Solutions to Exercise 6C Foundation questions

1a $v = -6\sin 2t$

$\ddot{x} = -12\cos 2t = -4x$

$36\sin^2 2t + 36\cos^2 2t = 36$

$v^2 = 36 - 36\cos^2 2t = 36 - 4x^2$

$v^2 = 4(9 - x^2)$

1b When $x = 2$,

$v = \pm\sqrt{4(9 - 2^2)}$

$= \pm 2\sqrt{5}$ m/s

$\ddot{x} = -4(2)$

$= -8$ m/s^2

2a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$

$v^2 = D - n^2(x - c)^2$

At each extreme of motion, the velocity is 0.

$0 = D - n^2a^2$

$D = n^2a^2$

$v^2 = -n^2((x - c)^2 - a^2)$

$c = 0$, $a = 5$ and $n = 3$

Thus, $v^2 = 9(25 - x^2)$

2b $v = \pm\sqrt{9(25 - 3^2)}$

$= \pm 12$ m/s

$\ddot{x} = -9(3)$

$= -27$ m/s^2

Uncorrected proofs

2c When $t = 0, x = 0$

$v = \pm\sqrt{9(25)}$

$= \pm 15$ m/s

3a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$

$v^2 = D - n^2(x - c)^2$

$c = 0$, and $n = \left|\sqrt{16}\right| = 4$

$v^2 = D - 4^2x^2$

$v = 24$ when $x = 0$

$576 = D - 16(0)$

Thus,

$v^2 = 576 - 16x^2$

$= 16(36 - x^2)$

3b Period $= \frac{2\pi}{4} = \frac{\pi}{2}$ and Amplitude $= 6$

3c When $x = 2$,

$|v| = \sqrt{16(36 - 2^2)}$

$= 16\sqrt{2}$ cm/s

$\ddot{x} = -16(2)$

$= -32$ cm/s^2

Uncorrected proofs

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4a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$

$v^2 = D - n^2(x-c)^2$

At each extreme of motion, the velocity is 0.

$0 = D - n^2a^2$

$D = n^2a^2$

$v^2 = -n^2((x-c)^2 - a^2)$

$c = 0, a = 6$ and $n = 2$

Thus, $v^2 = 4(36 - x^2)$

Therefore, the period is $\frac{2\pi}{2} = \pi$ and maximum speed is 6 cm/s.

4b i $x = 6\cos 2t$

4b ii $x = -6\cos 2t$

4b iii $x = 6\sin 2t$

4b iv $x = -6\sin 2t$

5a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$

$v^2 = D - n^2(x-c)^2$

$n = \left|\sqrt{256}\right| = 16$

$c = 0$

$v^2 = D - 16^2(x)^2$

$v = 0$ when $x = 2$

$0 = D - 16^2(4)$

Uncorrected proofs

Thus, $v^2 = 1024 - 16x^2 = 16(64 - x^2)$

When $x = 0$,

$|v| = \sqrt{16(64)}$

$= 32$ m/s

5b

$\ddot{x} = -\frac{1}{4}x$

$v^2 = D - n^2(x - c)^2$

$n = \left|\sqrt{\frac{1}{4}}\right| = \frac{1}{2}$

$c = 0$

$v^2 = D - \frac{1}{2}^2 (x)^2$

When $x = 0, v = 4$.

$16 = \frac{1}{4}a$

$a = 64$

$v^2 = \frac{1}{2}(64 - x^2)$

When $v = 0$,

$0 = \frac{1}{2}(64 - x^2)$

$0 = 64 - x^2$

$x^2 = 64$

$x = 8$ cm

6a When $x = 0, v = 4$

$n = \frac{\frac{2\pi}{\pi}}{2} = 4$

$16 = 4^2(a^2)$

Uncorrected proofs

$a = 1 \text{ m}$

6b

$n = \frac{2\pi}{3}$

$v^2 = \frac{2\pi}{3}^2 (2^2 - x^2)$

When $x = 0$,

$|v| = \sqrt{\frac{4\pi}{9}(4)}$

$= \frac{4\pi}{3} \text{ m/s}$

Uncorrected proofs

Solutions to Exercise 6C Development questions

7a Amplitude $a = \frac{AB}{2} = 10$ cm

$T = 8 = \frac{2\pi}{n}$

$n = \frac{\pi}{4}$

Maximum speed of particle in SHM is:

$an = 10 \times \frac{\pi}{4}$

$= \frac{5\pi}{2}$ cm/s

Assuming the particle started from O (though, it won't affect the maximum speed or acceleration):

$x = 10 \sin \frac{\pi}{4} t$

$v = \frac{5\pi}{2} \cos \frac{\pi}{4} t$

$\ddot{x} = -\frac{5}{8}\pi^2 \sin \frac{\pi}{4} t$

Maximum acceleration is $\frac{5}{8}\pi^2$ cm/s^2

7b $x = 6$ cm

Using the relations: $v^2 = n^2(a^2 - x^2)$ and $\ddot{x} = n^2(c - x)$

$v^2 = \left(\frac{\pi}{4}\right)^2 (10^2 - 6^2)$

$v = \pm 2\pi$ cm/s

$\ddot{x} = \pm \left(\frac{\pi}{4}\right)^2 (0 - 6)$

$= \pm \frac{3\pi^2}{8}$ cm/s^2

Uncorrected proofs

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8 $a = 5$ m; when $x = 2$, $\ddot{x} = \pm 4$

Using $\ddot{x} = n^2(c - x)$

$\ddot{x} = \pm 4 = n^2(-2)$

$n^2 = 2$

Using $v^2 = n^2(a^2 - x^2)$:

At $x = 0$, $v^2 = 2(5^2 - 0^2)$

$v_0 = 5\sqrt{2}$ m/s

At $x = 4$, $v^2 = 2(5^2 - 4^2)$

$v_4 = 3\sqrt{2}$ m/s

9 $T = \pi$; $v_{max} = an = 8$

$n = \dfrac{2\pi}{T} = 2$

$a = \dfrac{v_{max}}{n} = \dfrac{8}{2} = 4$

Using $v^2 = n^2(a^2 - x^2)$:

When $x = 3$,

$v^2 = 2^2(4^2 - 3^2)$

$v_3 = 2\sqrt{7}$ m/s

10 $x = 4$ cm; $v = 20$ cm/s; $\ddot{x} = -6\frac{2}{3}$ cm/s^2

Using the relations $v^2 = n^2(a^2 - x^2)$ and $\ddot{x} = n^2(c - x)$:

$400 = n^2(a^2 - 16)$ and $-6\frac{2}{3} = n^2(-4)$

$n^2 = \dfrac{5}{3}$

$400 = \dfrac{5}{3}(a^2 - 16)$

$a = \sqrt{\left(400 \times \frac{3}{5}\right) + 16}$

$= 16$ cm

Uncorrected proofs

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11 Initial speed $= |v_{max}| = 10\sqrt{3}$ cm/s

$x = a\sin nt$

$v = an\cos nt$

$v_{max} = an$

$n = \dfrac{10\sqrt{3}}{a}$

$x_m = \dfrac{a}{2}$

$v^2 = n^2(a^2 - x^2)$

$v^2 = \dfrac{300}{a^2}\left(a^2 - \dfrac{a^2}{4}\right)$

$= 300 \times \dfrac{3}{4}$

$= 225$

$v = \pm 15$ cm/s

Therefore speed is 15 cm/s.

12 $v_0 = v_{max} = V = an$

$x = a\sin nt$ and $v = an\cos nt$

$v^2 = n^2(a^2 - x^2)$

The particle comes to rest when $v = 0$.

$0 = n^2a^2 - n^2x^2$

$V^2 = n^2x^2$ (as $V = an$)

$x = \pm\dfrac{V}{n}$

Hence proved.

13a Amplitude $= a$ and $\ddot{x} = -n^2x$

$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2x$

Uncorrected proofs

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Integrating both sides with dx

$\frac{1}{2}v^2 = -n^2 \int x \ dx$

$v^2 = -2n^2 \int x \ dx$

$v^2 = -n^2x^2 + C$

At $x = a, \ v = 0$,

As the particle is in SHM,

$0 = -n^2a^2 + C$

$C = n^2a^2$

Therefore

$v^2 = n^2a^2 - n^2x^2$

$= n^2(a^2 - x^2)$

13b i $x = 0$

$v_0^2 = n^2(a^2 - 0^2)$

$v_0 = \pm an$

Speed $= |v_0| = an$

13b ii $x = \frac{a}{2}$

$v_m^2 = n^2\left(a^2 - \frac{a^2}{4}\right)$

$v_m = \pm\frac{\sqrt{3}}{2}an$

Speed $= |v_m| = \frac{1}{2}\sqrt{3}an$

$\ddot{x} = -n^2x$

$= -n^2\left(\frac{a}{2}\right)$

$= -\frac{1}{2}an^2$

Uncorrected proofs

14a $v^2 = -9x^2 + 18x + 27$

$= -9(x^2 - 2x - 3)$

$= -9\left((x^2 - 2x + 1) - 4\right)$

$= -3^2((x-1)^2 - 2^2)$

$= 3^2(2^2 - (x-1)^2)$

which is of the form $v^2 = n^2(a^2 - (x-c)^2)$

where $n = 3, a = 2$ and $c = 1$.

So, the amplitude is 2.

Period is:

$T = \frac{2\pi}{n} = \frac{2\pi}{3}$

The centre of the motion is $x = 1$.

14b i $v^2 = 80 + 64x - 16x^2$

$= 16(5 + 4x - x^2)$

$= 16\left(9 - (x^2 - 4x + 4)\right)$

$= 4^2(3^2 - (x-2)^2)$

which is of the form $v^2 = n^2(a^2 - (x-c)^2)$

where $n = 4, a = 3$ and $c = 2$.

So, amplitude is 3.

Period is:

$T = \frac{2\pi}{n} = \frac{\pi}{2}$

Centre of motion is $x = 2$.

14bii $v^2 = -9x^2 + 108x - 180$

$= 9(-20 + 12x - x^2)$

$= 9\left(16 - (x^2 - 12x + 36)\right)$

$= 3^2(4^2 - (x-6)^2)$

Uncorrected proofs

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which is of the form $v^2 = n^2(a^2 - (x-c)^2)$

where $n = 3, a = 4$ and $c = 6$.

So, amplitude is 4.

Period is:

$$T = \frac{2\pi}{n} = \frac{2\pi}{3}$$

Centre of motion is $x = 6$.

14b iii $v^2 = -2x^2 - 8x - 6$

$= 2(-x^2 - 4x - 3)$

$= 2\left(1 - (x^2 + 4x + 4)\right)$

$= \left(\sqrt{2}\right)^2(1^2 - (x+2)^2)$

which is of the form $v^2 = n^2(a^2 - (x-c)^2)$

where $n = \sqrt{2}, a = 1$ and $c = -2$.

So, amplitude is 1.

Period is:

$$T = \frac{2\pi}{n} = \sqrt{2}\pi$$

Centre of motion is $x = -2$.

14b iv $v^2 = 8 - 10x - 3x^2$

$$= 3\left(\frac{8}{3} - \frac{10}{3}x - x^2\right)$$

$$= 3\left(\frac{8}{3} + \frac{25}{9} - \left(x^2 + \frac{10}{3}x + \frac{25}{9}\right)\right)$$

$$= 3\left(\frac{49}{9} - \left(x + \frac{5}{3}\right)^2\right)$$

$$= \left(\sqrt{3}\right)^2\left(\left(\frac{7}{3}\right)^2 - \left(x + \frac{5}{3}\right)^2\right)$$

which is of the form $v^2 = n^2(a^2 - (x-c)^2)$

Uncorrected proofs

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where $n = \sqrt{3}, a = \frac{7}{3}$ and $c = -\frac{5}{3}$.

So, amplitude is $\frac{7}{3}$ or $2\frac{1}{3}$.

Period is:

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{3}} = \frac{2\sqrt{3}}{3}\pi$$

Centre of motion is $x = -\frac{5}{3}$.

15a i $x = \sin^2 5t$

Using $\cos 2y = 1 - 2\sin^2 y \Rightarrow \sin^2 y = \frac{1}{2} - \frac{1}{2}\cos 2y$

$x = \frac{1}{2} - \frac{1}{2}\cos 10t$

$v = \frac{dx}{dt} = \dot{x} = 5\sin 10t$

$\frac{d^2x}{dt^2} = \ddot{x} = 50\cos 10t$

$\ddot{x} = 50\cos 10t + 50 - 50$

$$= -100\left(\left(\frac{1}{2} - \frac{1}{2}\cos 10t\right) - \frac{1}{2}\right)$$

$$= -10^2\left(x - \tfrac{1}{2}\right)$$

Hence proved.

15aii $x = \sin^2 5t$

$\sin 2y = 2\sin y \,.\cos y$

$\dot{x} = \frac{dx}{dt} = 2\sin 5t \,.\cos 5t \;.5 = 5\sin 10t$

$\ddot{x} = \frac{d^2x}{dt^2} = 50\cos 10t$

$\ddot{x} = 50\cos 10t + 50 - 50$

$$= -100\left(\left(\frac{1}{2} - \frac{1}{2}\cos 10t\right) - \frac{1}{2}\right)$$

Uncorrected proofs

$= -10^2\left(\sin^2 5t - \frac{1}{2}\right)$

$= -10^2\left(x - \frac{1}{2}\right)$

Hence proved.

15b Centre of the motion is:

$x = c => x = \frac{1}{2}$

Range of the motion is:

$\left(\frac{1}{2} - \frac{1}{2}\right) \leq x \leq \left(\frac{1}{2} + \frac{1}{2}\right) = 0 \leq x \leq 1$

Period is:

$T = \frac{2\pi}{n} = \frac{2\pi}{10} = \frac{\pi}{5}$ minutes

16a $\ddot{x} = -9(x - 7)$; $a = 7$cm

Centre of the motion is the line: $x = c => x = 7$

So, the particle oscillates in the range:

$(7 - a) \leq x \leq (7 + a) = 0 \leq x \leq 14$

That is, the origin is a point of extremity for this motion.

The velocity of a particle in SHM at the extremes is zero.

Hence, for this motion, the velocity at the origin is zero.

16b $\ddot{x} = -9(x - 7)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x - 7)$

Integrating both sides w.r.t x

$\frac{v^2}{2} = -9\int(x - 7)dx$

$v^2 = -9(x^2 - 14x) + C$

At $x = 0$, $v = 0$ so $C = 0$

Therefore

Uncorrected proofs

$v^2 = -9(x^2 - 14x)$

$= -9\big((x^2 - 14x + 49) - 49\big)$

$= -9((x-7)^2 - 7^2)$

$v = v_{max}$ for $(x - c) = 0$

$v_{max}^2 = -9((7-7)^2 - 7^2)$

$v_{max} = \pm 21$ cm/s

Maximum speed $= |v_{max}| = 21$ cm/s

16c $\ddot{x} = -9(x - 7)$

At origin, $x = 0$

$\ddot{x} = -9(-7) = 63$ cm/s^2

Observe that the acceleration at origin of the particle is towards the positive direction and at its maximum value.

Thus, even if the particle has zero velocity at the origin, the restoring acceleration of the particle is maximum and makes it move away from the origin.

17a $x = 4\cos 3t - 6\sin 3t$

$\cos\theta = \dfrac{4}{\sqrt{4^2 + 6^2}}$ and $\sin\theta = \dfrac{6}{\sqrt{4^2 + 6^2}}$

$x = \sqrt{52}(\cos\theta \cos 3t - \sin\theta \sin 3t)$

$x = \sqrt{52}\cos(3t - \theta)$

$\dfrac{dx}{dt} = \dot{x} = -3\sqrt{52}\sin(3t - \theta)$

$\dfrac{d^2x}{dt^2} = \ddot{x} = -9\sqrt{52}\cos(3t - \theta)$

$\ddot{x} = -9(\sqrt{52}\cos(3t - \theta))$

$\ddot{x} = -9x$

Hence, the acceleration is proportional to the displacement but in the opposite direction and the motion is simple harmonic.

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17b $\ddot{x} = -9x = -3^2x$

$x = a\cos nt\,; \quad a = \sqrt{52} = 2\sqrt{13}; \quad n = 3$

Amplitude is $2\sqrt{13}$.

$T = \dfrac{2\pi}{n} = \dfrac{2\pi}{3}$

Maximum speed $= |v_{max}| = an = 6\sqrt{13}$

Required magnitude of acceleration

$= |\ddot{x}|$

$= 9\left(\dfrac{a}{2}\right)$

$= 9\left(\dfrac{2\sqrt{13}}{2}\right)$

$= 9\sqrt{13}$

18a $x = 3 + \sin 4t + \sqrt{3}\cos 4t$

$\cos\alpha = \dfrac{1}{\sqrt{1^2 + \sqrt{3}^2}} = \dfrac{1}{2}$ and $\sin\alpha = \dfrac{\sqrt{3}}{2}$

$x = 3 + 2(\sin 4t\cos\alpha + \sin\alpha\cos 4t)$

$x = 3 + 2(\sin(4t + \alpha))$

$\dot{x} = 8\cos(4t + \alpha)$

$\ddot{x} = -32\sin(4t + \alpha)$

$= -16(2\sin(4t + \alpha))$

$= -16\big((3 + 2\sin(4t + \alpha)) - 3\big)$

$= -16(x - 3)$

Hence proved.

The centre of the motion is $x = 3$.

Period is:

$T = \dfrac{2\pi}{n} = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

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18b From the solution in part a:

$x = 3 + \sin 4t + \sqrt{3}\cos 4t$

$\cos\alpha = \dfrac{1}{\sqrt{1^2 + \sqrt{3}^2}} = \dfrac{1}{2}$ and $\sin\alpha = \dfrac{\sqrt{3}}{2}$

$x = 3 + 2(\sin 4t \cos\alpha + \sin\alpha \cos 4t)$

$= 3 + 2(\sin(4t + \alpha))$

$= 3 + 2\sin(4t + \alpha)$

$\alpha = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

18c When the particle is at the centre in SHM its speed is the maximum.

So, speed at centre is:

$an = 2 \times 4 = 8$

$x = c => x = 3$

$3 = 3 + 2\sin\left(4t + \dfrac{\pi}{3}\right)$

$\sin\left(4t + \dfrac{\pi}{3}\right) = 0$

For first three times:

$4t + \dfrac{\pi}{3} = \pi, 2\pi, 3\pi$

$t = \dfrac{\pi}{6}, \dfrac{5\pi}{12}, \dfrac{2\pi}{3}$

19a $x = 10 + 8\sin 2t + 6\cos 2t$

$\cos\alpha = \dfrac{8}{\sqrt{8^2 + 6^2}} = \dfrac{8}{10} = \dfrac{4}{5}$ and $\sin\alpha = \dfrac{6}{10} = \dfrac{3}{5}$ and $\alpha = \tan^{-1}\dfrac{3}{4}$

$x = 10 + 10(\cos\alpha \sin 2t + \sin\alpha \cos 2t)$

$x = 10 + 10\sin(2t + \alpha)$

$\dot{x} = 20\cos(2t + \alpha)$

$\ddot{x} = -40\sin(2t + \alpha)$

Uncorrected proofs

$= -4\big((10 + 10\sin(2t + \alpha)) - 10\big)$

$= -4(x - 10)$

$= -n^2(x - c)$ where $n = 2$ and $c = 10$

Hence, the motion is SHM.

The centre of the motion is: $x = c \Rightarrow x = 10$

Period is:

$T = \dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$

Since $x = 10 + 10\sin(2t + \alpha)$ is of the form $x = x_0 + a\sin(nt + \alpha)$, amplitude is 10.

19b The particle first reaches origin when $x = 0$.

$0 = 10 + 10\sin(2t + \alpha)$

$\sin(2t + \alpha) = -1$

$(2t + \alpha) = \sin^{-1}(-1)$

$2t + \alpha = \dfrac{3\pi}{2}$

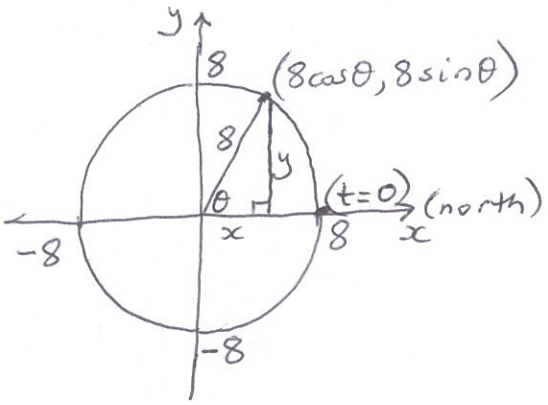
$t = \dfrac{3\pi}{4} - \dfrac{1}{2}\tan^{-1}\dfrac{3}{4}$

Uncorrected proofs

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6C

(20) The positive x-axis points north.



(a) $\frac{x}{8} = \cos\theta$ so $x = 8\cos\theta$

similarly, $y = 8\sin\theta$.

$\frac{d\theta}{dt} = 1$ rev/min

$= 2\pi$ rad/min

so $\theta = 2\pi t + C$.

when $t=0$, $\theta = 0$

so $C = 0$

so $\theta = 2\pi t$.

so $x = 8\cos 2\pi t$ and $y = 8\sin 2\pi t$.

(b) $\dot{x} = -16\pi \sin 2\pi t$ and $\dot{y} = 16\pi \cos 2\pi t$

so $\ddot{x} = -32\pi^2 \cos 2\pi t$ and $\ddot{y} = -32\pi^2 \sin 2\pi t$

$= -4\pi^2 x$. $= -4\pi^2 y$.

(c) (i) $\frac{y}{x} = \tan\theta = \frac{1}{\sqrt{3}}$

so $\theta = \frac{\pi}{6}$ or $\frac{7\pi}{6}$.

(ii) $\frac{\dot{x}}{\dot{y}} = -\tan\theta = -\sqrt{3}$

so $\theta = \frac{\pi}{3}$ or $\frac{4\pi}{3}$.

(iii) $-16\pi \sin\theta = 16\pi \cos\theta$

$\tan\theta = -1$

$\theta = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$.

6C

(21)(a) $\int \frac{d(\frac{1}{2}v^2)}{dx}\,dx = \int -n^2 x\,dx$

$\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + C$

When $v=0$, $x = \pm a$

so $C = \frac{1}{2}n^2a^2$

so $v^2 = n^2(a^2-x^2)$.

(b) When $x = x_1$, $|v| = v_1$

so $v_1^2 = n^2(a^2 - x_1^2)$ (1)

When $x = x_2$, $|v| = v_2$

so $v_2^2 = n^2(a^2 - x_2^2)$ (2)

From (1), $a^2 = \frac{v_1^2}{n^2} + x_1^2$

Sub. into (2): $v_2^2 = n^2\left(\frac{v_1^2}{n^2} + x_1^2 - x_2^2\right)$

$v_2^2 - v_1^2 = n^2(x_1^2 - x_2^2)$

$n = \sqrt{\frac{v_2^2 - v_1^2}{x_1^2 - x_2^2}}$

so $T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}}$.

(2) ÷ (1): $\frac{v_2^2}{v_1^2} = \frac{a^2 - x_2^2}{a^2 - x_1^2}$

$a^2v_2^2 - v_2^2x_1^2 = a^2v_1^2 - v_1^2x_2^2$

$a^2(v_1^2 - v_2^2) = v_1^2x_2^2 - v_2^2x_1^2$

$a = \sqrt{\frac{v_1^2x_2^2 - v_2^2x_1^2}{v_1^2 - v_2^2}}$ (answer in text is wrong)

(c) $a = \sqrt{\frac{1024-324}{64-36}} = 5$ cm

$T = 2\pi\sqrt{\frac{16-9}{64-36}} = 2\pi\sqrt{\frac{1}{4}} = \pi$ seconds

$|v|_{max} = na$

$= 2 \times 5$

$= 10$ cm/s

6C

(22) Max speed $= na = V$.

$$v^2 = n^2(a^2 - x^2)$$
$$= V^2 - n^2x^2$$

so when $x = \frac{1}{2}a$,

$$v^2 = V^2 - \frac{1}{4}n^2a^2$$
$$= V^2 - \frac{1}{4}V^2$$
$$= \frac{3}{4}V^2$$

so $v = \pm\frac{\sqrt{3}}{2}V$.

Also, when $v = \frac{1}{2}V$,

$$\frac{1}{4}V^2 = V^2 - n^2x^2$$
$$n^2x^2 = \frac{3}{4}V^2$$
$$\frac{V^2}{a^2}x^2 = \frac{3}{4}V^2$$
$$x^2 = \frac{3}{4}a^2$$

so $x = \pm\frac{\sqrt{3}}{2}a$.

$$v^2 = V^2 - n^2x^2$$
$$= V^2 - \frac{V^2}{a^2}x^2$$

so $|v| = V\sqrt{1 - \frac{x^2}{a^2}}$

and

$$n^2x^2 = V^2 - v^2$$
$$\frac{V^2}{a^2}x^2 = V^2 - v^2$$
$$x^2 = \frac{a^2}{V^2}(V^2 - v^2)$$

so $|x| = a\sqrt{1 - \frac{v^2}{V^2}}$.

6C

(23)(a) $x = \sin t - 2\sin(t-\alpha)$

$\dot{x} = \cos t - 2\cos(t-\alpha)$

$\ddot{x} = -\sin t - 2\sin(t-\alpha)$

$= -x$

Thus, the acceleration has the form $\ddot{x} = -n^2 x$, so the motion is SHM with $T = \frac{2\pi}{n} = 2\pi$.

(b) $x = \sin t - 2\sin t\cos\alpha + 2\cos t\sin\alpha$

$= (1 - 2\cos\alpha)\sin t + (2\sin\alpha)\cos t$

$= A\sin(t+\theta)$,

where $A^2 = (1-2\cos\alpha)^2 + (2\sin\alpha)^2$

$= 1 - 4\cos\alpha + 4(\cos^2\alpha + \sin^2\alpha)$

$= 5 - 4\cos\alpha$

so the amplitude (i.e. the maximum value of x) is $A = \sqrt{5-4\cos\alpha}$.

The maximum possible value of A occurs when $\cos\alpha = -1$.

So the max value is $\sqrt{9} = 3$ and it occurs when $\alpha = \pi$.

When $\alpha = \pi$, $x = \sin t - 2\sin(t-\pi)$

$= \sin t - 2(-\sin t)$

$= 3\sin t$.

The minimum possible value of A occurs when $\cos\alpha = 1$.

So the min value is $\sqrt{1} = 1$ and it occurs when $\alpha = 0$.

When $\alpha = 0$, $x = \sin t - 2\sin t$

$= -\sin t$.

6C

(23)(c) The balls are level when $x_1 = x_2$.

So $\sin t = 2\sin(t-\alpha)$

$\sin t = 2\sin t\cos\alpha - 2\cos t\sin\alpha$

$\tan t = 2\tan t\cos\alpha - 2\sin\alpha$

$$\tan t = \frac{2\sin\alpha}{2\cos\alpha - 1}$$

$$= \frac{4T}{1+T^2} \div \left(\frac{2-2T^2}{1+T^2} - 1\right), \text{ where } T = \tan\frac{\alpha}{2}$$

$$= \frac{4T}{2-2T^2-(1+T^2)}$$

$$= \frac{4T}{1-3T^2}$$

The balls will be level twice in the interval $0 \le t < 2\pi$; once in the interval $0 \le t \le \frac{\pi}{2}$ and once in the interval $\pi < t \le \frac{3\pi}{2}$.

(d) When $t = 0$, $x = A$

so $2\sin\alpha = \sqrt{5-4\cos\alpha}$

$4\sin^2\alpha = 5 - 4\cos\alpha$

$4(1-\cos^2\alpha) = 5 - 4\cos\alpha$

$4\cos^2\alpha - 4\cos\alpha + 1 = 0$

$(2\cos\alpha - 1)^2 = 0$

$\cos\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

When $\alpha = \frac{\pi}{3}$, $x = \sin t - 2\sin\left(t - \frac{\pi}{3}\right)$

$= \sin t - 2\left(\frac{1}{2}\sin t - \frac{\sqrt{3}}{2}\cos t\right)$

$= \sqrt{3}\cos t$.

When $\alpha = \frac{5\pi}{3}$, $x = \sin t - 2\left(\frac{1}{2}\sin t + \frac{\sqrt{3}}{2}\cos t\right)$

$= -\sqrt{3}\cos t$.