Solutions to Exercise 6A Foundation questions

1a i $v = 6$

$$v = \frac{dx}{dt}$$

$$\frac{1}{v} = \frac{dt}{dx}$$

$$\frac{1}{6} = \frac{dt}{dx}$$

$$t = \frac{1}{6}x + C$$

Substituting $x = 1$ when $t = 0$,

$$0 = \frac{1}{6}(1) + C$$

$$C = -\frac{1}{6}$$

$$t = \frac{1}{6}(x - 1)$$

1a ii $v = 6$

$$v = \frac{dx}{dt}$$

$$x = 6t + C$$

Substituting $x = 1$ when $t = 0$,

$$1 = C$$

$$\therefore x = 6t + 1$$

1b i $v = -6x^{-2}$

$$v = \frac{dx}{dt}$$

$$-6x^2 = \frac{dt}{dx}$$

$$t = -\frac{1}{18}x^3 + C$$

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Substituting $x = 1$ when $t = 0$,

$$0 = -\frac{1}{18}(1) + C$$

$$\therefore C = \frac{1}{18}$$

$$\therefore t = \frac{1}{18}(1 - x^3)$$

1b ii $v = -6x^{-2}$

$$t = \frac{1}{18}(1 - x^3)$$

$$18t = 1 - x^3$$

$$x^3 = 1 - 18t$$

$$\therefore x = (1 - 18t)^{\frac{1}{3}}$$

1c i $v = -6x^3$

$$-6x^3 = \frac{dx}{dt}$$

$$t = \frac{1}{12}x^{-2} + C$$

Substituting $x = 1$ when $t = 0$

$$0 = \frac{1}{12} + C$$

$$\therefore C = -\frac{1}{12}$$

$$\therefore t = \frac{1}{12}x^{-2} - \frac{1}{12} = \frac{1}{12}(x^{-2} - 1)$$

1c ii $v = -6x^3$

$$t = \frac{1}{12}(x^{-2} - 1)$$

$$x^{-2} = 12t + 1$$

$$x = (12t + 1)^{-\frac{1}{2}}$$

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1d i $v = e^{-2x}$

$e^{-2x} = \dfrac{dx}{dt}$

$t = \dfrac{1}{2}e^{2x} + C$

Substituting $x = 1$ when $t = 0$

$0 = \dfrac{1}{2}e^{2} + C$

$\therefore C = -\dfrac{1}{2}e^{2}$

$\therefore t = \dfrac{1}{2}(e^{2x} - e^{2})$

1d ii $v = e^{-2x}$

$t = \dfrac{1}{2}(e^{2x} - e^{2})$

$2t + e^{2} = e^{2x}$

$\therefore x = \dfrac{1}{2}\ln(2t + e^{2})$

1e i $v = 1 + x^{2}$

$1 + x^{2} = \dfrac{dx}{dt}$

$\dfrac{1}{1 + x^{2}} = \dfrac{dt}{dx}$

$t = \tan^{-1} x + C$

Substituting $x = 1$ when $t = 0$

$0 = \tan^{-1} 1 + C$

$\therefore C = -\dfrac{\pi}{4}$

$\therefore t = \tan^{-1} x - \dfrac{\pi}{4}$

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1e ii $v = 1 + x^2$

$t = \tan^{-1} x - \frac{\pi}{4}$

$t + \frac{\pi}{4} = \tan^{-1} x$

$x = \tan\ (t + \frac{\pi}{4})$

1f i $v = \cos^2 x$

$\cos^2 x = \frac{dx}{dt}$

$\frac{1}{\cos^2 x} = \frac{dt}{dx}$

$t = \tan x + C$

Substituting $x = 1$ when $t = 0$

$0 = \tan 1 + C$

$\therefore C = -\tan 1$

$\therefore t = \tan x - \tan 1$

1f ii $v = \cos^2 x$

$t = \tan x - \tan 1$

$t + \tan 1 = \tan x$

$x = \tan^{-1}(t + \tan 1)$

2a $v = 6$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}6^2\right)$

$= \frac{d}{dx}(18) = 0$

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2b $v = -6x^{-2}$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}(-6x^{-2})^2\right)$$
$$= \frac{d}{dx}(18x^{-4})$$
$$= -72x^{-5}$$

2c $v = -6x^3$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}(-6x^3)^2\right)$$
$$= \frac{d}{dx}(18x^6)$$
$$= 108x^5$$

2d $v = e^{-2x}$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}(e^{-2x})^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2} \times e^{-4x}\right)$$
$$= -2e^{-4x}$$

2e $v = 1 + x^2$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$
$$= \frac{d}{dx}\left(\frac{1}{2}(1 + x^2)^2\right)$$

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$$= \frac{d}{dx}\left(\frac{1}{2}(1 + x^4 + 2x^2)\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2} + \frac{1}{2}x^4 + x^2\right)$$

$$= 2x + 2x^3$$

$$= 2x(1 + x^2)$$

2f $v = \cos^2 x$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(\cos^2 x)^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(\cos^4 x)\right)$$

$$= -2\cos^3 x \sin x$$

3a $\ddot{x} = 6x^2$

From above

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^2$$

$$\because \frac{d}{dx}(v^2) = 12x^2$$

$$\therefore v^2 = 4x^3 + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = 0 + C$$

$$\because C = 0$$

$$\therefore v^2 = 4x^3$$

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3b

$$\ddot{x} = \frac{1}{e^x}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{e^x}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{e^x}$$

$$\therefore v^2 = -\frac{2}{e^x} + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = -2 + C$$

$$\because C = 2$$

$$\therefore v^2 = -\frac{2}{e^x} + 2$$

$$= 2(1 - e^{-x})$$

3c

$$\ddot{x} = \frac{1}{2x + 1}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2x + 1}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{2x + 1}$$

$$\therefore v^2 = \ln|2x + 1| + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = \ln|1| + C$$

$$\because C = 0$$

$$\therefore v^2 = \ln|2x + 1|$$

3d

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$$\ddot{x} = \frac{1}{4 + x^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{4 + x^2}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{4 + x^2}$$

$$\therefore v^2 = \tan^{-1}\frac{x}{2} + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = \tan^{-1} 0 + C$$

$$\because C = 0$$

$$\therefore v^2 = \tan^{-1}\frac{x}{2}$$

4a

$\ddot{x} = \frac{2}{v^2}$ and when $t = 0, v = 0$

$$\frac{dv}{dt} = \frac{2}{v^2}$$

$$\frac{dt}{dv} = \frac{v^2}{2}$$

$$t = \frac{1}{6}v^3 + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $t(0) = 0$

$$0 = 0 + C$$

$$\because C = 0$$

$$\therefore t = \frac{1}{6}v^3$$

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4b $\ddot{x} = v^2$ and when $t = 0, v = \frac{1}{2}$

$$\frac{dv}{dt} = v^2$$

$$\frac{dt}{dv} = \frac{1}{v^2}$$

$$t = -\frac{1}{v} + C$$

for some constant C.

Now apply the conditions $v(0) = \frac{1}{2}$ and $t(0) = 0$

$$0 = -2 + C$$

$$\because C = 2$$

$$\therefore t = 2 - \frac{1}{v}$$

4c $\ddot{x} = 2 + v$ and when $t = 0, v = 1$

$$\frac{dv}{dt} = 2 + v$$

$$\frac{dt}{dv} = \frac{1}{2 + v}$$

$$t = \ln|2 + v| + C$$

for some constant C.

Now apply the conditions $v(0) = 1$ and $t(0) = 0$

$$0 = \ln|2 + 1| + C$$

$$\because C = -\ln|3|$$

$$\therefore t = \ln|2 + v| - \ln|3| = \ln\left|\frac{2 + v}{3}\right|$$

5a

$\ddot{x} = \frac{v^2}{4}$ and when $x = 0, v = 1$

$$v\frac{dv}{dx} = \frac{v^2}{4}$$

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$$\frac{dx}{dv} = \frac{4}{v}$$

$$x = 4\ln|v| + C$$

for some constant C.

Now apply the conditions $v(0) = 1$ and $x(0) = 0$

$$0 = 0 + C$$

$$\because C = 0$$

$$\therefore x = 4\ln|v|$$

5b

$\ddot{x} = \frac{3}{v}$ and when $x = 0, v = 6$

$$v\frac{dv}{dx} = \frac{3}{v}$$

$$\frac{dx}{dv} = \frac{v^2}{3}$$

$$x = \frac{v^3}{9} + C$$

for some constant C.

Now apply the conditions $v(0) = 6$ and $x(0) = 0$

$$0 = 24 + C$$

$$\because C = -24$$

$$\therefore x = \frac{v^3}{9} - 24$$

5c $\ddot{x} = 2 + v$ and when $x = 0, v = 0$

$$v\frac{dv}{dx} = 2 + v$$

$$\frac{dx}{dv} = \frac{v}{2 + v}$$

$$x = 2 + v - 2\ln|2 + v| + C$$

for some constant C.

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Now apply the conditions $v(0) = 6$ and $x(0) = 0$

$0 = 2 - 2\ln|2| + C$

$\therefore C = 2\ln|2| - 2$

$\therefore x = v + 2\ln|2| - 2\ln|2 + v|$

$= v + 2\ln\left|\frac{2}{2+v}\right|$

6a From Newton's second law, after dividing through by the mass,

$\frac{dv}{dt} = 3t - 2.$

Integrating, $v = \frac{3}{2}t^2 - 2t + C$

At $= 0$, $v = 0$, so $C = 0$ and

$v = \frac{3}{2}t^2 - 2t$

$\frac{dx}{dt} = \frac{3}{2}t^2 - 2t$

Integrating, $\frac{dx}{dt} = \frac{3}{2}t^2 - 2t$

$x = \frac{1}{2}t^3 - t^2 + D$

At $t = 0, x = 0$, so $D = 0$ and

$x = \frac{1}{2}t^3 - t^2$

Hence at $t = 4, x = 16$

6b From Newton's second law, after dividing through by the mass,

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{3}x + \frac{2}{3}$

$\frac{d}{dx}(v^2) = \frac{8}{3}x + \frac{4}{3}$

Integrating, $v^2 = \frac{4}{3}x^2 + \frac{4}{3}x + C$

At $= 0$, $v = 0$, so $C = 0$

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$$v^2 = \frac{4}{3}x^2 + \frac{4}{3}x$$

Hence at $x = 3$, $v = 4$

6c

From Newton's second law, after dividing through by the mass,

$$\frac{dv}{dt} = \frac{1}{0.25v + 0.5}$$

$$\frac{dt}{dv} = 0.25v + 0.5$$

Integrating, $t = \frac{1}{8}v^2 + \frac{1}{2}v + C$

At $= 0$, $v = 0$, so $C = 0$

$$t = \frac{1}{8}v^2 + \frac{1}{2}v$$

Hence at $v = 4$, $t = 4$

6d

From Newton's second law, after dividing through by the mass,

$$v\frac{dv}{dx} = \frac{1}{0.5v + 1}$$

$$\frac{dv}{dx} = \frac{1}{0.5v^2 + v}$$

$$\frac{dx}{dv} = 0.5v^2 + v$$

Integrating, $x = \frac{1}{6}v^3 + \frac{1}{2}v^2 + C$

At $= 0$, $x = 0$, so $C = 0$

$$x = \frac{1}{6}v^3 + \frac{1}{2}v^2$$

Hence at $v = 3$, $x = 9$

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Solutions to Exercise 6A Development questions

7 $F_1 = 12\hat{\imath} + 23\hat{\jmath}$

$F_2 = 9\hat{\imath} - 7\hat{\jmath}$

$F_3 = -5\hat{\imath} + 14\hat{\jmath}$

$F_r = F_1 + F_2 + F_3$

$F_r = 16\hat{\imath} + 30\hat{\jmath}$

$F = ma; \quad m = 2$

$a_r = \dfrac{16\hat{\imath} + 30\hat{\jmath}}{2}$

$= 8\hat{\imath} + 15\hat{\jmath}$

$|a_r| = \sqrt{8^2 + 15^2}$

$= 17 \text{ m/s}^2$

$\widehat{a_r} = \tan^{-1}\left(\frac{15}{8}\right)$

The acceleration is 17 m/s^2 at an angle of $\tan^{-1}\left(\frac{15}{8}\right)$ above the horizontal.

8a $OA = 20 \text{ N at } 32°$

$OA = 20\cos 32°\,\hat{\imath} + 20\sin 32°\,\hat{\jmath}$

$= 16.96\hat{\imath} + 10.6\hat{\jmath}$

$OB = 15 \text{ N at } (180 - 54)°$

$OB = 15\cos(180 - 54)°\,\hat{\imath} + 15\sin(180 - 54)°\,\hat{\jmath}$

$= -8.81\hat{\imath} + 12.13\hat{\jmath}$

8b $F_r = OA + OB$

$= 16.96\hat{\imath} + 10.6\hat{\jmath} + (-8.81\hat{\imath} + 12.13\hat{\jmath})$

$= 8.14\hat{\imath} + 22.73\hat{\jmath}$

$|F_r| = \sqrt{8.14^2 + 22.73^2}$

$= 24.14 \ldots$

$\div 24 \text{ N}$

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8c Direction of F_r

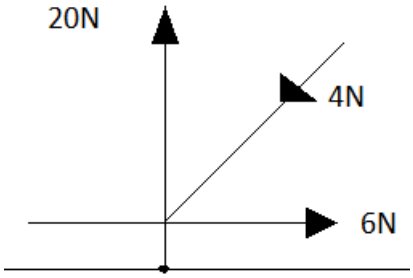
$= \widehat{F_r}$

$= \tan^{-1}\left(\frac{22.73}{8.14}\right)$

$= 70.3 \ldots °$

$\div 70°$ above the horizontal

9a $F_1 = 20\,\hat{j}; \qquad F_2 = 6\,\hat{\imath}; \qquad F_3 = -2\sqrt{2}\,\hat{\imath} - 2\sqrt{2}\,\hat{j}$



$F_r = F_1 + F_2 + F_3$

$= 20\,\hat{j} + 6\,\hat{\imath} + (-2\sqrt{2}\,\hat{\imath} - 2\sqrt{2}\,\hat{j})$

$= (6 - 2\sqrt{2}\,)\hat{\imath} + (20 - 2\sqrt{2})\hat{j}$

9b $|F_r| = \sqrt{(6 - 2\sqrt{2}\,)^2 + (20 - 2\sqrt{2})^2}$

$= 17.462 \ldots$

$\div 17.5$ N

$\widehat{F_r} = \tan^{-1}\left(\frac{20 - 2\sqrt{2}}{6 - 2\sqrt{2}}\right)$

$\div 79.5°$

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10 Initial velocity $= u$

Final velocity $= v$

Time $= t$

Acceleration $= a = \ddot{x}$

Displacement $= s$

$$a = \ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{vdv}{ds}$$

$$ads = vdv$$

$$as = \left[\frac{v^2}{2}\right]_u^v = \frac{v^2 - u^2}{2}$$

From time 0 to t: $dx = (s - 0)$ and $d(v^2) = v^2 - u^2$

Hence $a = \dfrac{\frac{1}{2}(v^2 - u^2)}{s - 0}$

Rearranging gives:

$$v^2 = u^2 + 2as$$

11a Using the result from question 10 and $u = 20, a = g$, with the distance as variable x, we get:

$$v^2 = 400 - 20x$$

At the greatest height, the velocity of the ball will be zero for a moment after which it will start descending back to the ground. So, $v = 0$.

$$x_{max} = H = \frac{400 - 0}{20} = 20 \text{ metres}$$

11b When the ball is rising, from the first and second laws of motion, the ball tries to maintain its inertial state of going up with a velocity of 20 m/s while being acted upon by a gravitational force pulling it towards the ground with an acceleration of 10 m/s^2. Hence, when the ball is rising, it follows the equation:

$$v = \sqrt{400 - 20x}.$$

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11c $v = \sqrt{400 - 20x}$

$\frac{dx}{dt} = \sqrt{400 - 20x}$ as $v = \frac{dx}{dt}$

$\int dt = \int \frac{dx}{\sqrt{400 - 20x}}$

$t = \frac{-1}{10}\sqrt{400 - 20x} + C$

At $t = 0, x = 0$. So, $C = 2$.

$t = 2 - \frac{\sqrt{400 - 20x}}{10}$

$x = 20t - 5t^2$

To find time taken to reach maximum height, put $x = x_{max} = 20$.

$20 = 20t - 5t^2$

$t^2 - 4t + 4 = 0$

$(t - 2)(t - 2) = 0$

$t = 2$

Therefore $T_{max} = 2$ seconds.

12a $u = 1$ km/s $= 1000$ m/s; $a = \ddot{x} = -kv^2 = \frac{vdv}{dx}$; $x = 100$ m; $v = 10$ m/s

$\frac{vdv}{dx} = -kv^2$

$\frac{dv}{v} = -kdx$

$\ln v = -kx + C$

Using the given information of 0 m and after 100 m, we get:

$\ln 1000 = C$

so $C = \ln 1000 = 3 \ln 10$

Also

$\ln 10 = -k(100) + \ln 1000$

$k = \frac{\ln 100}{100}$

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$$= \frac{2 \ln 10}{100}$$

$$= \frac{\ln 10}{50}$$

Therefore

$$\ln v = -\frac{\ln 10}{50} x + 3 \ln 10$$

$$x = 150 - \frac{50 \ln v}{\ln 10}$$

At $v = 1$ m/s,

$$x = 150 - \frac{50(0)}{\ln 10}$$

$$= 150 \text{ metres}$$

12b $u = 1$ km/s $= 1000$ m/s; $a = \ddot{x} = -kv^2 = \frac{dv}{dt}$; $t = 1s$; $v = 10$ m/s

$$-kv^2 = \frac{dv}{dt}$$

$$-kdt = \frac{dv}{v^2}$$

$$-kt = -\frac{1}{v} + C$$

At $t = 0, v = 1000$ and at $t = 1, v = 10$

$$C = \frac{1}{1000} \text{ and } k = \left(\frac{1}{10} - \frac{1}{1000}\right) = \frac{99}{1000}$$

Therefore $\frac{99}{1000} t = \frac{1}{v} - \frac{1}{1000}$

For $v = 1$;

$$t = \frac{1000}{99}\left(1 - \frac{1}{1000}\right)$$

$$= \frac{999}{99}$$

$$= 10\frac{1}{11} \text{ seconds}$$

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13 $\ddot{x} = e^{-x}; \quad u = 2; \quad x = 0 \text{ at } t = 0$

$$\ddot{x} = v\frac{dv}{dx} = e^{-x}$$

$$vdv = e^{-x}dx$$

$$\frac{v^2}{2} = -e^{-x} + C$$

Using initial conditions:

$$\frac{2^2}{2} = -e^0 + C$$

$$2 = -1 + C$$

$$C = 3$$

Therefore

$$\frac{v^2}{2} = 3 - e^{-x}$$

The particle has an acceleration of e^{-x} which is always greater than zero and the particle has an initial velocity of 2. So, from there onwards, the particle can only accelerate and increase the velocity. Hence, the velocity can never fall below 2. After infinite amount of time, the particle will reach a limiting velocity of $\sqrt{6}$ m/s^2

14a $v = 6 - 2x; \quad X_i = (0, 0, 0)$

$$a = \frac{dv}{dt} = v\frac{dv}{dx}$$

$$a = (6 - 2x)(-2)$$

$$= 4x - 12$$

At origin (X_i), $x = 0$.

Therefore $a = \ddot{x} = -12$.

14b $v = 6 - 2x = \frac{dx}{dt}$

$$dt = \frac{dx}{6 - 2x}$$

$$t = \frac{1}{2}\int \frac{dx}{3 - x}$$

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$= -\frac{1}{2}\ln(3-x) + C$

At $t = 0, x = 0$, so

$C = \frac{1}{2}\ln 3$

Therefore

$t = -\frac{1}{2}\ln(3-x) + \frac{1}{2}\ln 3$

$t = -\frac{1}{2}\ln\left(1 - \frac{x}{3}\right)$

$-2t = \ln\left(1 - \frac{x}{3}\right)$

$e^{-2t} = 1 - \frac{x}{3}$

$x = 3(1 - e^{-2t})$

14c As $t \to \infty$, the particle reaches (tends towards) the point $x = 3$.

15a $x_i = 0; \quad u = 0; \quad a = 2(1+v); \quad v = 20$

$a = \frac{dv}{dt} = 2(1+v)$

$2dt = \frac{dv}{(1+v)}$

$2t = \ln(1+v) + C$

Using initial conditions:

$0 = \ln(1+0) + C$

$C = 0$

Therefore

$t = \frac{1}{2}\ln(1+v)$

When $v = 20$,

$t = \frac{1}{2}\ln 21$

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$\div 1.52$ seconds

15b $a = \frac{vdv}{dx} = 2(1+v)$

$2dx = \frac{v}{1+v}\,dv$

$= \left(1 - \frac{1}{1+v}\right)dv$

$2x + C = v - \ln(1+v)$

Using initial conditions, $C = 0$

Therefore

$2x = v - \ln(1+v)$

$x = \frac{1}{2}v - \frac{1}{2}\ln(1+v)$

When $v = 20$,

$x = 10 - \frac{1}{2}\ln(21)$

$x \div 8.48$ m

16a Mass $= m$; $v_i = u$; $x_i = 0$; $v = u + \frac{x}{k}$

$F = ma$ and $a = v\frac{dv}{dx}$

$a = v\frac{dv}{dx}$

$= \left(u + \frac{x}{k}\right)\left(\frac{1}{k}\right)$

$= \frac{v}{k}$

$F = ma$

$= \frac{mv}{k}$

$= \frac{m}{k}v$

Hence, $F \propto v$ and constant of proportionality is $\frac{m}{k}$.

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16b i $v_A = 3u$

$$v_A = u + \frac{x_A}{k} = 3u$$

$$x_A = 2uk$$

16b ii $v = \frac{dx}{dt} = u + \frac{x}{k}$

$$dt = \frac{dx}{u + \frac{x}{k}}$$

$$t + C = k \ln\left(u + \frac{x}{k}\right)$$

Using initial conditions,

$$0 + C = k \ln\left(u + \frac{0}{k}\right)$$

$$C = k \ln u$$

Therefore

$$t = k \ln\left(u + \frac{x}{k}\right) - k \ln u$$

$$t = k \ln\left(1 + \frac{x}{uk}\right)$$

At point A, $x = 2uk$

$$t_A = k \ln\left(1 + \frac{2uk}{uk}\right)$$

$$= k \ln 3$$

17a $m = 0.5$ kg; $F = \left(x - \frac{1}{2}\right)$ N; $x_i = +5$; $u = 0$

$$F = ma$$

$$x - \frac{1}{2} = m\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\frac{v^2}{4} = \int \left(x - \frac{1}{2}\right) dx$$

$$= \frac{x^2 - x}{2} + C$$

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Using initial conditions,

$$0 = \frac{5^2 - 5}{2} + C$$

$$C = -10$$

Therefore

$$v^2 = 2(x^2 - x - 20)$$

$$= 2(x + 4)(x - 5)$$

As $v^2 \geq 0$, the particle cannot be at a position where $x < 5$. Hence the particle can never be at the origin.

17b $v = 2\sqrt{5}; \quad v^2 = 20$

$$20 = 2(x^2 - x - 20)$$

$$x^2 - x - 30 = 0$$

$$(x + 5)(x - 6) = 0$$

$$x = -5 \text{ or } 6$$

As $x \geq 5$; $x = 6$ is the position where the velocity is $2\sqrt{5}$. The particle moves forwards with increasing velocity.

18a $m = 2$ kg; $F = 6x^2$ N; $x_i = 1$; $u_i = -\sqrt{2}$ m/s

$$F = ma$$

$$6x^2 = m\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$v^2 = 6\int x^2\,dx$$

$$= 2x^3 + C$$

Using initial conditions,

$$\left(-\sqrt{2}\right)^2 = 2(1) + C$$

$$C = 0$$

Therefore

$$v^2 = 2x^3$$

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18b $v = \frac{dx}{dt}$ and $v = \sqrt{2x^3}$

$$\frac{dx}{dt} = \sqrt{2x^3}$$

$$dt = \frac{1}{\sqrt{2}} \times \frac{dx}{x^{\frac{3}{2}}}$$

$$dt = \frac{1}{\sqrt{2}} x^{-\frac{3}{2}}\, dx$$

$$t + C = \frac{\frac{1}{\sqrt{2}} x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}$$

$$t + C = -\sqrt{2}\, x^{-\frac{1}{2}}$$

$$t + C = -\sqrt{\frac{2}{x}}$$

Using initial conditions:

$$0 + C = -\sqrt{\frac{2}{1}}$$

$$C = -\sqrt{2}$$

Therefore

$$t - \sqrt{2} = -\sqrt{\frac{2}{x}}$$

$$x = \frac{2}{\left(t - \sqrt{2}\right)^2}$$

The particle starts at position $x = 1$ and moves away from the origin and then it disappears at time $t = \sqrt{2}$.

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19a $\ddot{x} = 3(1 - x^2)$; $u = 4$; $x_i = 0$

$$\ddot{x} = 3(1 - x^2) = \frac{d}{dx}\left(\frac{v^2}{2}\right)$$

$$\frac{v^2}{2} = 3\int (1 - x^2)\, dx$$

$$= 3x - x^3 + C$$

Using initial conditions:

$$\frac{4^2}{2} = 0 - 0 + C$$

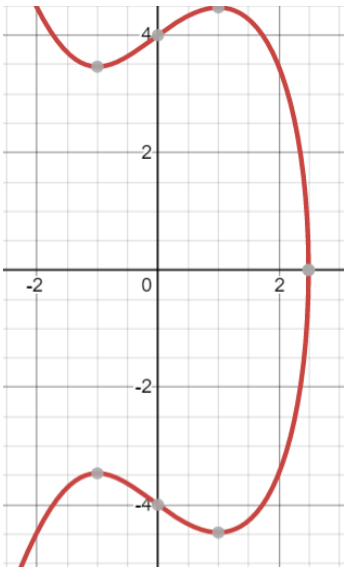
$$C = 8$$

Therefore

$$\frac{v^2}{2} = 3x - x^3 + 8$$

$$v^2 = 6x - 2x^3 + 16$$

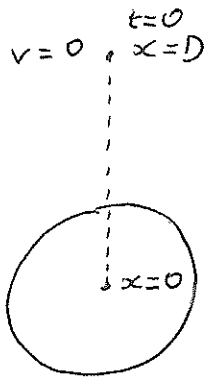
19b $v^2 = 6x - 2x^3 + 16$



From the sketch, we can see that the particle does change direction.

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6A



(20) $\ddot{x} = \frac{d(\frac{1}{2}v^2)}{dx} = -kx^{-2}$

$$\int \frac{d(\frac{1}{2}v^2)}{dx}\,dx = \int -kx^{-2}\,dx$$

$$\frac{1}{2}v^2 = \frac{-kx^{-1}}{-1} + C$$

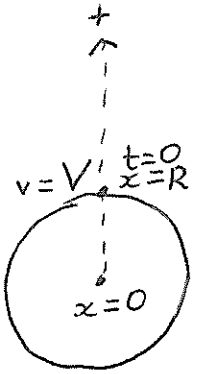
When $t=0$, $x=D$ and $v=0$

so $C = -kD^{-1}$

so $\frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{D}$

$$v^2 = 2k\left(\frac{1}{x} - \frac{1}{D}\right)$$

So the speed is $\sqrt{2k\left(\frac{1}{x} - \frac{1}{D}\right)} = \sqrt{\frac{2k(D-x)}{Dx}}$.



(21) (a) $\ddot{x} = \frac{-k}{x^2}$

When $t=0$, $x=R$ and $\ddot{x} = -g$.

So $-g = \frac{-k}{R^2}$

so $k = gR^2$.

(b) From (20),

$$\frac{1}{2}v^2 = \frac{k}{x} + C$$

$$= \frac{gR^2}{x} + C$$

When $x=R$, $v=V$

so $C = \frac{1}{2}V^2 - gR$

so $\frac{1}{2}v^2 = \frac{gR^2}{x} + \frac{1}{2}V^2 - gR$

$$v^2 = \frac{2gR^2}{x} + V^2 - 2gR$$

At $v=0$, $x=H$ (the greatest height)

so $0 = \frac{2gR^2}{H} + V^2 - 2gR$

$$\frac{2gR^2}{H} = 2gR - V^2$$

$$H = \frac{2gR^2}{2gR - V^2}$$

(c) H does not exist if $V^2 \geq 2gR$

i.e. $V^2 \geq 2 \times 0.0098 \times 6400$ km/s

so $V \geq 11.2$ km/s

6A

(22)(a) $v = 8 - 3e^{-2t}$

$\ddot{x} = 6e^{-2t}$

$= 2(8-v)$

$v \cdot \frac{dv}{dx} = 2(8-v)$

$\frac{dv}{dx} = \frac{2(8-v)}{v}$

$\frac{dx}{dv} = \frac{v}{2(8-v)}$

$2\,dx = \frac{v}{8-v}\,dv$

so $\int 2\,dx = \int \frac{v}{8-v}\,dv$.

(b) Let the required distance be D m.

$\int_0^D 2\,dx = \int_0^7 \frac{v}{8-v}\,dv$

$= \int_0^7 \frac{-(8-v)+8}{8-v}\,dv$

$= \int_0^7 \left(\frac{8}{8-v} - 1\right) dv$

so $[2x]_0^D = [-8\ln|8-v| - v]_0^7$

$2D = 8\ln 8 - 7$

$D = 12\ln 2 - 3.5$

≈ 4.82 m