Solutions to Exercise 6D Foundation questions

1a Let direction of motion be positive, mass $= M$ kg.

So braking force $= -\frac{2}{3}M$ N, hence

braking acceleration $= \frac{-\frac{2}{3}M}{M} = -\frac{2}{3}$ m/s^2

and resistive force $= -\frac{1}{180}Mv^2$ N, hence

resistive acceleration $= \frac{-\frac{1}{180}Mv^2}{M} = -\frac{1}{180}v^2$ m/s^2

Therefore

$$\ddot{x} = -\frac{2}{3} - \frac{1}{180}v^2$$

$$\frac{dv}{dt} = -\frac{2}{3} - \frac{1}{180}v^2$$

$$= \frac{-120 - v^2}{180}$$

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{-120 - v^2}{180v}$$

$$\frac{dx}{dv} = \frac{180v}{-120 - v^2}$$

$$= \frac{-90 \times 2v}{120 + v^2}$$

$$x = -90\ln(120 + v^2) + C \qquad \text{(since } 120 + v^2 > 0\text{)}$$

When $x = 0, v = 288$ km/h $= 80$ m/s.

$$0 = -90\ln(120 + 80^2) + C$$

$$C = 90\ln(120 + 80^2)$$

Therefore

$$x = -90\ln(120 + v^2) + 90\ln(120 + 80^2)$$

Uncorrected proofs

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$$= 90 \ln\left(\frac{120 + 80^2}{120 + v^2}\right)$$

1b When $v = 0$,

$$x = 90 \ln\left(\frac{120 + 80^2}{120 + 0^2}\right)$$

$= 360$ metres

Therefore the car travels 360 metres to bring it to rest.

2a Let direction of motion be positive, mass $= 10\,000$ kg.

So propelling force $= 10\,000$ N, hence

propelling acceleration $= \frac{10\,000}{10\,000} = 1$ m/s^2

and resistive force $= -100v^2$ N, hence

resistive acceleration $= \frac{-100v^2}{10\,000} = -\frac{1}{100}v^2$ m/s^2

Therefore

$$\ddot{x} = 1 - \frac{1}{100}v^2$$

Maximum speed is when $\ddot{x} = 0$.

$$1 - \frac{1}{100}v^2 = 0$$

$$v^2 = 100$$

$$|v| = 10 \text{ m/s}$$

Hence the maximum speed is 10 m/s or 36 km/h.

2b From part a:

$$\frac{dv}{dt} = 1 - \frac{1}{100}v^2$$

$$= \frac{100 - v^2}{100}$$

Uncorrected proofs

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{100 - v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{100 - v^2}$$

$$= \frac{-50 \times -2v}{100 - v^2}$$

$x = -50\ln(100 - v^2) + C$ (since $|v| < 10$ m/s so $100 - v^2 > 0$)

When $x = 0, v = 0$ m/s.

$0 = -50\ln 100 + C$

$C = 50\ln 100$

Therefore

$x = -50\ln(100 - v^2) + 50\ln 100$

$= 50\ln\left(\frac{100}{100 - v^2}\right)$

2c When $x = 50$ m,

$50 = 50\ln\left(\frac{100}{100 - v^2}\right)$

$\ln\left(\frac{100}{100 - v^2}\right) = 1$

$\frac{100}{100 - v^2} = e^1$

$100 = e^1(100 - v^2)$

$\frac{100}{e} = 100 - v^2$

$v^2 = 100 - \frac{100}{e}$

$|v| = \sqrt{100 - 100e^{-1}}$ m/s

Percentage of maximum speed

$= \frac{\sqrt{100 - 100e^{-1}}}{10} \times 100\%$

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$= 79.506 \ldots \%$

$\div 80\%$

After 50 m, the monorail is travelling at a speed that is about 80% of its maximum speed.

3a Let direction of motion be positive, mass $= m$ kg.

So propelling force $= \frac{P}{v}$ N, hence

propelling acceleration $= \frac{\frac{P}{v}}{m} = \frac{P}{mv}$ m/s^2

and resistive force $= -kv^2$ N, hence

resistive acceleration $= \frac{-kv^2}{m}$ m/s^2

Therefore

$$\ddot{x} = \frac{P}{mv} - \frac{k}{m}v^2$$

$$\ddot{x} = \frac{P - kv^3}{mv}$$

Maximum speed of u m/s occurs when $\ddot{x} = 0$.

$$0 = \frac{P - ku^3}{mu}$$

$$P - ku^3 = 0$$

$$k = \frac{P}{u^3}$$

3b From part a:

$$\ddot{x} = \frac{dv}{dt} = \frac{P - kv^3}{mv}$$

$$\frac{dv}{dt} = \frac{P - \frac{P}{u^3}v^3}{mv}$$

$$= \frac{Pu^3 - Pv^3}{mu^3v}$$

Uncorrected proofs

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{Pu^3 - Pv^3}{mu^3v^2}$$

$$= \frac{P}{m}\left(\frac{u^3 - v^3}{u^3v^2}\right)$$

$$= \frac{P}{m}\left(\frac{u^3}{u^3v^2} - \frac{v^3}{u^3v^2}\right)$$

$$= \frac{P}{m}\left(\frac{1}{v^2} - \frac{1}{u^3}\right)$$

3c From part b:

$$\frac{dv}{dx} = \frac{Pu^3 - Pv^3}{mu^3v^2}$$

$$\frac{dx}{dv} = \frac{mu^3v^2}{Pu^3 - Pv^3}$$

$$= \frac{-\frac{1}{3P}mu^3 \times -3Pv^2}{Pu^3 - Pv^3}$$

$$x = -\frac{1}{3P}mu^3 \ln(Pu^3 - Pv^3) + C \qquad (\text{since } u > v \text{ so } Pu^3 - Pv^3 > 0)$$

Distance travelled when speed changes from $\frac{1}{3}u$ to $\frac{2}{3}u$

$$= \left(-\frac{1}{3P}mu^3 \ln\left(Pu^3 - P\left(\frac{2}{3}u\right)^3\right) + C\right) - \left(-\frac{1}{3P}mu^3 \ln\left(Pu^3 - P\left(\frac{1}{3}u\right)^3\right) + C\right)$$

$$= -\frac{1}{3P}mu^3 \ln\left(Pu^3 - \frac{8}{27}Pu^3\right) + \frac{1}{3P}mu^3 \ln\left(Pu^3 - \frac{1}{27}Pu^3\right)$$

$$= \frac{1}{3P}mu^3 \ln\left(\frac{Pu^3 - \frac{1}{27}Pu^3}{Pu^3 - \frac{8}{27}Pu^3}\right)$$

$$= \frac{1}{3P}mu^3 \ln\left(\frac{1 - \frac{1}{27}}{1 - \frac{8}{27}}\right)$$

$$= \frac{1}{3P}mu^3 \ln\left(\frac{\frac{26}{27}}{\frac{19}{27}}\right)$$

Uncorrected proofs

$$= \frac{1}{3P} mu^3 \ln \frac{26}{19}$$

$$= \frac{mu^3 \ln \frac{26}{19}}{3P} \text{ metres}$$

3d

Maximum braking force $= -B$ N, hence

maximum braking acceleration $= -\frac{B}{m}$ m/s^2

and resistive force $= -kv^2 = -\frac{Pv^2}{u^3}$ N, hence

resistive acceleration $= -\frac{Pv^2}{mu^3}$ m/s^2

Therefore

$$\ddot{x} = -\frac{B}{m} - \frac{Pv^2}{mu^3}$$

$$\ddot{x} = \frac{-Bu^3 - Pv^2}{mu^3}$$

$$\frac{dv}{dt} = \frac{-Bu^3 - Pv^2}{mu^3}$$

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{-Bu^3 - Pv^2}{mu^3 v}$$

$$\frac{dx}{dv} = \frac{mu^3 v}{-Bu^3 - Pv^2}$$

$$= \frac{-mu^3 v}{Bu^3 + Pv^2}$$

$$= \frac{-\frac{1}{2P} mu^3 \times 2Pv}{Bu^3 + Pv^2}$$

$$x = -\frac{1}{2P} mu^3 \ln(Bu^3 + Pv^2) + C \qquad (\text{since } Bu^3 + Pv^2 > 0)$$

Minimum distance travelled when speed changes from u to 0

Uncorrected proofs

$$= \left(-\frac{1}{2P}mu^3 \ln(Bu^3 + 0) + C\right) - \left(-\frac{1}{2P}mu^3 \ln(Bu^3 + Pu^2) + C\right)$$

$$= -\frac{1}{2P}mu^3 \ln(Bu^3) + \frac{1}{2P}mu^3 \ln(Bu^3 + Pu^2)$$

$$= \frac{1}{2P}mu^3 \ln\left(\frac{Bu^3 + Pu^2}{Bu^3}\right)$$

$$= \frac{1}{2P}mu^3 \ln\left(1 + \frac{Pu^2}{Bu^3}\right)$$

$$= \frac{mu^3}{2P}\ln\left(1 + \frac{P}{Bu}\right) \text{ metres}$$

Uncorrected proofs

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Solutions to Exercise 6D Development questions

4a The negative sign of the forces indicates that the forces are acting in the direction opposite to the actual motion of the parts. The force in the spring is resistive, i.e., resisting the motion completely. The force in the dashpot is viscous force of resistance which helps slowing the motion down.

4b The forces acting on the door are:

$m\ddot{y} = -3m\dot{y} - 2my$

$\ddot{y} + 3\dot{y} + 2y = 0$

4c $y = f(t)$ and $y = g(t)$ are solutions of $\ddot{y} + 3\dot{y} + 2y = 0$.

$f''(t) + 3f'(t) + 2f(t) = 0 \quad (1)$

$g''(t) + 3g'(t) + 2g(t) = 0 \quad (2)$

$y = Af(t) + Bg(t)$ where A, B are constants

$\dot{y} = Af'(t) + Bg'(t)$

$\ddot{y} = Af''(t) + Bg''(t)$

Substituting the above values of y and its derivatives into $\ddot{y} + 3\dot{y} + 2y$:

$\ddot{y} + 3\dot{y} + 2y$

$= Af''(t) + Bg''(t) + 3\big(Af'(t) + Bg'(t)\big) + 2\big(Af(t) + Bg(t)\big)$

$= A\big(f''(t) + 3f'(t) + 2f(t)\big) + B\big(g''(t) + 3g'(t) + 2g(t)\big)$

$= A(0) + B(0)$ from (1) and (2)

$= 0$

Hence proved.

4d $y = e^{kt}$ is a solution of the differential equation.

$\dot{y} = ke^{kt}$ and $\ddot{y} = k^2e^{kt}$

Substituting the above values of y and its derivatives in the equation of motion:

$\ddot{y} + 3\dot{y} + 2y = 0$

$k^2e^{kt} + 3ke^{kt} + 2e^{kt} = 0$

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$(k^2 + 3k + 2)e^{kt} = 0$

$(k + 1)(k + 2)e^{kt} = 0$

$k = -1 \text{ or } -2$

Possible values of the constant k are $k = -1$ and -2.

4e At $t = 0$, $y = 0$ and $\frac{dy}{dt} = 1$

$y = Ae^{-2t} + Be^{-t}$

$\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t}$

At $t = 0$,

$0 = A(e^0) + B(e^0) => A = -B$

$1 = -2A - B$

$=> 1 = -2A + A$

$A = -1 \quad \& \quad B = 1$

Therefore

$y = e^{-t} - e^{-2t}$

5a The driving force is: F

Resistive force is: kv^2

$m\ddot{x} = (F - kv^2)$ … denotes the motion of the submarine or the net force on the submarine

Now, $\ddot{x} = \frac{dv}{dt}$

Hence,

$\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$

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5b

$$\frac{dv}{dt} = \frac{vdv}{dx} = vv' \qquad (1)$$

$$\int \frac{vv'}{a+bv^2}dx = \frac{1}{2b}\log(a+bv^2) + C \qquad (2)$$

Using (1) and (2):

$$vv' = \frac{1}{m}(F - kv^2)$$

Integrate both sides w.r.t x between the limits v_1 and v_2

$$\int_{v_1}^{v_2} \frac{vv'}{F-kv^2}dx = \frac{1}{m}\int dx$$

$$\left[-\frac{1}{2k}\ln(F-kv^2)\right]_{v_1}^{v_2} = \frac{x}{m}$$

$$x = \frac{m}{2k}[-\ln(F-kv_2^2) - (-\ln(F-kv_1^2))]$$

$$x = \frac{m}{2k}\ln\frac{F-kv_1^2}{F-kv_2^2}$$

6a $\quad m = 1; \quad f_r = v + v^3; \quad v_0 = Q$

Since, there is only one resistive force: $m\ddot{x} = -(v+v^3)$

$$\frac{dv}{dt} = -(v+v^3)$$

$$\int dt = -\int_Q^v \frac{dv}{v+v^3}$$

$$\frac{1}{v+v^3} = \frac{1}{v} - \frac{1}{2}\left(\frac{2v}{1+v^2}\right)$$

$$\int dt = -\int_Q^v \left(\frac{1}{v} - \frac{1}{2}\left(\frac{2v}{1+v^2}\right)\right)dv$$

$$t = -\left[\ln v - \frac{1}{2}\ln(1+v^2)\right]_Q^v$$

$$= -\left[\ln\frac{v}{Q} - \frac{1}{2}\ln\frac{1+v^2}{1+Q^2}\right]$$

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$$= -\left[\frac{1}{2}\ln\frac{v^2}{Q^2} - \frac{1}{2}\ln\frac{1+v^2}{1+Q^2}\right]$$

$$= \frac{1}{2}\ln\left(\frac{Q^2}{v^2} \times \frac{1+v^2}{1+Q^2}\right)$$

6b $t = \frac{1}{2}\ln\left(\frac{Q^2}{v^2} \times \frac{1+v^2}{1+Q^2}\right)$

$$\frac{1+v^2}{v^2} = \frac{1+Q^2}{Q^2}e^{2t}$$

$$\frac{v^2}{1+v^2} = \frac{Q^2}{1+Q^2}e^{-2t}$$

$$1 - \frac{1}{1+v^2} = \frac{Q^2}{1+Q^2}e^{-2t}$$

$$1+v^2 = \frac{1}{1-\frac{Q^2}{1+Q^2}e^{-2t}}$$

$$v^2 = \frac{\frac{Q^2}{1+Q^2}e^{-2t}}{1-\frac{Q^2}{1+Q^2}e^{-2t}} = \frac{\left(\frac{Q^2}{1+Q^2}\right)}{\frac{e^{2t}(1+Q^2)-Q^2}{1+Q^2}}$$

$$v^2 = \frac{Q^2}{e^{2t}(1+Q^2)-Q^2}$$

6c $v^2 = \frac{Q^2}{e^{2t}(1+Q^2)-Q^2}$

Therefore $v^2 \propto e^{-2t}$

As $t \to \infty$, $\quad v^2 \to 0$

$Q > 0$ initially; which means that the speed decreases from some positive value to zero with passage of time.

Hence, v is always positive.

Uncorrected proofs

6d $\frac{dv}{dt} = -(v + v^3) = v\frac{dv}{dx}$

$v\frac{dv}{dx} = -v(1 + v^2)$

$\int dx = -\int_Q^v \frac{dv}{1 + v^2}$

Substituting $v = \tan a$

$dv = \sec^2 a\, da$ and $(1 + \tan^2 b) = \sec^2 b$

$x = -[\tan^{-1} v]_Q^v$

$x = -[\tan^{-1} v - \tan^{-1} Q]$

$x = \tan^{-1} Q - \tan^{-1} v$

$\lim_{t\to\infty} x = \lim_{t\to\infty}(\tan^{-1} Q - \tan^{-1} v)$

$\lim_{t\to\infty} x = \tan^{-1} Q - \lim_{t\to\infty}\left(\tan^{-1}\sqrt{\frac{Q^2}{e^{2t}(1+Q^2)-Q^2}}\right)$

$\lim_{t\to\infty} x = \tan^{-1} Q - \lim_{t\to\infty} \tan^{-1} 0$

$\lim_{t\to\infty} x = \tan^{-1} Q$

6e $x = \tan^{-1} Q - \tan^{-1} v$

Let $\tan^{-1} Q = q$ and $\tan^{-1} v = r$

So, $Q = \tan q$ and $v = \tan r$

$\tan(q - r) = \frac{\tan q - \tan r}{1 + \tan q \tan r}$

$\tan(q - r) = \frac{Q - v}{1 + Qv}$

$(q - r) = \tan^{-1}\frac{Q - v}{1 + Qv}$

$x = \tan^{-1} Q - \tan^{-1} v$

$= \tan^{-1}\frac{Q - v}{1 + Qv}$

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7a

$$\frac{1}{(2-v)(3+v)} = \frac{A}{2-v} + \frac{B}{3+v}$$

$$(3A + Av) + (2B - Bv) = 1$$

$$3A + 2B = 1$$

$$(A - B)v = 0$$

$$A = B$$

$$5A = 1$$

$$A = B = \frac{1}{5}$$

7b i

$$\frac{dv}{dt} = \frac{10^4}{m}(6 - v - kv^2); \quad m = 4.5 \times 10^6; \quad v_{max} = 2; \quad F = 10^4(6 - v)$$

$$\ddot{x} = \frac{10^4}{m}(6 - v - kv^2)$$

When the speed reaches its maximum value, the acceleration is zero.

$$0 = \frac{10^4}{m}(6 - v_{max} - kv_{max}^2)$$

$$6 - 2 - 4k = 0$$

$$k = 1$$

7b ii

$$\frac{dv}{dt} = \frac{10^4}{m}(6 - v - v^2)$$

$$\frac{10^4}{m}\int dt = \int \frac{dv}{6 - v - v^2} = \frac{1}{5}\int \left(\frac{1}{2-v} + \frac{1}{3+v}\right) dv$$

$$\frac{10^4}{m}t = \frac{1}{5}\ln\frac{3+v}{2-v} + C$$

At $t = 0$, $v = 0$.

So $C = -\frac{1}{5}\ln\frac{3}{2}$

Uncorrected proofs

At $v = 1.5$,

$$t = \frac{m}{5 \times 10^4}\left(\ln\left(\frac{3 + 1.5}{2 - 1.5}\right) - \ln\frac{3}{2}\right)$$

$t = 161.25$ seconds

$t = 2$ minutes and 41 seconds

8a For the first 3 seconds, the equation of motion is:

$M\ddot{x} = -2M$

$\ddot{x} = -2\ \text{m/s}^2$

$v_{landing} = 72\text{m/s}^2$

Using: $v = u + at$ and $x = ut + \frac{1}{2}at^2$ at $t = 3$

$v_3 = v_{landing} + \ddot{x}t$

$= 72 - 3 \times 2$

$= 66\ \text{m/s}$

$x_3 = 72 \times 3 - \frac{1}{2} \times 2 \times 3^2$

$= 207\text{m}$

8b After the first 3 seconds, the equation of motion is:

$$\ddot{x} = v\frac{dv}{dx} = -2 - \frac{v^2}{10^4}$$

$$dx = -\frac{10^4 v}{20000 + v^2}dv$$

$$\int dx = -5000\int \frac{2v}{20000 + v^2}dv$$

$$x = -5000[\ln(20000 + v^2)] + C$$

At $t = 3$, $x = 207$ and $v = 66$.

$$207 = -5000\ln(20000 + 66^2) + C$$

$$C = 207 + 5000\ln(20000 + 66^2)$$

Therefore

Uncorrected proofs

$$x = 207 + 5000 \ln\left(\frac{20000 + 66^2}{20000 + v^2}\right)$$

8c

$$x = 207 + 5000 \ln\left(\frac{20000 + 66^2}{20000 + v^2}\right)$$

This equation is valid till the limit of $v = 36$

$$x = 207 + 5000 \ln\left(\frac{20000 + 66^2}{20000 + 36^2}\right)$$

$$= 207 + 671.3$$

$$= 878.3 \text{ m}$$

8d Equation of motion after the speed has reduced to 36 is:

$$\ddot{x} = -2 = v\frac{dv}{dx}$$

$$dx = -\frac{v}{2}dv$$

So, the distance covered by the plane till the speed reduces to 7 m/s is:

$$x = 878 + \int_{36}^{7}\left(-\frac{v}{2}\right)dv$$

$$= 878 + \int_{7}^{36}\frac{v}{2}dv$$

$$= 878 + 311.75$$

$$= 1190 \text{ m}$$

9a $F = mP; \quad f_r = mkv; \quad u = V_i$

Equation of the motion of the box:

$$m\ddot{x} = (F - f_r) = mP - mkv$$

$$a = P - kv$$

When net force is zero, $a = 0$.

$$P - kv = 0$$

Uncorrected proofs

$$v = \frac{P}{k} = V_0$$

9b

$$\frac{dv}{dt} = (P - kv)$$

$$dt = \frac{dv}{P - kv}$$

$$\int dt = \int \frac{dv}{P - kv}$$

$$t = -\frac{1}{k}\ln(P - kv) + C$$

$$= -\frac{1}{k}\ln\left(1 - \frac{v}{V_0}\right) + C$$

At $t = 0, \quad v = V_i$

$$C = \frac{1}{k}\ln(P - kV_i)$$

$$= \frac{1}{k}\ln\left(1 - \frac{V_i}{V_0}\right)$$

Therefore

$$t = -\frac{1}{k}\ln\frac{1 - \frac{v}{V_0}}{1 - \frac{V_i}{V_0}}$$

$$\left(1 - \frac{V_i}{V_0}\right)e^{-kt} = 1 - \frac{v}{V_0}$$

$$v = V_0\left(1 - \left(1 - \frac{V_i}{V_0}\right)e^{-kt}\right)$$

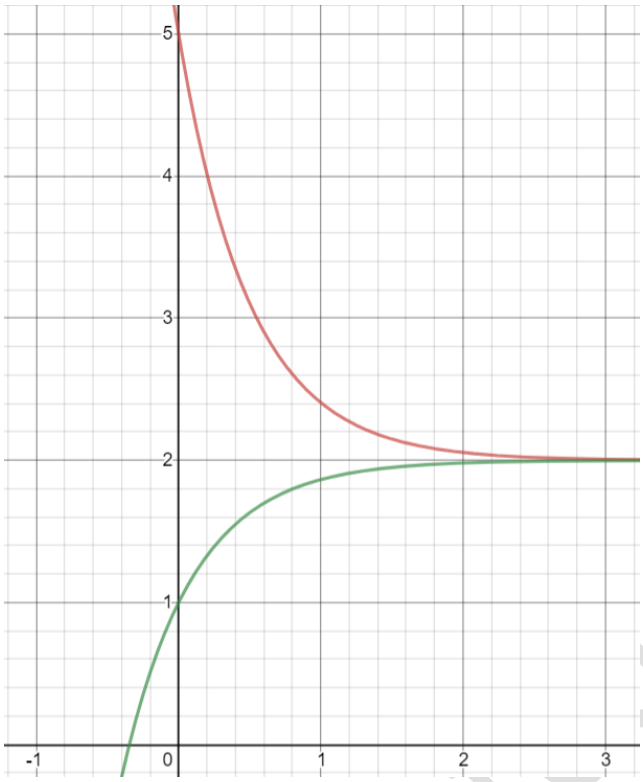
$$\lim_{t\to\infty} v = \lim_{t\to\infty} V_0\left(1 - \left(1 - \frac{V_i}{V_0}\right)e^{-kt}\right) = V_0(1 - 0)$$

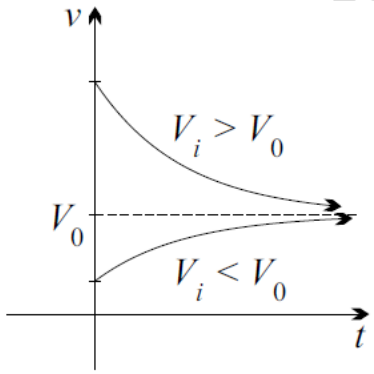
$$\lim_{t\to\infty} v = V_0$$

Uncorrected proofs

9c $V_i > V_0 =$ Red Curve; $V_i = 5$; $V_0 = 2$

$V_i < V_0 =$ Green Curve $V_i = 1$; $V_0 = 2$





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9d $v_1 = \frac{1}{3}V_0; \quad v_2 = \frac{2}{3}V_0$

$$\int dt = \int_{\frac{V_0}{3}}^{\frac{2V_0}{3}} \frac{dv}{P - kv}$$

$$t = -\frac{1}{k}[\ln(P - kv)]_{\frac{V_0}{3}}^{\frac{2V_0}{3}}$$

$$= \left[\frac{1}{k}\ln\left(1 - \frac{v}{V_0}\right)\right]_{\frac{2V_0}{3}}^{\frac{V_0}{3}}$$

$$= \frac{1}{k}\left(\ln\left(1 - \frac{1}{3}\right) - \ln\left(1 - \frac{2}{3}\right)\right)$$

$$= \frac{1}{k}\ln 2$$

10a

$$\frac{d^2x}{dt^2} = \frac{5 - 2x}{x^3}; \quad x_0 = 1; \quad v_0 = 0$$

$$\frac{d^2x}{dt^2} = \frac{5 - 2}{1} = 3\ (> 0\ ) \text{ at } x = 1$$

At $x = 1$ the acceleration is positive. So, the particle will start to move in positive x direction.

10b

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{5 - 2x}{x^3}$$

$$\frac{v^2}{2} = \int \frac{5 - 2x}{x^3}dx$$

$$v^2 = 2\int \frac{5 - 2x}{x^3}dx$$

$$= 10\int \frac{dx}{x^3} - 4\int \frac{dx}{x^2}$$

$$= -5x^{-2} + 4x^{-1} + C$$

$$= \frac{4}{x} - \frac{5}{x^2} + C$$

Uncorrected proofs

At $t = 0,\ x = 1$ and $v = 0$.

$0 = 4 - 5 + C$

$C = 1$

Therefore

$v^2 = \frac{4}{x} - \frac{5}{x^2} + 1$

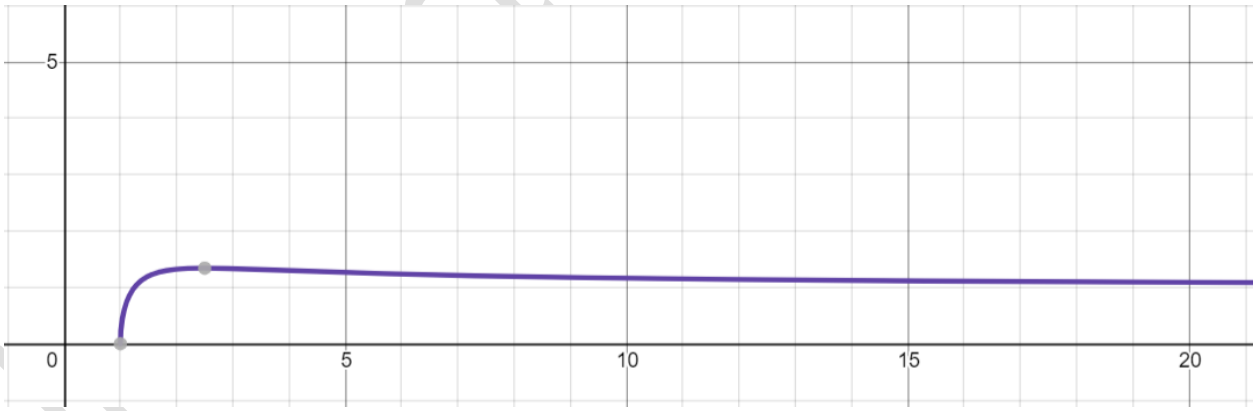
$v^2 = \frac{x^2 + 4x - 5}{x^2}$

$v = \pm\frac{\sqrt{x^2 + 4x - 5}}{x}$

$v = \frac{\sqrt{x^2 + 4x - 5}}{x}$ for $x \geq 1$

10c

Graph of $v = \frac{\sqrt{x^2 + 4x - 5}}{x}$. Consider $x \geq \frac{5}{2}$



As can be seen from the above graph, the velocity approaches 1 from above.

Unedited proofs

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11a i $x_0 = 0; \quad v_0 = V_0\ (> 0)$

$$v\frac{dv}{dx} = -kv^{\frac{3}{2}}$$

$$\int dx = -\frac{1}{k}\int_{V_0}^{v}\frac{dv}{\sqrt{v}}$$

$$x = -\left[\frac{1}{k}\times 2\sqrt{v}\right]_{V_0}^{v}$$

$$\frac{kx}{2} = \left[\sqrt{v}\right]_{v}^{V_0}$$

$$= \sqrt{V_0} - \sqrt{v}$$

$$= \left(1 - \sqrt{\frac{v}{V_0}}\right)\sqrt{V_0}$$

$$\sqrt{\frac{v}{V_0}} = 1 - \frac{kx}{2\sqrt{V_0}}$$

11a ii

$$\frac{kx}{2} = \sqrt{V_0} - \sqrt{v}$$

$$x_{max} = \frac{2\sqrt{V_0}}{k} \quad \text{and} \quad x_{min} = 0$$

$$0 \le x \le \frac{2\sqrt{V_0}}{k}$$

When particle stops moving, $v = 0$.

$$x = \frac{2\sqrt{V_0}}{k}$$

Uncorrected proofs

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11b i

$$\sqrt{\frac{v}{V_0}} = 1 - \frac{kx}{2\sqrt{V_0}} \quad \text{from part a}$$

$$v = \frac{dx}{dt} = V_0\left(1 - \frac{kx}{2\sqrt{V_0}}\right)^2$$

Let $\frac{k}{2\sqrt{V_0}} = C$

$$\frac{dx}{dt} = V_0(1 - Cx)^{-2}$$

$$\int dt = V_0 \int (1 - Cx)^{-2}\, dx$$

$$t = \frac{V_0}{C}(1 - Cx)^{-1} + D$$

At $t = 0, x = 0$, we get:

$$D = -\frac{V_0}{C}$$

$$\frac{C}{V_0}t = (1 - Cx)^{-1} - 1$$

$$\frac{C}{V_0}t = \frac{1}{1 - Cx} - 1$$

$$\frac{V_0 + Ct}{V_0} = \frac{1}{1 - Cx}$$

$$1 - Cx = \frac{V_0}{V_0 + Ct}$$

$$x = \frac{1}{C}\left(1 - \frac{V_0}{V_0 + Ct}\right)$$

$$= \frac{2\sqrt{V_0}}{k}\left(1 - \frac{V_0}{V_0 + \frac{kt}{2\sqrt{V_0}}}\right)$$

Uncorrected proofs

$$x = \frac{2\sqrt{V_0}}{k}\left(1 - \frac{2}{2 + \frac{kt}{\sqrt{V_0^3}}}\right)$$

11b ii $\lim\limits_{t\to\infty} x$

$$= \lim_{t\to\infty} \frac{2\sqrt{V_0}}{k}\left(1 - \frac{2}{2 + \frac{kt}{\sqrt{V_0^3}}}\right)$$

$$= \frac{2\sqrt{V_0}}{k}(1 - 0)$$

$$= \frac{2\sqrt{V_0}}{k}$$

Uncorrected proofs

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6D

(12)(a) $\frac{dv}{dt} = 1 - v^{3/2}$

$\frac{dv}{du} \cdot \frac{du}{dt} = 1 - u^3$

$2u \cdot \frac{du}{dt} = 1 - u^3$

$\frac{dt}{du} = \frac{2u}{1-u^3}$

$u = \sqrt{v}$

$v = u^2$

$\frac{dv}{du} = 2u$

$v^{3/2} = v\sqrt{v}$

$= u^2 \cdot u$

$= u^3$

(b) Let $\frac{2u}{(1-u)(1+u+u^2)} = \frac{A}{1-u} + \frac{Bu+C}{1+u+u^2}$.

Then $2u = A(1+u+u^2) + (Bu+C)(1-u)$.

Put $u=1$.

$2 = 3A$

$A = \frac{2}{3}$

Put $u=0$.

$0 = \frac{2}{3} + C$

$C = -\frac{2}{3}$.

Put $u=-1$.

$-2 = \frac{2}{3} + \left(-B - \frac{2}{3}\right)(2)$

$-2 = \frac{2}{3} - 2B - \frac{4}{3}$

$2B = \frac{4}{3}$

$B = \frac{2}{3}$

So $\frac{dt}{du} = \frac{2}{3} \cdot \frac{1}{1-u} + \frac{2}{3} \cdot \frac{u-1}{1+u+u^2}$

$= \frac{2}{3}\left(\frac{1}{1-u} + \frac{\frac{1}{2}(1+2u) - \frac{3}{2}}{1+u+u^2}\right)$

$= \frac{2}{3}\left(\frac{1}{1-u} + \frac{1}{2} \cdot \frac{1+2u}{1+u+u^2} - \frac{3}{2} \cdot \frac{1}{\left(\frac{1}{2}+u\right)^2 + \frac{3}{4}}\right)$

6D

(12)(c) $t = \int \frac{dt}{du}\,du$

$$= \frac{2}{3}\left(-\ln|1-u| + \frac{1}{2}\ln(1+u+u^2) - \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2(u+\frac{1}{2})}{\sqrt{3}}\right)\right) + C$$

$$= \frac{2}{3}\left(\frac{1}{2}\ln\left(\frac{1+u+u^2}{(1-u)^2}\right) - \sqrt{3}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right)\right) + C$$

When $t=0$, $v = u^2 = 0$, so $u = 0$.

So $C = -\frac{2}{3}\left(\frac{1}{2}\ln 1 - \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$$= \frac{2}{3}\cdot\sqrt{3}\cdot\frac{\pi}{6}$$

$$= \frac{\pi\sqrt{3}}{9}.$$

So, replacing u with $\sqrt{v}$,

$$t = \frac{2}{3}\left(\frac{1}{2}\ln\left(\frac{1+\sqrt{v}+v}{(1-\sqrt{v})^2}\right) - \sqrt{3}\left(\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \frac{\pi}{6}\right)\right)$$

Now, $\tan\left(\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \tan^{-1}\frac{1}{\sqrt{3}}\right)$

$$= \frac{\frac{2\sqrt{v}+1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \frac{2\sqrt{v}+1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}}\cdot\frac{3}{3}$$

$$= \frac{\sqrt{3}(2\sqrt{v}+1) - \sqrt{3}}{3 + 2\sqrt{v} + 1}$$

$$= \frac{2\sqrt{3v}}{2(2+\sqrt{v})}$$

$$= \frac{\sqrt{3v}}{2+\sqrt{v}}$$

So $\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \frac{\pi}{6} = \tan^{-1}\left(\frac{\sqrt{3v}}{2+\sqrt{v}}\right)$

and the result is proven.