Solutions to Exercise 5H Chapter review

1 $\underset{\sim}{a} = 6\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 6^2 + (-3)^2 + 2^2$

$|\underset{\sim}{a}|^2 = 36 + 9 + 4$

$|\underset{\sim}{a}|^2 = 49$

$|\underset{\sim}{a}| = 7$

$\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

$= \dfrac{1}{7}\left(6\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} + 2\underset{\sim}{k}\right)$

$= \dfrac{6}{7}\underset{\sim}{\imath} - \dfrac{3}{7}\underset{\sim}{\jmath} + \dfrac{2}{7}\underset{\sim}{k}$

2a $A = 3\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} - 6\underset{\sim}{k}$

$B = -2\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 1\underset{\sim}{k}$

$\overrightarrow{AB} = -2\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 1\underset{\sim}{k} - \left(3\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} - 6\underset{\sim}{k}\right)$

$= -5\underset{\sim}{\imath} - 4\underset{\sim}{\jmath} + 7\underset{\sim}{k}$

2b $A = 3\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} - 6\underset{\sim}{k}$

$B = -2\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 1\underset{\sim}{k}$

$\overrightarrow{BA} = 3\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} - 6\underset{\sim}{k} - \left(-2\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 1\underset{\sim}{k}\right)$

$= 5\underset{\sim}{\imath} + 4\underset{\sim}{\jmath} - 7\underset{\sim}{k}$

2c Distance $AB = |\overrightarrow{AB}|$

$\overrightarrow{AB} = -5\underset{\sim}{\imath} - 4\underset{\sim}{\jmath} + 7\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right|^2 = (-5)^2 + (-4)^2 + 7^2$

$\left|\overrightarrow{AB}\right|^2 = 25 + 16 + 49$

$\left|\overrightarrow{AB}\right| = \sqrt{90} = 3\sqrt{10}$

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3 $A = (6, 12, 7)$

$B = (10, 2, -15)$

$C = (-4, 1, 5)$

$D = (-2, -4, -6)$

$\overrightarrow{AB} = (10 - 6)\underset{\sim}{i} + (2 - 12)\underset{\sim}{j} + (-15 - 7)\underset{\sim}{k}$

$= 4\underset{\sim}{i} - 10\underset{\sim}{j} - 22\underset{\sim}{k}$

$\overrightarrow{CD} = (-2 - (-4))\underset{\sim}{i} + (-4 - 1)\underset{\sim}{j} + (-6 - 5)\underset{\sim}{k}$

$= 2\underset{\sim}{i} - 5\underset{\sim}{j} - 11\underset{\sim}{k}$

If $\overrightarrow{AB} || \overrightarrow{CD}$, then $\overrightarrow{AB} = a\overrightarrow{CD}$, for $a \in \mathbb{R}$

$\overrightarrow{AB} = a\overrightarrow{CD}$

This holds for $a = 2$.

So $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

4 $A = (2, 3, -1)$

$B = (5, -1, 1)$

$C = (-4, 11, -5)$

For A, B and C to be collinear $\overrightarrow{AB} || \overrightarrow{BC} || \overrightarrow{CA}$

$\overrightarrow{AB} = (5 - 2)\underset{\sim}{i} + (-1 - 3)\underset{\sim}{j} + (1 - (-1))\underset{\sim}{k}$

$= 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{BC} = (-4 - 5)\underset{\sim}{i} + (11 - (-1))\underset{\sim}{j} + (-5 - 1)\underset{\sim}{k}$

$\overrightarrow{BC} = -9\underset{\sim}{i} + 12\underset{\sim}{j} - 6\underset{\sim}{k}$

$\overrightarrow{CA} = (2 - (-4))\underset{\sim}{i} + (3 - 11)\underset{\sim}{j} + (-1 - (-5))\underset{\sim}{k}$

$\overrightarrow{CA} = 6\underset{\sim}{i} - 8\underset{\sim}{j} + 4\underset{\sim}{k}$

If $\overrightarrow{AB} || \overrightarrow{BC}$, then $\overrightarrow{AB} = a\overrightarrow{BC}$, for $a \in \mathbb{R}$

$\overrightarrow{AB} = -3\overrightarrow{BC}$

$\overrightarrow{BC} = -\frac{2}{3}\overrightarrow{CA}$

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$\overrightarrow{AB} = 2\overrightarrow{CA}$

So A, B and C are collinear.

5a $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

$= \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$= 16 + 9 + 25$

$= 50$

5b $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$\underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{b}|^2$

$= \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$= 36 + 4 + 4$

$= 44$

5c $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$= 24 - 6 - 10$

$= 8$

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5d $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= |\underset{\sim}{a}^2| + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}^2|$

$= 50 + 16 + 44$

$= 110$

6 $\underset{\sim}{a} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\underset{\sim}{d} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}$

$\overrightarrow{CD} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular if:

$\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$

LHS $= \overrightarrow{AB} \cdot \overrightarrow{CD}$

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$$= \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$

$$= 10 - 4 - 6$$

$$= 0$$

$$= \text{RHS}$$

So $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

7 $\underset{\sim}{a} = (\lambda + 4)\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + (\lambda - 4)\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= ((\lambda + 4)\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + (\lambda - 4)\underset{\sim}{j} + \underset{\sim}{k})$

$= 2(\lambda + 4) + 2(\lambda - 4) + 4$

$= 2\lambda + 8 + 2\lambda - 8 + 4$

$= 4\lambda + 4$

For $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be perpendicular, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

$4\lambda + 4 = 0$

$\lambda = -1$

8 $\underset{\sim}{a} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 4 + 9 + 4$

$|\underset{\sim}{a}|^2 = 4 + 9 + 4$

$= 17$

$|\underset{\sim}{a}| = \sqrt{17}$

$|\underset{\sim}{b}|^2 = 1 + 4 + 1$

$= 6$

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$|\underset{\sim}{b}| = \sqrt{6}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$= -2 + 6 + 2$$

$$= 6$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{6}{\sqrt{102}}$$

9 $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}) \cdot (4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= 8 - 3 + 6$$

$$= 11$$

$$\underset{\sim}{b} \cdot \underset{\sim}{b} = \left(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= 16 + 9 + 4$$

$$= 29$$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$$

$$= \frac{11}{29}(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{44}{29}\underset{\sim}{i} - \frac{33}{29}\underset{\sim}{j} - \frac{22}{29}\underset{\sim}{k}$$

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10a $P = (2, 3, 1)$

$A = (1, 0, -2)$

$B = (0, -1, 1)$

$$\overrightarrow{AP} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

10b $\overrightarrow{AP} = \underset{\sim}{p}$

$$= \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$\overrightarrow{AB} = \underset{\sim}{b}$

$$= \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$\underset{\sim}{b} \cdot \underset{\sim}{p} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$= -1 - 3 + 9$$

$$= 5$$

$$\underset{\sim}{b} \cdot \underset{\sim}{b} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= 1 + 1 + 9$$

$$= 11$$

$$Proj_{\underset{\sim}{b}} \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{5}{11} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

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10c $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

$$d = \left|Proj_{\underset{\sim}{b}}\underset{\sim}{p} - p\right| = \left|\frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b} - \underset{\sim}{p}\right|$$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{5}{11}\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$d = \left|\frac{5}{11}\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}\right|$$

$$= \left\|\begin{bmatrix} -\frac{5}{11} \\ -\frac{5}{11} \\ \frac{15}{11} \end{bmatrix} - \begin{bmatrix} \frac{11}{11} \\ \frac{33}{11} \\ \frac{33}{11} \end{bmatrix}\right\|$$

$$= \left\|\begin{bmatrix} \frac{16}{11} \\ \frac{38}{11} \\ -\frac{18}{11} \end{bmatrix}\right\|$$

$$= \sqrt{\left(\frac{16}{11}\right)^2 + \left(\frac{38}{11}\right)^2 + \left(-\frac{18}{11}\right)^2}$$

$$= \sqrt{\frac{256}{121} + \frac{1444}{121} + \frac{324}{121}}$$

$$= \frac{\sqrt{2024}}{11}$$

$$= \frac{2\sqrt{506}}{11} \text{ units}$$

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11 $X = (-5, 7, 3)$

$Y = (5, -2\ 6)$

$Z = (3, -5, -4)$

$$\overrightarrow{YX} = \begin{bmatrix} -5-5 \\ 7-(-2) \\ 3-6 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \\ -3 \end{bmatrix}$$

$$\overrightarrow{YZ} = \begin{bmatrix} 3-5 \\ -5-(-2) \\ -4-6 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -10 \end{bmatrix}$$

$$\overrightarrow{YX} \cdot \overrightarrow{YZ} = \begin{bmatrix} -10 \\ 9 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \\ -10 \end{bmatrix}$$

$$\overrightarrow{XY} \cdot \overrightarrow{YZ} = (-10 \times -2) + (9 \times -3) + (-3 \times -10)$$
$$= 20 - 27 + 30$$
$$= 23$$

$$\left|\overrightarrow{XY}\right|^2 = (-10)^2 + 9^2 + (-3)^2$$
$$= 100 + 81 + 9$$
$$= 190$$

$$\left|\overrightarrow{XY}\right| = \sqrt{190}$$

$$\left|\overrightarrow{YZ}\right|^2 = (-2)^2 + (-3)^2 + (-10)^2$$
$$= 4 + 9 + 100$$
$$= 113$$

$$\left|\overrightarrow{YZ}\right| = \sqrt{113}$$

$$\cos \angle XYZ = \frac{\overrightarrow{XY} \cdot \overrightarrow{YZ}}{\left|\overrightarrow{XY}\right|\left|\overrightarrow{YZ}\right|}$$

$$\cos \angle XYZ = \frac{23}{\sqrt{190} \times \sqrt{113}}$$

$$\angle XYZ \div 81°$$

Uncorrected proofs

12a For a rhombus with verticies A, B, C, D with respective position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ and diagonals of length a and b:

$$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}$$

$$\underset{\sim}{c} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\underset{\sim}{d} = \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix}$$

The midpoint of $AB = M$

$$\overrightarrow{OM} = \frac{1}{2}\left(\begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix}$$

The midpoint of $BC = N$

$$\overrightarrow{ON} = \frac{1}{2}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix}$$

The midpoint of $CD = P$

$$\overrightarrow{OP} = \frac{1}{2}\left(\begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}\right)$$

Uncorrected proofs

$$= \begin{bmatrix} \frac{3}{4}a \\ -\frac{1}{4}b \end{bmatrix}$$

The midpoint of $DA = Q$

$$\overrightarrow{OQ} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix} \right)$$

$$= \begin{bmatrix} \frac{1}{4}a \\ -\frac{1}{4}b \end{bmatrix}$$

So the sides of the quadrilateral $MNPQ$ will be:

$$\overrightarrow{MN} = \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix}$$

$$\overrightarrow{NP} = \begin{bmatrix} \frac{3}{4}a \\ -\frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}b \end{bmatrix}$$

$$\overrightarrow{PQ} = \begin{bmatrix} \frac{1}{4}a \\ -\frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{3}{4}a \\ -\frac{1}{4}b \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ 0 \end{bmatrix}$$

$$\overrightarrow{QM} = \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{1}{4}a \\ -\frac{1}{4}b \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}b \end{bmatrix}$$

For a rectangle, opposite sides are equal in length and parallel, and adjacent sides are at right-angles.

$\overrightarrow{MN} = -\overrightarrow{PQ}$

$\overrightarrow{NP} = -\overrightarrow{QM}$

So $\overrightarrow{MN}$ and$\overrightarrow{PQ}$ are equal in length and parallel and $\overrightarrow{NP}$ and $\overrightarrow{QM}$ are equal in length and parallel.

For adjacent sides $\overrightarrow{MN}$ and $\overrightarrow{NP}$ to be perpendicular, $\overrightarrow{MN}.\overrightarrow{NP} = 0$:

Uncorrected proofs

$$\overrightarrow{MN}.\overrightarrow{NP} = \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix}.\begin{bmatrix} 0 \\ -\frac{1}{2}b \end{bmatrix}$$

$$= 0$$

So $\overrightarrow{MN}$ and $\overrightarrow{NP}$ are perpendicular.

Therefore opposite sides are parallel and equal in length and adjacent sides are at right angles. Hence $NPQM$ is a rectangle.

12b For a rectangle with vertices A, B, C, D with respective position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ and length a and width b:

$$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

$$\underset{\sim}{c} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\underset{\sim}{d} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

The midpoint of $AB = M$

$$\overrightarrow{OM} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \frac{1}{2}b \end{bmatrix}$$

The midpoint of $BC = N$

$$\overrightarrow{ON} = \frac{1}{2}\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 0 \\ b \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}a \\ b \end{bmatrix}$$

The midpoint of $CD = P$

$$\overrightarrow{OP} = \frac{1}{2}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a \\ \frac{1}{2}b \end{bmatrix}$$

The midpoint of $DA = Q$

$$\overrightarrow{OQ} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix}$$

So the sides of the quadrilateral $MNPQ$ will be:

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$$\overrightarrow{MN} = \begin{bmatrix} \frac{1}{2}a \\ b \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{1}{2}b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}$$

$$\overrightarrow{NP} = \begin{bmatrix} a \\ \frac{1}{2}b \end{bmatrix} - \begin{bmatrix} \frac{1}{2}a \\ b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix}$$

$$\overrightarrow{PQ} = \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix} - \begin{bmatrix} a \\ \frac{1}{2}b \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix}$$

$$\overrightarrow{QM} = \begin{bmatrix} 0 \\ \frac{1}{2}b \end{bmatrix} - \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}$$

For a rhombus all sides are equal in length and opposite sides are parallel.

$\overrightarrow{MN} = -\overrightarrow{PQ}$

$\overrightarrow{NP} = -\overrightarrow{QM}$

So $\overrightarrow{MN}$ and $\overrightarrow{PQ}$ are parallel (and equal in length) and $\overrightarrow{NP}$ and $\overrightarrow{QM}$ are parallel (and equal in length).

$$|\overrightarrow{MN}| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

Similarly,

$$|\overrightarrow{NP}| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(-\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

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$$|\overrightarrow{PQ}| = \sqrt{\left(-\frac{1}{2}a\right)^2 + \left(-\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

$$|\overrightarrow{QM}| = \sqrt{\left(-\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

So all sides are equal in length.

Hence $MNPQ$ is a rhombus.

13 $y = 2x + 3$

For $x = 0, y = 3$

So

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

For $x = 1, y = 5$

So

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda(\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda\left(\begin{bmatrix} 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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14 $\underset{\sim}{r} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x = 2 + 3\lambda \quad (1)$

$y = -4 + \lambda$

$\lambda = y + 4$

Substituting for λ in (1):

$x = 2 + 3(y + 4)$

$x = 2 + 3y + 12$

$x - 3y = 14$

15 $\underset{\sim}{r} = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$x = -6 + 2\lambda \quad (1)$

$y = 4 + \lambda$

So $\lambda = y - 4 \quad (2)$

$z = 3 - 2\lambda \quad (3)$

So $\lambda = \dfrac{3 - z}{2} \quad (4)$

Substituting (2) into (1) to obtain an equation in terms of x and y:

$x = -6 + 2(y - 4)$

$x = -6 + 2y - 8$

$x - 2y = -14 \quad (5)$

Substituting (2) into (3) to obtain an equation in terms of z and y:

$z = 3 - 2(y - 4)$

$z = 3 - 2y + 8$

$z + 2y = 11 \quad (6)$

Substituting (4) into (1) to obtain an equation in terms of x and z:

$x = -6 + 2\left(\dfrac{3 - z}{2}\right)$

$x = -6 + 3 - z$

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$x + z = -3 \qquad (7)$

For ℓ intersecting the xy-plane, $z = 0$.

Substituting $z = 0$ into (6):

$0 + 2y = 11$

$y = \frac{11}{2}$

So (5) becomes:

$x - 2\left(\frac{11}{2}\right) = -14$

$x - 11 = -14$

$x = -3$

So ℓ intersects the xy-plane at:

$\left(-3, \frac{11}{2}, 0\right)$

For ℓ intersecting the yz-plane, $x = 0$

Substituting $x = 0$ into (5):

$0 - 2y = -14$

$y = 7$

So (6) becomes:

$z + 2(7) = 11$

$z + 14 = 11$

$z = -3$

So ℓ intersects the yz-plane at:

$(0, 7, -3)$

For ℓ intersecting the xz-plane, $y = 0$

Substituting $y = 0$ into (6):

$z + 2(0) = 11$

$z = 11$

So (7) becomes:

$x + 11 = -3$

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$x = -14$

So ℓ intersects the xy-plane at:

$(-14, 0, 11)$

16a $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}$

$x = 6 - 5\lambda$

So $\lambda = \dfrac{6 - x}{5}$ (1)

$y = -4 + 2\lambda$

So $\lambda = \dfrac{y + 4}{2}$ (2)

$z = -3 + 7\lambda$

So $\lambda = \dfrac{z + 3}{7}$ (3)

Consider the point $(-4, 0, 13)$. If it lies on the line then the values for λ should be the same.

Substituting $x = -4$ into (1):

$\lambda = \dfrac{6 - (-4)}{5} = 2$

Substituting $y = 0$ into (2):

$\lambda = \dfrac{0 + 4}{2} = 2$

Substituting $z = 13$ into (3):

$\lambda = \dfrac{13 + 3}{7} = \dfrac{16}{7}$

As all the values for λ are not the same, $(-4, 0, 13)$ is not a point on the line $\underset{\sim}{r}$.

Uncorrected proofs

16b $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}$

$x = 6 - 5\lambda$

So $\lambda = \dfrac{6 - x}{5}$ (1)

$y = -4 + 2\lambda$

So $\lambda = \dfrac{y + 4}{2}$ (2)

$z = -3 + 7\lambda$

So $\lambda = \dfrac{z + 3}{7}$ (3)

Consider the point $(16, -8, -17)$. If it lies on the line then the values for λ should be the same.

Substituting $x = 16$ into (1):

$\lambda = \dfrac{6 - 16}{5} = -2$

Substituting $y = -8$ into (2):

$\lambda = \dfrac{-8 + 4}{2} = -2$

Substituting $z = -17$ into (3):

$\lambda = \dfrac{-17 + 3}{7} = -2$

As all the values for λ are the same, $(16, -8, -17)$ is a point on the line $\underset{\sim}{r}$.

17 $\underset{\sim}{p} = \underset{\sim}{\imath} + \underset{\sim}{\jmath} - \underset{\sim}{k}$

$Q = (2, -1, 2)$

$\underset{\sim}{q} = 2\underset{\sim}{\imath} - \underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\overrightarrow{PQ} = (2 - 1)\underset{\sim}{\imath} + (-1 - 1)\underset{\sim}{\jmath} + (2 - (-1))\underset{\sim}{k}$

$\overrightarrow{PQ} = \underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{p} + \lambda\overrightarrow{PQ}$

$\underset{\sim}{r} = \underset{\sim}{\imath} + \underset{\sim}{\jmath} - \underset{\sim}{k} + \lambda(\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 3\underset{\sim}{k})$

Uncorrected proofs

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18a $\underset{\sim}{r}_1 = \begin{bmatrix}6\\5\\3\end{bmatrix} + \lambda\begin{bmatrix}2\\1\\4\end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix}-3\\7\\2\end{bmatrix} + \mu\begin{bmatrix}5\\-4\\-7\end{bmatrix}$

For a point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$:

$\begin{bmatrix}6\\5\\3\end{bmatrix} + \lambda\begin{bmatrix}2\\1\\4\end{bmatrix} = \begin{bmatrix}-3\\7\\2\end{bmatrix} + \mu\begin{bmatrix}5\\-4\\-7\end{bmatrix}$

$6 + 2\lambda = -3 + 5\mu$

$\lambda = -\frac{9}{2} + \frac{5}{2}\mu$

$5 + \lambda = 7 - 4\mu$

Substituting λ we get

$5 + \left(-\frac{9}{2} + \frac{5}{2}\mu\right) = 7 - 4\mu$

$\frac{1}{2} - 7 = -4\mu - \frac{5}{2}\mu$

$-\frac{13}{2} = -\frac{13}{2}\mu$

$\mu = 1$

For λ

$\lambda = -\frac{9}{2} + \frac{5}{2}(1)$

$\lambda = -2$

So:

$x = 6 + 2\lambda$

$x = 6 + 2(-2)$

$x = 2$

$y = 5 + \lambda$

$y = 5 + (-2)$

$y = 3$

$z = 3 + 4\lambda$

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$z = 3 + 4(-2)$

$z = -5$

So $\underset{\sim}{r}_1$ intersects $\underset{\sim}{r}_2$ at point:

$(2, 3, -5)$

18b $\underset{\sim}{r}_1 = -7\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} + 7\underset{\sim}{k} + \lambda(2\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} - 4\underset{\sim}{k})$

$\underset{\sim}{r}_2 = 9\underset{\sim}{\imath} - 4\underset{\sim}{\jmath} - 16\underset{\sim}{k} + \mu(4\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} - 5\underset{\sim}{k})$

For a point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$:

$-7\underset{\sim}{\imath} - \underset{\sim}{\jmath} + 7\underset{\sim}{k} + \lambda(2\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} - 4\underset{\sim}{k}) = 9\underset{\sim}{\imath} - 4\underset{\sim}{\jmath} - 16\underset{\sim}{k} + \mu(4\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} - 5\underset{\sim}{k})$

$-7 + 2\lambda = 9 + 4\mu$

$2\lambda = 16 + 4\mu$

$\lambda = 8 + 2\mu$

$-1 + 3\lambda = -4 - 3\mu$

Substituting λ we get

$-1 + 3(8 + 2\mu) = -4 - 3\mu$

$27 = -9\mu$

$\mu = -3$

For λ

$\lambda = 8 + 2(-3)$

$\lambda = 2$

So:

$x = -7 + 2\lambda$

$x = -7 + 2(2)$

$x = -3$

$y = -1 + 3\lambda$

$y = -1 + 3(2)$

$y = 5$

Uncorrected proofs

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$z = 7 - 4\lambda$

$z = 7 - 4(2)$

$z = 13$

So $\underset{\sim}{r}_1$ intersects $\underset{\sim}{r}_2$ at point:

$(-3, 5, -1)$

19a $\underset{\sim}{c} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$r = \sqrt{7}$

$|\underset{\sim}{r} - \underset{\sim}{c}|^2 = r^2$

$(x-3)^2 + (y+4)^2 + (z-2)^2 = 7$

19b $\underset{\sim}{c} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$r = \sqrt{7}$

$|\underset{\sim}{r} - \underset{\sim}{c}| = r$

$\left|\underset{\sim}{r} - \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}\right| = \sqrt{7}$

20 $P = (5, 1, 4)$

$\underset{\sim}{p} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

$\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right| = 7$

$r = 7$

$\underset{\sim}{c} = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

$r = 7$

$\overrightarrow{CP} = \underset{\sim}{p} - \underset{\sim}{c}$

Uncorrected proofs

$$= \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$$

$$\left|\overrightarrow{CP}\right|^2 = 7^2 + 4^2 + 3^2$$

$$\left|\overrightarrow{CP}\right|^2 = 49 + 16 + 9$$

$$\left|\overrightarrow{CP}\right|^2 = 74$$

$$\overrightarrow{CP} = \sqrt{74}$$

So as $\overrightarrow{CP} > r$ the point $(5, 1, 4)$ lies outside of the sphere $\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right| = 7$.

21 $x^2 + y^2 + z^2 - 4x - 10y + 12z + 41 = 0$

$(x^2 - 4x) + (y^2 - 10y) + (z^2 + 12z) = -41$

$(x^2 - 4x + 4) + (y^2 - 10y + 25) + (z^2 + 12z + 36) = 24$

$(x - 2)^2 + (y - 5)^2 + (z + 6)^2 = 24$

Centre is $(2, 5, -6)$

$r^2 = 24$

$r = \sqrt{24}$

$= 2\sqrt{6}$

Radius is $2\sqrt{6}$ units.

So vector equation is

$$\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix}\right| = 2\sqrt{6}$$

Uncorrected proofs

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22 $(x+2)^2+(y-3)^2+(z+4)^2=125$

$$\left|\underset{\sim}{r}-\begin{bmatrix}-2\\3\\-4\end{bmatrix}\right|=125$$

$$\underset{\sim}{r}=\begin{bmatrix}4\\-5\\1\end{bmatrix}+\lambda\begin{bmatrix}2\\13\\-11\end{bmatrix}$$

$$\left|\begin{bmatrix}4\\-5\\1\end{bmatrix}+\lambda\begin{bmatrix}2\\13\\-11\end{bmatrix}-\begin{bmatrix}-2\\3\\-4\end{bmatrix}\right|=125$$

$$\left|\begin{bmatrix}2\lambda+6\\13\lambda-8\\-11\lambda+5\end{bmatrix}\right|=125$$

$$(2\lambda+6)^2+(13\lambda-8)^2+(-11\lambda+5)^2=125$$

$$4\lambda^2+24\lambda+36+169\lambda^2-208\lambda+64+121\lambda^2-110\lambda+25=125$$

$$294\lambda^2-294\lambda+125=125$$

$$294\lambda^2-294\lambda=0$$

$$\lambda(\lambda-1)=0$$

So the line intersects with the sphere for the values of λ where

$$\lambda=0,1$$

Substituting these values back into the line equation gives:

$$\underset{\sim}{r}=\begin{bmatrix}4\\-5\\1\end{bmatrix}+\lambda\begin{bmatrix}2\\13\\-11\end{bmatrix}$$

For $\lambda=0$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\-5\\1\end{bmatrix}$$

For $\lambda=1$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}4\\-5\\1\end{bmatrix}+\begin{bmatrix}2\\13\\-11\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\8\\-10\end{bmatrix}$$

The line intersects with the sphere at points:

Uncorrected proofs

$(4, -5, 1)$ and $(6, 8, -10)$

23a

$$\underset{\sim}{r} = (2t)\underset{\sim}{i} + \left(\frac{2}{1+t^2}\right)\underset{\sim}{j}$$

$$x = 2t$$

$$t = \frac{x}{2}$$

$$y = \frac{2}{1+t^2}$$

$$y = \frac{2}{1+\left(\frac{x}{2}\right)^2}$$

$$y = \frac{8}{4+x^2}$$

23b

$$\underset{\sim}{r} = \left(\frac{2t}{1+t^2}\right)\underset{\sim}{i} + \left(\frac{1-t^2}{1+t^2}\right)\underset{\sim}{j}$$

$$|\underset{\sim}{r}|^2 = \underset{\sim}{r} \cdot \underset{\sim}{r}$$

$$|\underset{\sim}{r}|^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2$$

$$= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{1}{(1+t^2)^2}(4t^2 + 1 - 2t^2 + t^4)$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2}$$

$$= 1$$

So we have a circle with radius 1.

$x^2 + y^2 = 1$ where $y \neq 1$

Uncorrected proofs

23c $\underset{\sim}{r} = \sin t \underset{\sim}{i} + \sin 2t \underset{\sim}{j}$

$x = \sin t$

$t = \sin^{-1} x$

$y = \sin 2t$

$= 2 \sin t \times \cos t$

$= 2 \sin(\sin^{-1} x) \times \cos(\sin^{-1} x)$

$y = \pm 2x\sqrt{1 - x^2}$ (as $\sin x$ and $\cos x$ can be both positive and negative)

Uncorrected proofs

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