Solutions to Exercise 6E Foundation questions

1a Resistive acceleration (acting downwards)

$$= \frac{\text{resistive force (acting downwards)}}{\text{mass}}$$

$$= -\frac{0.2v^2}{5}$$

g = acceleration due to gravity (acting downwards)

$$= -10$$

Hence

$$a = -10 - \frac{0.2v^2}{5}$$

$$\frac{dv}{dt} = \frac{-50 - 0.2v^2}{5}$$

$$= \frac{-250 - v^2}{25}$$

1b

$$\frac{dv}{dt} = \frac{-250 - v^2}{25}$$

$$\frac{dt}{dv} = \frac{25}{-250 - v^2}$$

$$= \frac{-25}{250 + v^2}$$

$$= -25 \times \frac{1}{\left(5\sqrt{10}\right)^2 + v^2}$$

$$t = -25 \times \frac{1}{5\sqrt{10}} \tan^{-1} \frac{v}{5\sqrt{10}} + C$$

$$= -\frac{5}{\sqrt{10}} \tan^{-1} \frac{v}{5\sqrt{10}} + C$$

When $t = 0, v = 40$.

$$0 = -\frac{5}{\sqrt{10}} \tan^{-1} \frac{40}{5\sqrt{10}} + C$$

Uncorrected proofs

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$$C = \frac{5}{\sqrt{10}}\tan^{-1}\frac{8}{\sqrt{10}}$$

Therefore

$$t = -\frac{5}{\sqrt{10}}\tan^{-1}\frac{v}{5\sqrt{10}} + \frac{5}{\sqrt{10}}\tan^{-1}\frac{8}{\sqrt{10}}$$

Maximum height when $v = 0$:

$$t = -\frac{5}{\sqrt{10}}\tan^{-1}\frac{0}{5\sqrt{10}} + \frac{5}{\sqrt{10}}\tan^{-1}\frac{8}{\sqrt{10}}$$

$$= 0 + \frac{5}{\sqrt{10}}\tan^{-1}\frac{8}{\sqrt{10}}$$

$$= 1.888 \ldots$$

$$\div 1.9 \text{ seconds}$$

Maximum height is reached after about 1.9 seconds.

1c

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$\frac{-250 - v^2}{25} = v\frac{dv}{dy}$$

$$\frac{dv}{dy} = \frac{-250 - v^2}{25v}$$

$$\frac{dy}{dv} = \frac{25v}{-250 - v^2}$$

$$= \frac{-25v}{250 + v^2}$$

$$= -\frac{25}{2} \times \frac{2v}{250 + v^2}$$

$$y = -\frac{25}{2}\ln(250 + v^2) + C \qquad (\text{since } 250 + v^2 > 0)$$

When $y = 0, v = 40$.

$$0 = -\frac{25}{2}\ln(250 + 40^2) + C$$

Uncorrected proofs

$$C = \frac{25}{2}\ln 1850$$

Therefore

$$y = -\frac{25}{2}\ln(250 + v^2) + \frac{25}{2}\ln 1850$$

Maximum height is reached when $v = 0$.

$$y = -\frac{25}{2}\ln 250 + \frac{25}{2}\ln 1850$$

$$= \frac{25}{2}\ln\frac{1850}{250}$$

$$= 25.0185$$

$$\div 25 \text{ metres}$$

Maximum height, correct to the nearest metre, is 25 metres.

2a Resistive acceleration (acting downwards)

$$= \frac{\text{resistive force (acting downwards)}}{\text{mass}}$$

$$= -\frac{0.2v}{0.5}$$

g = acceleration due to gravity (acting downwards)

$$= -10$$

Hence

$$\ddot{x} = -10 - \frac{0.2v}{0.5}$$

$$= \frac{-5 - 0.2v}{0.5}$$

$$= \frac{-50 - 2v}{5}$$

2b

$$\ddot{x} = \frac{dv}{dt}, \quad \text{so}$$

$$\frac{dv}{dt} = \frac{-50 - 2v}{5}$$

Uncorrected proofs

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$$\frac{dt}{dv} = \frac{5}{-50 - 2v}$$

$$= \frac{-\frac{5}{2}}{25 + v}$$

$$t = -\frac{5}{2}\ln|25 + v| + C$$

When $t = 0$, $v = 40$.

$$0 = -\frac{5}{2}\ln|25 + 40| + C$$

$$C = \frac{5}{2}\ln 65$$

Therefore

$$t = -\frac{5}{2}\ln|25 + v| + \frac{5}{2}\ln 65$$

Maximum height is reached when $v = 0$.

$$t = -\frac{5}{2}\ln 25 + \frac{5}{2}\ln 65$$

$$= \frac{5}{2}(\ln 65 - \ln 25)$$

$$= \frac{5}{2}\ln\frac{65}{25}$$

$$= \frac{5}{2}\ln\frac{13}{5}$$

Hence the object takes $\frac{5}{2}\ln\frac{13}{5}$ seconds to reach its maximum height.

2c

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$\frac{-50 - 2v}{5} = v\frac{dv}{dy}$$

$$\frac{dv}{dy} = \frac{-50 - 2v}{5v}$$

Uncorrected proofs

$$\frac{dy}{dv} = \frac{5v}{-50 - 2v}$$

$$= \frac{-5v}{50 + 2v}$$

$$= -\frac{5}{2} + \frac{125}{50 + 2v} \qquad \text{(by long division)}$$

$$y = -\frac{5}{2}v + \frac{125}{2}\ln|50 + 2v| + C$$

When $y = 0, v = 40$.

$$0 = -\frac{5}{2}(40) + \frac{125}{2}\ln|50 + 80| + C$$

$$C = 100 - \frac{125}{2}\ln 130$$

Therefore

$$y = -\frac{5}{2}v + \frac{125}{2}\ln|50 + 2v| + 100 - \frac{125}{2}\ln 130$$

Maximum height reached when $v = 0$.

$$y = 0 + \frac{125}{2}\ln 50 + 100 - \frac{125}{2}\ln 130$$

$$= 100 + \frac{125}{2}\ln\frac{50}{130}$$

$$= 100 + \frac{125}{2}\ln\frac{5}{13}$$

Maximum height reached is $\left(100 + \frac{125}{2}\ln\frac{5}{13}\right)$ metres.

3a Resistive acceleration (acting upwards)

$$= \frac{\text{resistive force (acting upwards)}}{\text{mass}}$$

$$= \frac{0.1v^2}{100}$$

g = acceleration due to gravity (acting downwards)

$$= -9.8$$

Hence

Uncorrected proofs

$$\ddot{x} = -9.8 + \frac{0.1v^2}{100}$$

$$= -9.8 + \frac{v^2}{1000}$$

$$= \frac{-9800 + v^2}{1000}$$

Terminal velocity occurs when $\ddot{x} = 0$.

$$0 = \frac{-9800 + v^2}{1000}$$

$$0 = -9800 + v^2$$

$$9800 = v^2$$

$$|v| = \sqrt{9800}$$

$$= 70\sqrt{2}$$

$$\div 99$$

The terminal speed is about 99 m/s.

3b

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$v\frac{dv}{dy} = \frac{-9800 - v^2}{1000}$$

$$\frac{dv}{dy} = \frac{-9800 - v^2}{1000v}$$

$$\frac{dy}{dv} = \frac{1000v}{-9800 - v^2}$$

$$= \frac{-500 \times 2v}{9800 + v^2}$$

$y = -500\ln(9800 + v^2) + C$ (since $9800 + v^2 > 0$)

When $y = 0, v = 0.8 \times 70\sqrt{2} = 56\sqrt{2}$

$$0 = -500\ln\left(9800 + \left(56\sqrt{2}\right)^2\right) + C$$

$$C = 500\ln(16\,072)$$

Uncorrected proofs

Therefore

$y = -500\ln(9800 + v^2) + 500\ln(16\,072)$

When object is dropped, $v = 0$.

$y = -500\ln(9800) + 500\ln(16\,072)$

$= 500\ln\left(\frac{16\,072}{9800}\right)$

$= 247.348\ldots$

$\div 247$ metres

The point at which the object was dropped was about 247 metres above the ground.

Uncorrected proofs

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Solutions to Exercise 6E Development questions

4ai $g = 10; \quad u = 20; \quad f_r = \frac{1}{40}v^2$

The equation of motion of the object is:

$$\ddot{x} = v\frac{dv}{dx} = -\left(g + \frac{1}{40}v^2\right)$$

$$dx = -\frac{40v}{400 + v^2}dv$$

$$\int dx = -20\int \frac{2v}{400 + v^2}dv$$

$$x = -20\ln(400 + v^2) + C$$

At $x = 0, v = 20$.

$$C = 20\ln(800)$$

Therefore

$$x = 20\ln\left(\frac{800}{400 + v^2}\right)$$

At maximum height, $v = 0$

$$x = 20\ln\left(\frac{800}{400}\right)$$

$$= 20\ln 2 \text{ metres}$$

4a ii

$$\frac{dv}{dt} = \ddot{x} = -\left(g + \frac{1}{40}v^2\right)$$

$$\int dt = -\int \frac{40dv}{400 + v^2}$$

$$t = -40\int \frac{dv}{400 + v^2}$$

Using $v = \tan a$,

$dv = \sec^2 a\, da$ and $1 + \tan^2 a = \sec^2 a$

$$t = -\frac{40}{20}\tan^{-1}\frac{v}{20} + C$$

Uncorrected proofs

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$= -2\tan^{-1}\frac{v}{20} + C$

At $t = 0,\ \ v = 20.$

$0 = C - 2\tan^{-1} 1$

$C = \frac{\pi}{2}$

Therefore

$t = \frac{\pi}{2} - 2\tan^{-1}\frac{v}{20}$

At $v = 0, (x = x_{max})$

$t = \frac{\pi}{2} - 2\tan^{-1} 0$

$= \frac{\pi}{2}$ seconds

4b i Equation of motion of the object as it falls is:

$\ddot{x} = g - \frac{1}{40}v^2$

4b ii

$\ddot{x} = g - \frac{1}{40}v^2 = v\frac{dv}{dx}$

$dx = \frac{40v}{400 - v^2}dv$

$\int dx = -20\int \frac{-2v}{400 - v^2}dx$

$x = -20\ln(400 - v^2) + C$

Now, since the object is falling down, the new stationary point conditions will be $x = 0, v = 0$ as the direction of the motion is changed.

$0 = -20\ln 400 + C$

$C = 20\ln 400$

Therefore

$x = -20\ln(400 - v^2) + 20\ln 400$

Uncorrected proofs

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$$x = 20 \ln \frac{400}{400 - v^2}$$

When the object returns to starting point, which is now $x = 20 \ln 2$,

$$20 \ln 2 = 20 \ln \frac{400}{400 - v^2}$$

$$800 - 2v^2 = 400$$

$$v^2 = 200$$

Speed $= |v| = 10\sqrt{2}$ m/s

5a $v_i = V_0; \quad f_r = mkv;$

$$m\ddot{x} = (mg - mkv)$$

$$\ddot{x} = (g - kv)$$

$$\frac{dv}{dt} = (g - kv)$$

$$\int dt = \int \frac{dv}{g - kv}$$

$$t = -\frac{1}{k} \ln(g - kv) + C$$

At $t = 0, \ v = V_0$.

$$C = \frac{1}{k} \ln(g - kV_0)$$

Therefore

$$t = -\frac{1}{k} \ln(g - kv) + \frac{1}{k} \ln(g - kV_0)$$

$$t = \frac{1}{k} \ln \frac{g - kV_0}{g - kv}$$

5b

$$t = \frac{1}{k} \ln \frac{g - kV_0}{g - kv} \quad \text{from part a}$$

$$e^{kt} = \frac{g - kV_0}{g - kv}$$

Uncorrected proofs

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$$kv = g - (g - kV_0)e^{-kt}$$

$$v = \frac{g}{k}(1 - e^{-kt}) + V_0 e^{-kt}$$

As $t \to \infty$,

$$\lim_{t\to\infty} v$$

$$= \lim_{t\to\infty} \left(\frac{g}{k}(1 - e^{-kt}) + V_0 e^{-kt}\right)$$

$$= \frac{g}{k}\left(1 - \frac{1}{T}\right) + V_0\left(\frac{1}{T}\right)$$

$$= \frac{g}{k} \quad \text{(as } T \text{ is very large)}$$

5c

$$v = \frac{g}{k} - \frac{g}{k}e^{-kt} + V_0 e^{-kt} = \frac{dx}{dt}$$

$$\int dx = \int \left(\frac{g}{k} + \left(V_0 - \frac{g}{k}\right)e^{-kt}\right) dt$$

$$x = \frac{g}{k}t + \left(V_0 - \frac{g}{k}\right)\left(-\frac{1}{k}e^{-kt}\right) + C$$

At $t = 0, x = 0$.

$$C = \left(V_0 - \frac{g}{k}\right)\left(\frac{1}{k}\right)$$

$$x = \frac{g}{k}t + \frac{kV_0 - g}{k^2}(1 - e^{-kt})$$

5d From part c, the object will fall as:

$$x = \frac{g}{k}t + \frac{kV_0 - g}{k^2}(1 - e^{-kt})$$

Terminal velocity is:

$$\frac{g}{k} = V_0 = 20$$

$$k = \frac{1}{2}$$

Uncorrected proofs

$$x_1 = \frac{g}{k}t + \frac{(kV_0 - g)}{k^2}(1 - e^{-kt})$$

$$= 20t + 0 \qquad \text{as} \frac{kV_0 - g}{k^2} = \frac{0}{k^2} = 0$$

The other object falling simply under gravity will fall as:

$$\ddot{x} = (g - kv)$$

$$t = -\frac{1}{k}\ln(g - kv) + C$$

At $t = 0,\ v = 0.$

$$C = \frac{1}{k}\ln g$$

Therefore

$$t = -\frac{1}{k}\ln(g - kv) + \frac{1}{k}\ln g$$

$$t = \frac{1}{k}\ln\frac{g}{g - kv}$$

$$v = \frac{dx}{dt} = \frac{1}{k}(g - ge^{-kt}) = \frac{g}{k}(1 - e^{-kt})$$

$$x_2 = \frac{g}{k}t - \frac{g}{k^2}e^{-kt} + D$$

At $t = 0,\ x = 0.$

$$D = \frac{g}{k^2}$$

Therefore

$$x_2 = \frac{g}{k}t - \frac{g}{k^2}e^{-kt} + \frac{g}{k^2}$$

$$= \frac{g}{k}t + \frac{g}{k^2}(1 - e^{-kt})$$

$$= 20t + 40(1 - e^{-kt})$$

The distance between the objects after t seconds will be:

$$d = |x_1 - x_2|$$

$$= |20t - (20t + 40(1 - e^{-kt})|$$

$$= 40(1 - e^{-kt})$$

Uncorrected proofs

The limiting distance is when the stone thrown with terminal velocity hits the ground. As we don't know the height, we assume the time of descent is very large and $t \to \infty$.

$$\lim_{t\to\infty} d = \lim_{t\to\infty} 40(1 - e^{-kt}) = 40$$

6a $\quad m = 10; \quad k = \frac{1}{10}; \quad f_r = kv^2; \quad v_i = u; \quad g = 10$

At the highest point (point A), $v = 0$.

$$\ddot{mx} = -mg - kv^2 = \frac{dv}{dt}$$

$$\int dt = -\int_u^v \frac{dv}{g + \frac{k}{m}v^2}$$

$$= 10\sqrt{10}\int_v^u \frac{dv}{1000 + v^2}$$

$$t = \sqrt{10}\left[\tan^{-1}\frac{v}{10}\right]_v^u$$

$$= \sqrt{10}\left(\tan^{-1}\frac{u}{10\sqrt{10}} - \tan^{-1}\frac{v}{10\sqrt{10}}\right)$$

At point A:

$$t = \sqrt{10}\left(\tan^{-1}\frac{u}{10\sqrt{10}} - \tan^{-1}\frac{0}{10\sqrt{10}}\right)$$

$$= \sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$$

6b

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2$$

$$\int dx = -\frac{1}{2}\int \frac{2v\,dv}{g + \frac{k}{m}v^2}$$

$$x = -\frac{m}{2k}\ln\left(g + \frac{k}{m}v^2\right) + C$$

At $t = 0, x = 0$ and $v = u$.

Uncorrected proofs

$$C = 50\ln\left(\frac{1000+u^2}{1000}\right)$$

Therefore

$$x = 50\ln\left(\frac{1000+u^2}{1000+v^2}\right)$$

At maximum height, $v = 0$.

$$x = 50\ln\left(\frac{1000+u^2}{1000}\right) \text{ metres}$$

6c $v_f = \omega$

The object will fall a distance of OA.

$$v\frac{dv}{dx} = \left(g - \frac{k}{m}v^2\right)$$

$$\frac{1}{2}\int \frac{2vdv}{g-\frac{k}{m}v^2} = \int dx$$

$$x = -\frac{m}{2k}\ln\left(g - \frac{k}{m}v^2\right) + C$$

$$x = -50\ln\frac{(1000-v^2)}{1000} + C$$

As the direction of motion changes, new stationary point conditions will be:

$v = 0, x = 0$

$$C = 50\ln\frac{(1000-0)}{1000} = 0$$

Therefore

$$x = -50\ln\frac{1000-v^2}{1000}$$

$$e^{-\frac{x}{50}} = \frac{1000-v^2}{1000}$$

$$1000 - v^2 = (1000)e^{-\frac{x}{50}}$$

$$v^2 = 1000 - (1000)e^{-\frac{x}{50}}$$

Uncorrected proofs

At $x = 50\ln\dfrac{1000+u^2}{1000}$, $v = \omega$.

(This value of x is the original position of the object)

$$\omega^2 = 1000 - 1000 \times \frac{1000}{1000+u^2}$$

$$= 1000 \times \frac{1000+u^2-1000}{(1000+u^2)}$$

$$= \frac{1000u^2}{1000+u^2}$$

7a i The equation of motion is:

$$m\ddot{y} = mg - mkv$$

$$\ddot{y} = (g - kv)$$

Terminal velocity is when the object hits the ground and $\ddot{y} = 0$.

$$g - kV_T = 0$$

$$V_T = \frac{g}{k}$$

7a ii

$$\frac{dv}{dt} = (g - kv)$$

$$\int dt = \int \frac{1}{g-kv}\,dv$$

$$t = -\frac{1}{k}\ln(g-kv) + C$$

At $t = 0,\ v = 0$.

$$C = \frac{1}{k}\ln g$$

Therefore

$$t = -\frac{1}{k}\ln(g-kv) + \frac{1}{k}\ln g$$

$$t = \frac{1}{k}\ln\left(\frac{g}{g-kv}\right)$$

Uncorrected proofs

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$$v = \frac{1}{k}(g - ge^{-kt})$$
$$= V_T(1 - e^{-kt})$$

7b i The equation of motion for the second particle:

$$\ddot{y} = -(g + kv) = \frac{dv}{dt}$$
$$\int dt = -\int_U^0 \frac{dv}{g + kv}$$
$$= \int_0^U \frac{dv}{g + kv}$$
$$t = \left[\frac{1}{k}\ln(g + kv)\right]_0^U$$
$$= \frac{1}{k}\ln\left(\frac{g + kU}{g}\right)$$

7b ii When the second particle reaches its maximum height, $v_2 = 0$

The speed of the first particle is given by:

$v_1 = V(1 - e^{-kt})$

The time taken by second particle to reach maximum height:

$$T_2 = \frac{1}{k}\ln\left(\frac{g + kU}{g}\right)$$

Substituting the value of time in the equation for velocity:

$$v_1 = V\left(1 - e^{-k \times \frac{1}{k}\ln\frac{g+kU}{g}}\right)$$
$$= V\left(1 - \frac{g}{g + kU}\right)$$
$$= \frac{V}{g + kU}(g + kU - g)$$
$$= \frac{kVU}{g + kU}$$

Uncorrected proofs

$$= \frac{kVU}{k\left(\frac{g}{k} + U\right)}$$

$$= \frac{VU}{V + U} \quad \text{as } V = \frac{g}{K}$$

8a The terminal velocity for a particle falling under gravity is:

$\ddot{y} = (g - kv)$

At $\ddot{y} = 0$,

$g - kV_t = 0$

$V_t = \frac{g}{k}$

8b The initial velocity of projection for P_2 is:

$2V_t = \frac{2g}{k} = v_2(0)$

The equation of motion for P_2:

$\ddot{y} = -(g + kv) = \frac{dv}{dt}$

$\int dt = -\int_{v_2}^{0} \frac{dv}{g + kv}$

$= \int_{0}^{v_2} \frac{dv}{g + kv}$

$t = \left[\frac{1}{k}\ln(g + kv)\right]_0^{v_2}$

$= \frac{1}{k}\left\{\ln\left(g + \frac{2kg}{k}\right) - \ln(g + 0)\right\}$

$= \frac{1}{k}\ln\left(\frac{3g}{g + kv}\right)$

Uncorrected proofs

8c Velocity of P_1 is 30% of its terminal velocity:

$$v_1 = 0.3\frac{g}{k}$$

$$\frac{dv}{dt} = (g - kv)$$

$$t = \frac{1}{k}\ln\left(\frac{g}{g - kv}\right)$$

$$= \frac{1}{k}\ln\left(\frac{1}{1 - 0.3}\right)$$

$$= \frac{1}{k}\ln\frac{10}{7}$$

Substituting the value of time in the equation from part b:

$$t = \frac{1}{k}\ln\left(\frac{3g}{g + kv_2}\right)$$

$$= \frac{1}{k}\ln\frac{10}{7}$$

$$\frac{3\frac{g}{k}}{\frac{g}{k} + v_2} = \frac{10}{7}$$

$$\frac{21g}{k} = \frac{10g}{k} + 10v$$

$$v_2 = \frac{11g}{10k} \text{ m/s}$$

9a i

$$f(x) = x - \frac{g^2}{x} - 2g\ln\frac{x}{g} \qquad \text{where } x \geq g$$

$$f(g) = g - \frac{g^2}{g} - 2g\ln\left(\frac{g}{g}\right)$$

$$= g - g - 2g\ln 1$$

$$= 0$$

Uncorrected proofs

9a ii

$$f'(x) = \frac{d}{dx}f(x)$$

$$= 1 + \frac{g^2}{x^2} - 2g\left(\frac{1}{x}\right)$$

If $\frac{g}{x} = k$,

$1 + k^2 - 2k = (1 - k)^2$

Hence,

$$f'(x) = \left(1 - \frac{g}{k}\right)^2$$

9a iii $f'(x) = \left(1 - \frac{g}{x}\right)^2$

$f'(x) > 0 \quad \forall \quad (x \in R)$

Hence, $f(x)$ is an increasing function and as $f(g) = 0,\ f(x) > 0$ for $x > g$.

9b i

$$\frac{d^2y}{dt^2} = -(g + kv) = \frac{dv}{dt}$$

$$\int dt = -\int \frac{dv}{g + kv}$$

$$t = -\frac{1}{k}\ln(g + kv) + C$$

At $t = 0,\ v = V_0$.

$C = \frac{1}{k}\ln g + kV_0$

Therefore

$$t = \frac{1}{k}\ln\left(\frac{g + kV_0}{g + kv}\right)$$

$$g + kv = (g + kV_0)e^{-kt}$$

$$v = \frac{(g + kV_0)e^{-kt} - g}{k} = \frac{dy}{dt}$$

Uncorrected proofs

$$\int dy = \frac{1}{k}\int (g + kV_0)e^{-kt}\,dt - \frac{g}{k}\int dt$$

$$y = -\frac{1}{k^2}(g + kV_0)e^{-kt} - \frac{g}{k}t + D$$

At $t = 0, y = 0$.

$$D = \frac{1}{k^2}(g + kV_0)$$

Therefore

$$y = \frac{1}{k^2}(g + kV_0)(1 - e^{-kt}) - \frac{g}{k}t$$

$$k^2y = (g + kV_0)(1 - e^{-kt}) - gkt$$

9b ii At maximum height, $v = 0$.

$$t = \frac{1}{k}\ln\left(\frac{g + kV_0}{g + kv}\right)$$

$$T = \frac{1}{k}\ln\left(\frac{g + kV_0}{g + 0}\right)$$

$$T = \frac{1}{k}\ln\left(\frac{g + kV_0}{g}\right)$$

9b iii $t = 2T$

Substituting in the equation of part a i:

$$k^2y = (g + kV_0)(1 - e^{-kt}) - gkt$$

$$k^2y = (g + kV_0)\left(1 - \left(\frac{g}{g + kV_0}\right)^2\right) - gk\left(\frac{2}{k}\ln\frac{g + kV_0}{g}\right)$$

$$k^2y = g + kV_0 - \frac{g^2}{g + kV_0} - 2g\ln\frac{(g + kV_0)}{g}$$

9biv Let $(g + kV_0) = X$

From part a,

$$f(X) = X - \frac{g^2}{X} - 2g\ln\frac{X}{g}$$

Uncorrected proofs

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As proven in part a, $f(X)$ is an increasing function and $f(X) > 0$ for $X > g$.

For times $t = T$ and $t = 2T$, $f(X)$ is continuously increasing at a rate of $\left(1 - \frac{g}{X}\right)^2$. That is, faster than linear increase.

Hence, the interval of $(2T - T)$ is greater than $(T - 0)$.

So, the downward journey will take longer than the upward journey.

10a i Object of mass 1 kg dropped under gravity:

The forces acting on the object are gravity and air resistance.

Gravity pulls the object downwards while the air resistance resists the motion of the object. So, the equation of motion is:

$$\ddot{x} = g - kv$$

$$\ddot{x} = 10 - \frac{1}{10}v$$

10a ii

$$\ddot{x} = v\frac{dv}{dx} = 10 - \frac{1}{10}v$$

$$\frac{dv}{dx} = \frac{100 - v}{10v}$$

$$\int dx = 10\int \frac{v}{100 - v}dv$$

$$= 10\int \left(\frac{100}{100 - v} - 1\right)dv$$

$$x = -1000\ln(100 - v) - 10v + C$$

At $t = 0,\ v = 0,\ x = 0$.

$$C = 1000\ln 100$$

Therefore

$$x = -1000\ln\frac{100 - v}{100} - v$$

The speed of object when it is 40 m below the lookout is V.

$x = 40$ and $v = V$

Uncorrected proofs

$$40 = -1000 \ln\left(1 - \frac{V}{100}\right) - 10V$$

$$V + 100 \ln\left(1 - \frac{V}{100}\right) + 4 = 0$$

10b i After the parachute opens, the new equation of motion will be:

$$\ddot{x} = (g - kv^2)$$

$$= \frac{100 - v^2}{10}$$

$$= 10 - \frac{1}{10}v^2$$

10b ii

$$\ddot{x} = (g - kv^2) = \frac{100 - v^2}{10} = v\frac{dv}{dx}$$

$$\int dx = 5\int \frac{2v}{100 - v^2} dx$$

$$x = -5\ln(100 - v^2) + C$$

At the start of this second journey, $x = 0$ and $v = V$.

$$C = 5\ln(100 - V^2)$$

Therefore

$$x = 5\ln\left(\frac{100 - V^2}{100 - v^2}\right)$$

$$e^{\frac{x}{5}} = \frac{100 - V^2}{100 - v^2}$$

$$v^2 = 100 - (100 - V^2)e^{-\frac{x}{5}}$$

Uncorrected proofs

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10b iii

$$\ddot{x} = (g - kv^2) = \frac{100 - v^2}{10} = \frac{dv}{dt}$$

$$\int dt = 10 \int \frac{dv}{100 - v^2}$$

$$= 10 \int \frac{1}{(10 + v)(10 - v)} dv$$

$$= \frac{1}{2} \int \left(\frac{1}{10 + v} + \frac{1}{10 - v} \right) dv$$

$$t = \frac{1}{2}(\ln(10 + v) - \ln(10 - v)) + C$$

At the start of this new journey, $t = 0,\ v = V$.

$$C = -\frac{1}{2}(\ln(10 + V) - \ln(10 - V))$$

Therefore

$$t = \frac{1}{2} \ln \left(\frac{10 + v}{10 - v} \right) - \frac{1}{2} \ln \left(\frac{10 + V}{10 - V} \right)$$

$$= \frac{1}{2} \ln \frac{(10 + v)(10 - V)}{(10 - v)(10 + V)}$$

10b iv $V = 25.7$ m/s

The terminal velocity after the parachute opens is:

$$V_t = \sqrt{\frac{g}{k}} = \sqrt{100} = 10 \text{ m/s}$$

$$105\% \text{ of } V_t = \frac{105}{100} \times 10 = 10.5 \text{ m/s}$$

Using equation from part b iii with $v = 10.5$ and $V = 25.7$:

$$t = \frac{1}{2} \ln \frac{(10 + v)(10 - V)}{(10 - v)(10 + V)}$$

$$\doteqdot 1.4819$$

Uncorrected proofs

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11a $V = 30, \quad \theta = 45°, \quad k = \frac{1}{3}, \quad g = 10$

$\frac{d^2x}{dt^2} = -k\frac{dx}{dt}$ and $\frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}$

$\frac{dv}{dt} = -kv$

$\int dt = -\frac{1}{k}\int \frac{dv}{v}$

$t = -\frac{1}{k}\ln v + C$

At $t = 0, \quad v = V\cos 45° = 15\sqrt{2}$ m/s $= v_x$.

$C = \frac{1}{k}\ln v_x$

Therefore

$t = -\frac{1}{k}\ln v + \frac{1}{k}\ln v_x$

$t = \frac{1}{k}\ln\frac{v_x}{v}$

$v = v_x e^{-kt} = \frac{dx}{dt}$

$\int dx = v_x \int e^{-kt}\, dt$

$x = -\frac{1}{k}v_x e^{-kt} + D$

At $t = 0, \quad x = 0.$

$D = \frac{1}{k}v_x$

Therefore

$x = -\frac{1}{k}v_x e^{-kt} + \frac{1}{k}v_x$

$= \frac{1}{k}v_x(1 - e^{-kt})$

$= 45\sqrt{2}\left(1 - e^{-\frac{t}{3}}\right)$

Uncorrected proofs

11b

$$\frac{dv_y}{dt} = -(g + kv_y)$$

$$\frac{dv}{dt} = -(g + kv)$$

$$\int dt = -\int \frac{dv}{g + kv}$$

$$t = -\frac{1}{k}\ln(g + kv) + C$$

At $t = 0$, $v = V\sin 45° = v_y = 15\sqrt{2}$.

$$C = \frac{1}{k}\ln(g + kv_y)$$

Therefore

$$t = -\frac{1}{k}\ln(g + kv) + \frac{1}{k}\ln(g + kv_y)$$

$$t = \frac{1}{k}\ln\left(\frac{g + kv_y}{g + kv}\right)$$

$$v = \frac{(g + kv_y)e^{-kt} - g}{k} = \frac{dy}{dt}$$

$$\int dy = \frac{1}{k}\int (g + kv_y)e^{-kt}\,dt - \frac{g}{k}\int dt$$

$$y = -\frac{1}{k^2}(g + kv_y)e^{-kt} - \frac{g}{k}t + D$$

At $t = 0$, $y = 0$.

$$D = \frac{1}{k^2}(g + kv_y)$$

Therefore

$$y = -\frac{1}{k^2}(g + kv_y)e^{-kt} - \frac{g}{k}t + \frac{1}{k^2}(g + kv_y)$$

$$y = \frac{1}{k^2}(g + kv_y)(1 - e^{-kt}) - \frac{g}{k}t$$

$$y = 9\left(10 + 5\sqrt{2}\right)(1 - e^{-kt}) - 30t$$

$$y = 45\left(2 + \sqrt{2}\right)(1 - e^{-kt}) - 30t$$

Uncorrected proofs

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$$y = 45(2+\sqrt{2})\left(1-e^{-\frac{t}{3}}\right) - 30t \qquad \text{as } k = \frac{1}{3}$$

11c $y = 45(2+\sqrt{2})\left(1-e^{-\frac{t}{3}}\right) - 30t$ and

$$x = 45\sqrt{2}\left(1-e^{-\frac{t}{3}}\right)$$

$$t = 3\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

Substituting for t in the equation for y:

$$y = 45(2+\sqrt{2})\left(1-e^{-\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)}\right) - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

$$y = 45(2+\sqrt{2})\left(1-e^{\ln\left(\frac{45\sqrt{2}-x}{45\sqrt{2}}\right)}\right) - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

$$y = 45(2+\sqrt{2})\left(1-\frac{45\sqrt{2}-x}{45\sqrt{2}}\right) - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

$$y = 45(2+\sqrt{2})\left(1-\left(1-\frac{x}{45\sqrt{2}}\right)\right) - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

$$y = 45(2+\sqrt{2})\frac{x}{45\sqrt{2}} - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

$$y = 45\sqrt{2}(\sqrt{2}+1)\frac{x}{45\sqrt{2}} - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

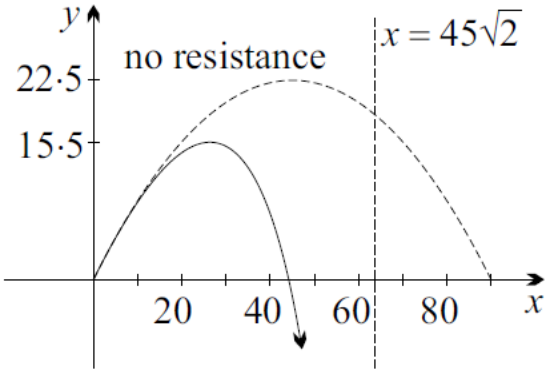
$$y = (1+\sqrt{2})x - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$

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11d Graph of

$$y = (1+\sqrt{2})x - 90\ln\left(\frac{45\sqrt{2}}{45\sqrt{2}-x}\right)$$



The range from the plot is approximately 44 m.

11e When there is no air resistance:

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

$$x = V\cos 45°\ t \quad \text{and} \quad y = V\sin 45°\ t - \frac{1}{2}gt^2$$

$$x = 15\sqrt{2}t \quad \text{and} \quad y = 15\sqrt{2}\,t - \frac{1}{2}gt^2$$

$$t = \frac{x}{15\sqrt{2}}$$

Substituting for t in equation for y:

$$y = 15\sqrt{2} \times \frac{x}{15\sqrt{2}} - \frac{1}{2} \times 10 \times \left(\frac{x}{15\sqrt{2}}\right)^2$$

$$y = x - \frac{5x^2}{450}$$

$$y = x - \frac{x^2}{90}$$

$$y = x\left(1 - \frac{x}{90}\right)$$

Uncorrected proofs

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6E

(12) $\ddot{y} = -(g + kv)$

When $\ddot{y} = 0$, $v = V_T$

so $0 = -g - kV_T$

so $V_T = \frac{-g}{k}$, as required.

From (9)(b), $t = \frac{1}{k} \ln\left(\frac{g + kV_0}{g + kv}\right)$ (1)

from which $v = \frac{g + kV_0}{k} e^{-kt} - \frac{g}{k}$. (2)

Then, after integrating with respect to t, the height is given by $y_h = \frac{g + kV_0}{k^2}(1 - e^{-kt}) - \frac{g}{k}t$. (3)

(See (9)(b)(i))

From (9)(b)(ii), the time T to reach the maximum height is $T = \frac{1}{k} \ln\left(\frac{g + kV_0}{g}\right)$. (4)

If, however, we consider the downwards motion separately, as in Worked Example 18, we have $y = 0$ as the starting point, downwards as the positive direction, and so y is the distance fallen, which we will call y_f.

Note that $t = T$ is the initial value of T.

From part (c), replacing t with $t - T$,

$$y_f = \frac{g}{k}\left((t - T) + \frac{1}{k}e^{-k(t-T)} - \frac{1}{k}\right)$$

$$= \frac{g}{k}(t - T) + \frac{g}{k^2} \cdot e^{kT} \cdot e^{-kt} - \frac{g}{k^2}.$$

From (4), $T = \frac{1}{k} \ln\left(\frac{g + kV_0}{g}\right)$, so $e^{kT} = \frac{g + kV_0}{g}$.

So $y_f = \frac{g}{k}t - \frac{g}{k^2} \ln\left(\frac{g + kV_0}{g}\right) + \frac{g + kV_0}{k^2} e^{-kt} - \frac{g}{k^2}$. (5)

6E

(12) (continued)

If the two approaches are consistent, adding y_h and y_f will give the maximum height H.

First find H.

When $t = T$, $y_h = H$.

So, substituting into (3),

$$H = \frac{g+kV_0}{k^2}\left(1 - e^{-kT}\right) - \frac{g}{k}T \qquad \left(\text{where } T = \frac{1}{k}\ln\left(\frac{g+kV_0}{g}\right)\right)$$

$$= \frac{g+kV_0}{k^2}\left(1 - \frac{g}{g+kV_0}\right) - \frac{g}{k^2}\ln\left(\frac{g+kV_0}{g}\right)$$

so $$H = \frac{V_0}{k} - \frac{g}{k^2}\ln\left(\frac{g+kV_0}{g}\right) \quad (6)$$

Finally, adding (3) and (5), for $t \geq T$, gives

$$y_h + y_f = \frac{g+kV_0}{k^2}\left(1 - e^{-kt}\right) - \frac{g}{k}t$$

$$+ \frac{g}{k}t - \frac{g}{k^2}\ln\left(\frac{g+kV_0}{g}\right) + \frac{g+kV_0}{k^2}e^{-kt} - \frac{g}{k^2}$$

$$= \cancel{\frac{g}{k^2}} - \cancel{\frac{g}{k^2}e^{-kt}} + \frac{V_0}{k} - \cancel{\frac{V_0}{k}e^{-kt}} - \cancel{\frac{g}{k}t}$$

$$+ \cancel{\frac{g}{k}t} - \frac{g}{k^2}\ln\left(\frac{g+kV_0}{g}\right) + \cancel{\frac{g}{k^2}e^{-kt}} + \cancel{\frac{V_0}{k}e^{-kt}} - \cancel{\frac{g}{k^2}}$$

$$= \frac{V_0}{k} - \frac{g}{k^2}\ln\left(\frac{g+kV_0}{g}\right)$$

$$= H.$$

So the two approaches are consistent.

6E

(13)(a)(i) $\frac{du}{dt} \approx \frac{u(t+\delta t)-u(t)}{\delta t}$

so $u(t+\delta t) \approx \delta t \cdot \dot{u} + u(t)$

$= u(t) + \delta t\left(-k(u^2+v^2)^{\frac{1}{2}} \cdot u(t)\right)$

$= u(t)\left(1 - k \cdot \delta t (u^2+v^2)^{\frac{1}{2}}\right)$

(ii) $\frac{dx}{dt} \approx \frac{x(t+\delta t)-x(t)}{\delta t}$

so $x(t+\delta t) = \delta t \cdot \dot{x} + x(t)$

$= \delta t \cdot u(t) + x(t)$

Also, $\frac{dy}{dt} \approx \frac{y(t+\delta t)-y(t)}{\delta t}$

so $y(t+\delta t) \approx \delta t \cdot \dot{y} + y(t)$

$= \delta t \cdot v(t) + y(t)$

Finally, $\frac{dv}{dt} \approx \frac{v(t+\delta t)-v(t)}{\delta t}$

so $v(t+\delta t) \approx \delta t \cdot \dot{v} + v(t)$

$= v(t) - \delta t\left(g + k(u^2+v^2)^{\frac{1}{2}} \cdot v(t)\right)$

$= v(t)\left(1 - k \cdot \delta t (u^2+v^2)^{\frac{1}{2}}\right) - g \cdot \delta t$

(b)(i) $u(0.1) \approx u(0)\left(1 - \frac{1}{90}(0.1)\left(u(0)^2+v(0)^2\right)^{\frac{1}{2}}\right)$

$= 15\sqrt{2}\left(1 - \frac{1}{900}(450+450)^{\frac{1}{2}}\right)$

$= 15\sqrt{2}\left(1 - \frac{1}{30}\right)$

≈ 20.51

The calculations in (b)(ii) and (c) are similar.

(d) See the answers at the back of the textbook.