parabola is

$y = -\frac{g\sec^2\alpha}{2V^2}x^2 + x\tan\alpha$

so $Y = -\frac{g\sec^2\alpha}{2V^2}X^2 + X\tan\alpha$ (*)

Now, $X = R\cos\beta$ and $Y = R\sin\beta$

so $\frac{Y}{X} = \tan\beta$.

Substituting into (*),

$\frac{Y}{X} = -\frac{g\sec^2\alpha}{2V^2}X + \tan\alpha$

$\tan\beta = -\frac{g}{2V^2\cos^2\alpha}X + \tan\alpha$

$X = \frac{2V^2\cos^2\alpha}{g}\left(\tan\alpha - \tan\beta\right)$.

6F (20)(b)(ii)

$$R = \frac{X}{\cos\beta}$$

$$= \frac{2V^2\cos^2\alpha(\tan\alpha - \tan\beta)}{g\cos\beta}$$

$$= \frac{2V^2\cos^2\alpha\left(\frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta}\right)}{g\cos\beta} \cdot \frac{\cos\beta}{\cos\beta}$$

$$= \frac{2V^2\cos\alpha(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{g\cos^2\beta}$$

$$= \frac{2V^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}$$

(iii) Using (a)(ii) with $x = \alpha$ and $\theta = \beta$,

$$R = \frac{V^2(\sin(2\alpha-\beta) - \sin\beta)}{g\cos^2\beta}.$$

If α varies such that $\alpha > \beta$ and β is constant,

$$R \leq \frac{V^2(1-\sin\beta)}{g(1-\sin^2\beta)} \quad (\text{since } -1 \leq \sin(2\alpha-\beta) \leq 1)$$

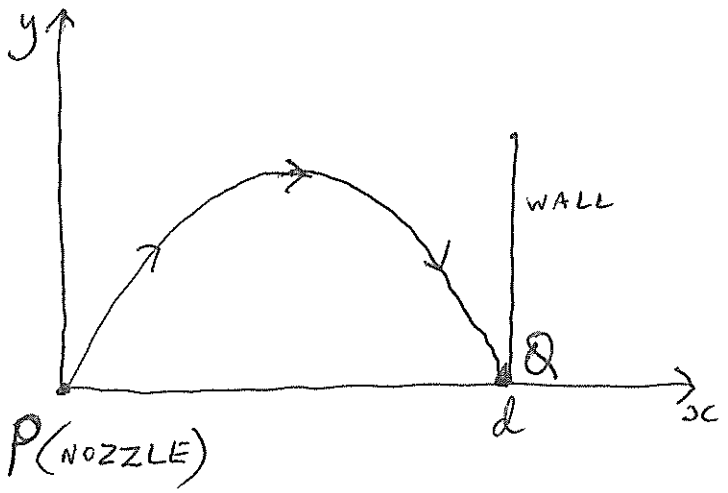
$$\text{so } R \leq \frac{V^2}{g(1+\sin\beta)}.$$

(iv) Given $\beta = 14^\circ$, the maximum value of R occurs when $\sin(2\alpha - 14^\circ) = 1$,

$$2\alpha - 14^\circ = 90^\circ$$

$$\alpha = 52^\circ.$$

6F (21)(a)



The horizontal range is $d = \frac{V^2 \sin 2\alpha}{g}$.

The maximum range is $d = \frac{V^2}{g}$, since $|\sin 2\alpha| \leq 1$.

So $V = \sqrt{gd}$ if the water can just reach the bottom of the wall.

Thus, if $V > \sqrt{gd}$, the water can hit the wall above ground level.

(b) $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{gt^2}{2}$

When $x = d$, $t = \frac{d}{V\cos\alpha}$

and so $y = d\tan\alpha - \frac{g}{2} \cdot \frac{d^2}{V^2\cos^2\alpha}$

$= d\tan\alpha - \frac{g}{2} \cdot \frac{d^2}{4gd\cos^2\alpha}$, since $V = 2\sqrt{gd}$

$= d\tan\alpha - \frac{d}{8\cos^2\alpha}$

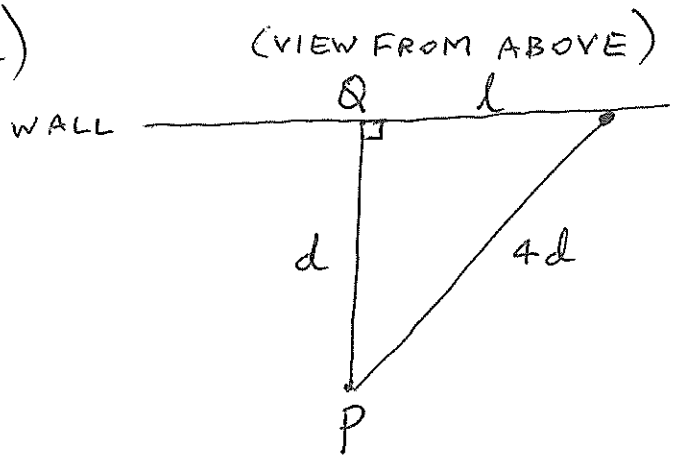
$= d\left(\tan\alpha - \frac{1}{8}(1+\tan^2\alpha)\right)$

$= \frac{d}{8}\left(-\tan^2\alpha + 8\tan\alpha - 1\right)$

$= \frac{d}{8}\left(15 - (\tan\alpha - 4)^2\right)$

so the greatest height that can be reached up the wall is $\frac{15d}{8}$, when $\alpha = \tan^{-1}4$.

6F (21)(b) (continued)



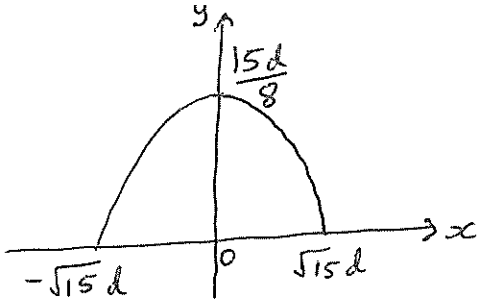
The maximum horizontal range given $V = 2\sqrt{gd}$ is $\frac{V^2}{g} = \frac{4gd}{g}$

$= 4d$.

Let l be the greatest horizontal distance from Q that the water can reach along the wall.

Then $l^2 = 16d^2 - d^2$

$l = \sqrt{15}\,d$.



Assuming that the portion of the wall that can be sprayed is a parabolic segment, the parabola has equation $y = A(\sqrt{15}\,d - x)(\sqrt{15}\,d + x)$, that is, $y = A(15d^2 - x^2)$, for some constant A.

When $x = 0$, $y = \frac{15d}{8}$, so $\frac{15d}{8} = A \cdot 15d^2$, so $A = \frac{1}{8d}$.

So the required area is $2\int_0^{\sqrt{15}\,d} \frac{1}{8d}\left(15d^2 - x^2\right) dx$

$$= \frac{1}{4d}\left[15d^2x - \frac{1}{3}x^3\right]_0^{\sqrt{15}\,d}$$

$$= \frac{1}{4d}\left(15\sqrt{15}\,d^3 - \frac{1}{3}\cdot 15\sqrt{15}\,d^3\right)$$

$$= \frac{1}{4d}\cdot 10\sqrt{15}\,d^3$$

$$= \frac{5\sqrt{15}}{2}d^2.$$

(Proving that the segment is parabolic is considered to be beyond the scope of the question.)