Solutions to Exercise 6B Foundation questions

1a The amplitude is 12.

The period is $\frac{2\pi}{\frac{\pi}{2}} = 4$

1b

$v = -6\pi \sin\frac{\pi}{2}t$

$\ddot{x} = -3\pi^2 \cos\frac{\pi}{2}t$

Thus, $\ddot{x} = -\left(\frac{\pi}{2}\right)^2 x$

$= -\frac{\pi^2}{4}x$

1c At $t = 0$, $x = 12\cos 0 = 12$ cm

At $t = 0$, $v = -6\pi \sin 0 = 0$ cm/s

1d Solving $x = 0$ gives

$0 = 12\cos\frac{\pi}{2}t$

$\cos\frac{\pi}{2}t = 0$

$\frac{\pi}{2}t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t =$ after 1 second

1e As the next time the particles hits the origin in at $t = 3$, it is 2 seconds between each visit.

2a The amplitude is 2.

The period is $\frac{2\pi}{4\pi} = \frac{1}{2}$ s

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2b Max and min are 2 and -2. The points of intersection with the x-axis are $\frac{\frac{1}{2}}{2} = \frac{1}{4}$.

$\therefore \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

2c $v = 8\pi \cos 4\pi t$

$\ddot{x} = -32\pi^2 \sin 4\pi t$

2d $x = 2 \sin 4\pi t$

$\ddot{x} = -32\pi^2 \sin 4\pi t$

$= -16\pi^2 (2 \sin 4\pi t)$

$= -16\pi^2 x$

2e When $v = 0$,

$0 = 8\pi \cos 4\pi t$

$\cos 4\pi t = 0$

$4\pi t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{1}{8}, \frac{3}{8}$

At $t = \frac{1}{8}$, $\ddot{x} = -32\pi^2$

At $t = \frac{3}{8}$, $\ddot{x} = 32\pi^2$

2f Greatest speed is when acceleration is 0.

When $\ddot{x} = 0$,

$0 = -32\pi^2 \sin 4\pi t$

$\sin 4\pi t = 0$

$4\pi t = 0, \pi$

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$t = \frac{1}{4}$

When $t = \frac{1}{4}$,

$v = 8\pi \cos \pi$

$= 8\pi$ m/s

3a When $t = 0, x = 0$

3b $v = 4 \cos \pi t$

$\ddot{x} = -4\pi \sin \pi t$

$= -\pi^2 \times \frac{4}{\pi} \sin \pi t$

$= -\pi^2 x$

3c The amplitude is $\frac{4}{\pi}$. The period is $\frac{2\pi}{\pi} = 2$ s

3d The maximum distance is the amplitude. $\therefore \frac{4}{\pi}$

Maximum speed $= \frac{4}{\pi} \times \pi = 4$ seconds

3e Max and Min of x at $\frac{4}{\pi}$ and $-\frac{4}{\pi}$.

When $x = 0$,

$0 = \frac{4}{\pi} \sin \pi t$

$\sin \pi t = 0$

$t = 1, 2$

$v = 4 \cos \pi t$

Max and Min of v at 4 and -4.

$\ddot{x} = -4\pi \sin \pi t$

Max and min are -4π and 4π

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3f When $v = -4$ m/s,

$-4 = 4\cos \pi t$

$\cos \pi t = -1$

$t = 1$

When $v = 4$ m/s,

$4 = 4\cos \pi t$

$\cos \pi t = 1$

$t = 2$

3g When $\ddot{x} = -4\pi$ m/s,

$-4\pi = -4\pi \sin \pi t$

$\sin \pi t = 1$

$t = \frac{1}{2}$

When $\ddot{x} = 4\pi$ m/s,

$4\pi = -4\pi \sin \pi t$

$\sin \pi t = -1$

$t = \frac{3}{2}$

4a

$n = \frac{2\pi}{\pi} = 2$

$a = \frac{4}{2} = 2$

$x = 2\sin 2t$

4b $a = 6$

$n = \frac{4}{6} = \frac{2}{3}$

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$$x = 6\sin\frac{2}{3}t$$

5a $x = b\sin nt + c\cos nt$

$v = bn\cos nt - cn\sin nt$

$\ddot{x} = -bn^2\sin nt - cn^2\cos nt$

$= -n^2x$

5b i When $t = 0, x = 3$

$3 = b\sin 0 + c\cos 0$

Thus, $c = 3$

When $t = 0, v = 0$

$0 = bn\cos 0 - cn\sin 0$

Thus, $b = 0$

So, $x = 3\cos nt$

5b ii

$n = \frac{2\pi}{1} = 2\pi$

When $t = 0, x = 5$

$5 = b\sin 0 + c\cos 0$

Thus, $c = 5$

When $t = 0, v = 0$

$0 = bn\cos 0 - cn\sin 0$

Thus, $b = 0$

So, $x = 5\cos 2\pi t$

$t = \frac{1}{4}$ as it takes 1 s to complete cycle of 2π so it takes $\frac{1}{4}$ s to return to origin.

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6a $x = a\sin\pi t$

$v = a\pi\cos\pi t$

$\ddot{x} = -a\pi^2\sin\pi t = -\pi^2 x$

6b

$a = \frac{4\pi}{\pi} = 4$

6c When $v = 2\pi$,

As this is slower than the starting velocity, this value is negative

$-2\pi = 4\pi\cos\pi t$

$-\frac{1}{2} = \cos\pi t$

$t = \frac{1}{3}, \frac{2}{3}$

7a $v = 6\sin 3t$

$\ddot{x} = 18\cos 3t$

A particle is stationary when the velocity is 0.

When $x = 10$,

$10 = 12 - 2\cos 3t$

$t = \frac{2\pi}{3}$

When $t = \frac{2\pi}{3}$,

$v = 6\sin 2\pi = 0$

7b $a = 2$

$t = \frac{2\pi}{3}$

Centre $x = 12$

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7c Moving between $10 \leq x \leq 14$

$$t = \frac{\frac{2\pi}{3}}{2} = \frac{\pi}{3}$$

7d When $x = 10$,

$$10 = 12 - 2\cos 3t$$

$$1 = \cos 3t$$

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

When $t = \frac{2\pi}{3}, \frac{4\pi}{3}$,

$$v = 0, \ddot{x} = 18 \text{ cm/s}^2$$

7e When $x = 12$,

$$12 = 12 - 2\cos 3t$$

$$0 = \cos 3t$$

$$t = \frac{\pi}{6}, \frac{\pi}{2}$$

When $t = \frac{\pi}{6}, \frac{\pi}{2}$

$$v = 6 \text{ cm/s}$$

$$\ddot{x} = 0 \text{ cm/s}^2$$

8a Amplitude $= 6$, period $= \frac{2\pi}{2}$, initial phase: $\frac{\pi}{2}$

8b

$$\dot{x} = 12\cos\left(2t + \frac{\pi}{2}\right)$$

$$\ddot{x} = -24\sin\left(2t + \frac{\pi}{2}\right)$$

$$= -2^2 x$$

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8c When $x = 0$,

$0 = 6\sin\left(2t + \frac{\pi}{2}\right)$

$\sin\left(2t + \frac{\pi}{2}\right) = 0$

$2t + \frac{\pi}{2} = \pi$ and $2t + \frac{\pi}{2} = 2\pi$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

When $t = \frac{\pi}{4}$, $v = -12$

When $t = \frac{3\pi}{4}$, $v = 12$

8d Velocity is maximum when acceleration is zero.

When $\ddot{x} = 0$,

$0 = -24\sin\left(2t + \frac{\pi}{2}\right)$

$2t + \frac{\pi}{2} = 2\pi$ and $2t + \frac{\pi}{2} = 4\pi$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$

When $t = \frac{3\pi}{4}, \frac{7\pi}{4}$, $x = 0$

8e As the period is π, $t = \pi, 2\pi$

$\dot{x} = 12\cos\left(2(\pi) + \frac{\pi}{2}\right) = 0$

$\ddot{x} = -24\sin\left(2(\pi) + \frac{\pi}{2}\right) = -24$

$\dot{x} = 12\cos\left(2(2\pi) + \frac{\pi}{2}\right) = 0$

$\ddot{x} = -24\sin\left(2(2\pi) + \frac{\pi}{2}\right) = -24$

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Solutions to Exercise 6B Development questions

9a $A = 120; \quad T = 24; \quad x_0 = 0; \quad v_0 > 0$

$$T = \frac{2\pi}{n} = 24$$

$$n = \frac{\pi}{12}$$

$$x = 120 \sin \frac{\pi}{12} t$$

$$v = 10\pi \cos \frac{\pi}{12} t$$

Maximum speed $= 10\pi$ m/s $(\div 31.4$ m/s$)$

9b i $x = +30 = 120 \sin \frac{\pi}{12} t$

$$t = \frac{12}{\pi} \sin^{-1} \frac{1}{4}$$

$\div 0.9652$ seconds

9b ii $x = -30 = 120 \sin \frac{\pi}{12} t$

$$t = \frac{12}{\pi} \sin^{-1} \left(-\frac{1}{4}\right)$$

$$= \frac{T}{2} + \frac{12}{\pi} \sin^{-1} \frac{1}{4}$$

$$= 12 + \frac{12}{\pi} \sin^{-1} \frac{1}{4}$$

$\div 12.97$ seconds

9c Half maximum speed $= \frac{10\pi}{2} = 5\pi$

Hence $v = 5\pi$ or -5π

When $v = 5\pi$,

$$5\pi = 10\pi \cos \frac{\pi}{12} t$$

$$t = \frac{12}{\pi} \cos^{-1} \frac{1}{2}$$

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$= \frac{12}{\pi} \times \frac{\pi}{3}$

$= 4$ seconds

When $v = -5\pi$,

$-5\pi = 10\pi \cos \frac{\pi}{12} t$

$t = \frac{12}{\pi} \cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{T}{2} - \frac{12}{\pi} \cos^{-1} \frac{1}{2}$

$= \frac{24}{2} - \frac{12}{\pi} \times \frac{\pi}{3}$

$= 12 - 4$

$= 8$ seconds

The speed of the particle will be the same when it reaches the same position again. That is, for every quarter of the SHM, the speed will be half the maximum speed after 4 seconds.

Hence, the speed will be half the maximum speed after 4 and 8 seconds and then after 16 and 20 seconds.

10a $T = \frac{\pi}{2}$; $x_0 = 4 = A$; $n = \frac{2\pi}{T} = 4$;

Particle at positive extreme initially and moving towards O.

$x = A \cos nt$

$= 4 \cos 4t$

$v = -16 \sin 4t$

10b i For $x = +2$,

$2 = 4 \cos 4t$

$t = \frac{1}{4} \cos^{-1} \frac{1}{2}$

$= \frac{1}{4} \times \frac{\pi}{3}$

$= \frac{\pi}{12}$ seconds

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10b ii For $x = -2$,

$-2 = 4\cos 4t$

$t = \frac{1}{4}\cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{T}{2} - \frac{1}{4}\cos^{-1}\frac{1}{2}$

$= \frac{\pi}{4} - \frac{\pi}{12}$

$= \frac{\pi}{6}$ seconds

10c $|v_{max}| = 16$ cm/s

For half maximum speed:

$v = -8$ and $v = 8$

When $v = -8$,

$-8 = -16\sin 4t$

$t = \frac{1}{4}\sin^{-1}\frac{1}{2}$

$= \frac{1}{4} \times \frac{\pi}{6}$

$= \frac{\pi}{24}$ seconds

When $v = 8$,

$8 = -16\sin 4t$

$t = \frac{1}{4}\sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{T}{2} - \frac{1}{4}\sin^{-1}\frac{1}{2}$

$= \frac{\pi}{4} - \frac{\pi}{24}$

$= \frac{5\pi}{24}$ seconds

Therefore the first two times are $\frac{\pi}{24}$ seconds and $\frac{5\pi}{24}$ seconds.

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11 $x = \sin^2 t$

Using $\cos 2\theta = 1 - 2\sin^2\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

So

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$x = \frac{1}{2} - \frac{1}{2}\cos 2t$$

$= x_0 - a\cos nt$ where $c = \frac{1}{2}, a = \frac{1}{2}$ and $n = 2$

Centre of the motion is the line $x_0 = \frac{1}{2}$.

Amplitude of the motion is $\frac{1}{2}$.

Range of SHM:

$$\left(\frac{1}{2} - \frac{1}{2}\right) \leq x \leq \left(\frac{1}{2} + \frac{1}{2}\right)$$

$0 \leq x \leq 1$

Period:

$$T = \frac{2\pi}{n} = \pi$$

12a $x = 3 - 2\cos^2 2t$

Using $\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1 + \cos 2\theta$

So

$2\cos^2 2t = 1 + \cos 4t$

$x = 3 - (1 + \cos 4t)$

$= 2 - \cos 4t$

which is of the form $x_0 - a\cos nt$ where $x_0 = 2, a = 1$ and $n = 4$

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12b Centre of the motion is the line $x_0 = 2$.

Amplitude is 1 cm.

Range of SHM:

$(2 - 1) \leq x \leq (2 + 1)$

$1 \leq x \leq 3$

Period:

$T = \frac{2\pi}{n} = \frac{\pi}{2}$ s

12c $v = \frac{dx}{dt} = 4 \sin 4t$

Maximum speed $= 4$ cm/s

The maximum speed occurs at the centre and as the particle starts at one of the extremes, it will occur after:

$\frac{T}{4} = \frac{\pi}{2} \times \frac{1}{4} = \frac{\pi}{8}$ s

13a $x = 2 + 3 \cos t + 3\sqrt{3} \sin t$

$\dot{x} = -3 \sin t + 3\sqrt{3} \cos t$

$\ddot{x} = -3 \cos t - 3\sqrt{3} \sin t$

$= -\left(3 \cos t + 3\sqrt{3} \sin t\right)$

$= -(x - 2)$

which is of the form

$\ddot{x} = -n^2(x - c)$ where $n = 1$ and $c = 2$.

Hence, the motion is simple harmonic.

13b Centre of the motion is the line $x = 2$.

13c Period of the motion is:

$T = \frac{2\pi}{n} = 2\pi$

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13d $3\cos t + 3\sqrt{3}\sin t$

$= 6\left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\sin t\right)$

$= 6\left(\cos\frac{\pi}{3}\cos t + \sin\frac{\pi}{3}\sin t\right)$ (since we are aiming for $0 < \theta < \frac{\pi}{2}$)

$= 6\left(\cos t\cos\frac{\pi}{3} + \sin t\sin\frac{\pi}{3}\right)$

Using $\cos(A - B) = \cos A\cos B + \sin A\sin B$,

$3\cos t + 3\sqrt{3}\sin t$

$= 6\cos\left(t - \frac{\pi}{3}\right)$

which is of the form

$A\cos(x - \theta)$ where $A = 6, x = t$ and $\theta = \frac{\pi}{3}$

13e $x = 2 + 6\cos\left(t - \frac{\pi}{3}\right)$

Amplitude is 6 cm.

Initial phase is: $\varphi = -\frac{\pi}{3}$

13f Interval of particle

= range of oscillation

$= (2 - 6) \leq x \leq (2 + 6)$

$= -4 \leq x \leq 8$

14a $v_0 = 3; \quad T = 4\pi; \quad x_0 = 6$

$x = b\sin nt + c\cos nt$

$T = \frac{2\pi}{n}$

$4\pi = \frac{2\pi}{n}$

$n = \frac{1}{2}$

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Hence

$x = b\sin\frac{t}{2} + c\cos\frac{t}{2}$

$v = \frac{dx}{dt}$

$= \frac{b}{2}\cos\frac{t}{2} - \frac{c}{2}\sin\frac{t}{2}$

Using initial conditions:

$x_0 = 6 = b\sin 0 + c\cos 0 = c$

Hence $c = 6$.

$v_0 = 3 = \frac{b}{2}\cos 0 - \frac{6}{2}\sin 0 = \frac{b}{2}$

Hence $b = 6$.

Therefore

$x = 6\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)$

The particle is at the origin for the first two times when:

$6\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right) = 0$

$\sin\frac{t}{2} + \cos\frac{t}{2} = 0$

$\sin\frac{t}{2} = -\cos\frac{t}{2}$

$\tan\frac{t}{2} = -1$

$\frac{t}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}$

So $t = \frac{3\pi}{2}$ and $\frac{7\pi}{2}$.

14b $T = 6; \quad x_0 = -2; \quad v_0 = \dot{x} = 3$

$T = 6 = \frac{2\pi}{n}$

$n = \frac{\pi}{3}$

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$x = b\sin\frac{\pi}{3}t + c\cos\frac{\pi}{3}t$

$v = b \times \frac{\pi}{3}\cos\frac{\pi}{3}t - c \times \frac{\pi}{3}\sin\frac{\pi}{3}t$

Using initial conditions:

$x_0 = -2 = c$

$v_0 = 3 = b \times \frac{\pi}{3}$

$b = \frac{9}{\pi}$

Therefore

$x = \frac{9}{\pi}\sin\frac{\pi}{3}t - 2\cos\frac{\pi}{3}t$

The particle is at the centre the first two times when $x = 0$:

$\frac{9}{\pi}\sin\frac{\pi}{3}t - 2\cos\frac{\pi}{3}t = 0$

$\frac{9}{\pi}\sin\frac{\pi}{3}t = 2\cos\frac{\pi}{3}t$

$\tan\frac{\pi}{3}t = \frac{2\pi}{9}$

$t = \frac{3}{\pi}\tan^{-1}\frac{2\pi}{9}$

$t \doteqdot 0.582$ and $\left(\frac{T}{2} + 0.582\right)$

$t \doteqdot 0.582$ and 3.582

15a $x = a\sin(nt + \alpha)$

$v_0 = 5; \quad x = 0; \quad T = 6$

$T = 6 = \frac{2\pi}{n}$

$n = \frac{\pi}{3}$

$x = a\sin\left(\frac{\pi}{3}t + \alpha\right)$

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$$v = \frac{dx}{dt}$$

$$= \frac{a\pi}{3}\cos(\frac{\pi}{3}t + \alpha)$$

Using initial conditions:

$x_0 = 0 = a\sin\alpha$

Since $a > 0, \alpha = 0$.

$$v_0 = 5 = \frac{a\pi}{3}\cos(0 + 0) = \frac{a\pi}{3}$$

$$a = \frac{15}{\pi}$$

15b $x = a\sin(nt + \alpha)$; $v = \frac{dx}{dt} = an\cos(nt + \alpha)$

$T = 3\pi$; $x_0 = -5$; $v_0 = 0$

$$T = 3\pi = \frac{2\pi}{n}$$

$$n = \frac{2}{3}$$

Using initial conditions:

$x_0 = a\sin(0 + \alpha) = -5$

$$v_0 = \frac{2a}{3}\cos(0 + \alpha) = 0$$

$$\frac{2a}{3}\cos\alpha = 0$$

$$\alpha = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

From $x_0 = -5 = a\sin\alpha$, we get:

$$a = 5 \text{ and } \alpha = \frac{3\pi}{2}$$

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15c $x = a\sin(nt + \alpha)$; $v = \frac{dx}{dt} = an\cos(nt + \alpha)$

$T = 2\pi$; $x_0 = 1$; $v_0 = -1$

$T = 2\pi = \frac{2\pi}{n}$

$n = 1$

Using initial conditions:

$x_0 = 1 = a\sin\alpha$

$v_0 = -1 = a\cos\alpha$

$\frac{x_0}{v_0} = -1 = \tan\alpha$

$\alpha = \frac{3\pi}{4}$

Hence

$1 = a\sin\frac{3\pi}{4}$

$a = \sqrt{2}$

16a $x = a\cos(2t + \alpha)$

$v = \frac{dx}{dt} = -2a\sin(2t + \alpha)$

$x_0 = 0$; $v_0 = 6$

Using initial conditions:

$x_0 = 0 = a\cos\alpha$

Since $a > 0$, $\alpha = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

$v_0 = 6 = -2a\sin\alpha$

$\alpha = -\frac{\pi}{2}$ and $a = 3$

16b $x = a\cos(2t + \alpha)$

$v = \frac{dx}{dt} = -2a\sin(2t + \alpha)$

$x_0 = 1$; $v = -2\sqrt{3}$

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Using initial conditions:

$x_0 = 1 = a\cos\alpha$

$v_0 = -2\sqrt{3} = -2a\sin\alpha$

$\sqrt{3} = a\sin\alpha$

$\frac{v_0}{x_0} = \tan\alpha = \sqrt{3}$

$\alpha = -\frac{5\pi}{6}$ or $\frac{\pi}{3}$

As $v_0 < 0$; $\alpha = \frac{\pi}{3}$

Substituting $\alpha = \frac{\pi}{3}$ into $\sqrt{3} = a\sin\alpha$ gives

$\sqrt{3} = a\sin\frac{\pi}{3}$

$\sqrt{3} = a \times \frac{\sqrt{3}}{2}$

$a = 2$

17 $x = a\cos\left(\frac{\pi}{8}t + \alpha\right)$

$x = 0$ at $t = 2$ and $v = -4$ at $t = 4$

$v = -a\frac{\pi}{8}\sin\left(\frac{\pi}{8}t + \alpha\right)$

Using given conditions $x = 0$ at $t = 2$:

$0 = a\cos\left(\frac{\pi}{4} + \alpha\right)$

$\cos\left(\frac{\pi}{4} + \alpha\right) = 0$

$\alpha = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

Using given conditions $v = -4$ at $t = 4$:

$-4 = -a\frac{\pi}{8}\sin\left(\frac{\pi}{2} + \alpha\right)$

$\frac{32}{\pi} = a\cos\alpha$ using $\sin\left(\frac{\pi}{2} + x\right) = \cos x$

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Since $\cos\frac{\pi}{4} > 0$ and $\cos\frac{5\pi}{4} < 0$; $\alpha = \frac{\pi}{4}$

Hence $a = \frac{32\sqrt{2}}{\pi}$

18 $T = 8\pi$; $x = a\sin(nt + \alpha)$; $t = 1, x_1 = 3$ and $v_1 = -1$

$T = 8\pi = \frac{2\pi}{n}$

$n = \frac{1}{4}$

$v = \frac{dx}{dt} = \frac{a}{4}\cos\left(\frac{t}{4} + \alpha\right)$

Using given conditions:

$x_1 = 3 = a\sin\left(\frac{1}{4} + \alpha\right)$ and $v_1 = -1 = \frac{a}{4}\cos\left(\frac{1}{4} + \alpha\right)$

$\frac{x_1}{v_1} = -3 = 4\tan\left(\frac{1}{4} + \alpha\right)$

$\frac{1}{4} + \alpha = \tan^{-1}\left(-\frac{3}{4}\right)$

As sine > 0 and cosine < 0; $\left(\frac{1}{4} + \alpha\right)$ in 2nd quadrant.

$\frac{1}{4} + \alpha = \pi - \tan^{-1}\left(\frac{3}{4}\right)$

$\alpha = -\frac{1}{4} + \pi - \tan^{-1}\left(\frac{3}{4}\right)$

$\alpha \div 2.248$

Hence

$a = \frac{3}{\sin\left(\frac{1}{4} + \alpha\right)}$

$= 5$

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19a $T = \frac{\pi}{2}; \quad x_0 = 3; \quad v_0 = 16$

$T = \frac{\pi}{2} = \frac{2\pi}{n}$

$n = 4$

$x = b\sin 4t + c\cos 4t$

$v = 4b\cos 4t - 4c\sin 4t$

Using initial conditions:

$3 = b\sin 0 + c\cos 0$

$c = 3$

and

$16 = 4b\cos 0 + 4c\sin 0$

$16 = 4b$

$b = 4$

Therefore $x = 4\sin 4t + 3\cos 4t$

19b $x = 3\cos 4t + 4\sin 4t$

$= 5\left(\frac{3}{5}\cos 4t + \frac{4}{5}\sin 4t\right)$ using triplet 3, 4, 5

$= 5(\cos\varepsilon\cos 4t + \sin\varepsilon\sin 4t)$

$= 5(\cos 4t\cos\varepsilon + \sin 4t\sin\varepsilon)$

$= 5\cos(4t - \varepsilon)$ using $\cos(A - B) = \cos A\cos B + \sin A\sin B$

where $\varepsilon = \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5} = \tan^{-1}\frac{4}{3} \div 37°$

This is of the form $x = a\cos(nt - \varepsilon)$ where $a = 5, n = 4$ and $\varepsilon = \tan^{-1}\frac{4}{3}$.

19c Amplitude of SHM is 5 m.

$v = -20\sin(4t - \varepsilon)$

So maximum speed is 20 m/s.

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19d When $x = 0$,

$0 = 4\sin 4t + 3\cos 4t$

$4\sin 4t = -3\cos 4t$

$\tan 4t = -\frac{3}{4}$

$t = \frac{T}{2} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

$= \frac{\pi}{4} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

Also, $0 = 5\cos(4t - \varepsilon)$

$\cos(4t - \varepsilon) = 0$

$4t - \varepsilon = \frac{\pi}{2}$

$4t - \tan^{-1}\frac{4}{3} = \frac{\pi}{2}$

$t = \frac{\pi}{8} + \frac{1}{4}\tan^{-1}\frac{4}{3}$

Since $\tan^{-1}\frac{4}{3} = \frac{\pi}{2} - \tan^{-1}\frac{3}{4}$,

$t = \frac{\pi}{8} + \frac{1}{4}\tan^{-1}\frac{4}{3}$ becomes

$t = \frac{\pi}{8} + \frac{1}{4}\left(\frac{\pi}{2} - \tan^{-1}\frac{3}{4}\right)$

$= \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

$= \frac{\pi}{4} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

Hence the two answers obtained are the same.

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20 $T = 24$ hours; Range $= 19° - 9° = 10°$C; $x_0 = 9$

$$c = \frac{19 + 9}{2} = 14°$$

$$A = c - 9 = 19 - c = 5$$

$n = \frac{2\pi}{T} = \frac{\pi}{12}$ hours; t will be in hours after 4: 00 am

$$x = 14 - 5\cos\left(\frac{\pi}{12}t\right)$$

20a When $x = 14°$,

$$14 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$$

$$\cos\left(\frac{\pi}{12}t\right) = 0$$

$$\frac{\pi}{12}t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$t = 6$ or 18

That is, 6 or 18 hours after 4:00 am, the temperature will be 14°.

In the given time interval, it will be 10:00 am.

20b When $x = 11°$,

$$11 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$$

$$\cos\left(\frac{\pi}{12}t\right) = \frac{3}{5}$$

$$t = \frac{12}{\pi}\cos^{-1}\frac{3}{5}$$

$= 3.542 \ldots$ hours after 4:00 am

$= 3$ hours 32.52 … minutes after 4:00 am

The temperature will be 11° at 7: 33 am.

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20c When $x = 17°$,

$$17 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$$

$$\cos\left(\frac{\pi}{12}t\right) = -\frac{3}{5}$$

$$t = \frac{12}{\pi}\cos^{-1}\left(-\frac{3}{5}\right)$$

$= 8.457 \ldots$ hours after 4:00 am

$= 8$ hours 27.479 … minutes after 4:00 am

The temperature will be 17° at 12: 27 am.

21 Range $= 16 - 10 = 6$ m

$$\frac{T}{2} = 7 \text{ hours } (16{:}00 - 9{:}00)$$

$$T = 14 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{7}$$

$$c = \frac{16 + 10}{2} = 13$$

$$A = 16 - c = c - 10 = 3$$

Therefore

$$x = 13 - 3\cos\left(\frac{\pi}{7}t\right)$$

When $x = 12$,

$$12 = 13 - 3\cos\left(\frac{\pi}{7}t\right)$$

$$\cos\left(\frac{\pi}{7}t\right) = \frac{1}{3}$$

$$t = \frac{7}{\pi}\cos^{-1}\frac{1}{3}$$

$= 2.742 \ldots$ hours after 9:00 am and 2.742 … hours before 10:00 pm

$\div$ 2 hours 45 minutes after 9:00 am and 2 hours 45 minutes before 10:00 pm

Hence, the allowed timings are after 11:45 am and before 8:15 pm.

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22 Consider a SHM with time period T, amplitude A, frequency n and phase angle α.

Say, the equation for displacement of the particle is: $x = A\sin(nt + \alpha)$

So, the velocity of particle is given by: $v = An\cos(nt + \alpha)$

In an SHM, a particle travels a total distance of $4A$ in time period T.

So, the average velocity is:

$$v_{avg} = \frac{4A}{T}$$

Also, since $T = \frac{2\pi}{n}$,

$$v_{avg} = \frac{2An}{\pi}$$

The maximum velocity of particle in SHM is: $v_{max} = An$

Ratio of average speed to maximum speed is

$$\left(\frac{2An}{\pi}\right) : (An)$$

$$= \frac{2}{\pi} : 1$$

$$= 2 : \pi$$

Hence proved.

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6B

(23)(a) $x = 4\sin\left(3t + \frac{\pi}{6}\right) + 2\sin 3t$

$\dot{x} = 12\cos\left(3t + \frac{\pi}{6}\right) + 6\cos 3t$

$\ddot{x} = -36\sin\left(3t + \frac{\pi}{6}\right) - 18\sin 3t$

$= -9x$

so $\ddot{x}$ is of the form $\ddot{x} = -n^2 x$,

so the motion is SHM.

(b) $x = 4\sin 3t\cos\frac{\pi}{6} + 4\cos 3t\sin\frac{\pi}{6} + 2\sin 3t$

$= 2\sqrt{3}\sin 3t + 2\cos 3t + 2\sin 3t$

$= 2(\sqrt{3}+1)\sin 3t + 2\cos 3t$

$= A\sin(3t + \alpha)$,

where $A^2 = 4(\sqrt{3}+1)^2 + 4$

$= 4(4 + 2\sqrt{3}) + 4$

$= 20 + 8\sqrt{3}$

so the amplitude A is $\sqrt{4(5+2\sqrt{3})}$

$= 2\sqrt{5+2\sqrt{3}}$.

6B

(24) Let $x = a\cos(nt + \alpha)$.

When $t=0$, $x=1$ so $a\cos\alpha = 1$ (1)

When $t=1$, $x=5$ so $a\cos(n+\alpha) = 5$ (2)

When $t=2$, $x=5$ so $a\cos(2n+\alpha) = 5$ (3)

We must show that $n = \cos^{-1}\frac{3}{5}$.

Expanding (2): $a\cos n\cos\alpha - a\sin n\sin\alpha = 5$

Using (1): $\cos n - a\sin n\sin\alpha = 5$ (4)

Expanding (3): $a\cos 2n\cos\alpha - a\sin 2n\sin\alpha = 5$

Using (1): $\cos 2n - a\sin 2n\sin\alpha = 5$

$2\cos^2 n - 1 - 2a\sin n\cos n\sin\alpha = 5$

$\cos^2 n - a\sin n\cos n\sin\alpha = 3$

$\cos n(\cos n - a\sin n\sin\alpha) = 3$

Using (4): $\cos n(\cos n + 5 - \cos n) = 3$

so $\cos n = \frac{3}{5}$

$n = \cos^{-1}\frac{3}{5}$, as required.