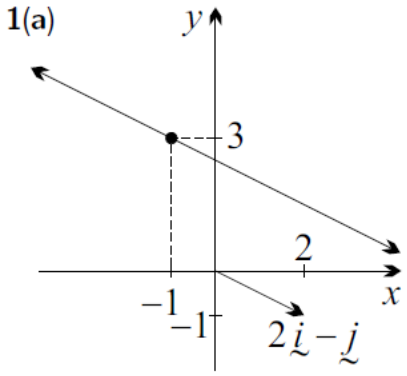
Solutions to Exercise 5F Foundation questions

1a The line intersects the point $A = (-1, 3)$ and is in the direction $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$.



1b $A = (-1, 3)$

$\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j}), \lambda \in \mathbb{R}$

1c $\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$

$m = -\frac{1}{2}$

1d $\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$

So

$x = 2\lambda - 1$

$y = 3 - \lambda$

$x = 2(3 - y) - 1$

$3 - y = \frac{1}{2}x + \frac{1}{2}$

$y = -\frac{1}{2}x + \frac{5}{2}$

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Chapter 5 worked solutions – Vectors

2a $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$x = 3$

$y = \frac{2}{3}(3) - 4$

$y = -2$

so

$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

2b $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$

$m = \frac{b_2}{b_1}$

$m = \frac{2}{3}$

$\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

2c $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

$\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{r} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \lambda(3\underset{\sim}{i} + 2\underset{\sim}{j}), \lambda \in \mathbb{R}$

3a i For $Ax + By = 0$,

directional vector $\underset{\sim}{b}$ is:

$\underset{\sim}{b} = \begin{bmatrix} -B \\ A \end{bmatrix}$ or $\underset{\sim}{b} = \begin{bmatrix} B \\ -A \end{bmatrix}$

So for

$x - 3y + 12 = 0$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

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3a ii $x - 3y + 12 = 0$

For the x-intercept:

$x - 3(0) + 12 = 0$

$x = -12$

x-intercept $= (-12, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} -12 \\ 0 \end{bmatrix}$

For the y-intercept:

$0 - 3y + 12 = 0$

$-3y = -12$

$y = 4$

y-intercept $= (0, 4)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

3a iii $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

Using the y-intercept,

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3b i $x + 3y = 6$

$x + 3y - 6 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

x-intercept is

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$x + 3(0) - 6 = 0$

$x = 6$

x-intercept $= (6, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

3b ii $y = 3$

$y - 3 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

y-intercept is

$y = 3$

y-intercept $= (3, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3b iii $x = -5$

$x + 5 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

x-intercept is $x = -5$

x-intercept $= (-5, 0)$

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So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$x = -3 + \lambda$

$y = 5 + 4\lambda$

So

$\lambda = x + 3$

Substituting this back into the equation for y gives:

$y = 5 + 4(x + 3)$

$y = 5 + 4x + 12$

$y = 4x + 17$

4b $\underset{\sim}{r} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + \lambda(-2\underset{\sim}{i} + 3\underset{\sim}{j})$

$x = 5 - 2\lambda$

$y = 2 + 3\lambda$

$\dfrac{y - 2}{3} = \lambda$

Substituting this back into the equation for x gives:

$x = 5 - 2\left(\dfrac{y - 2}{3}\right)$

$x - 5 = \dfrac{4 - 2y}{3}$

$3x - 15 = 4 - 2y$

So

$3x + 2y = 19$

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5a $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(2, -8)$

$\begin{bmatrix} 2 \\ -8 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$2 = -4 + 3\lambda$

$6 = 3\lambda$

$2 = \lambda$

For y-equation:

$-8 = 2 - 5\lambda$

$10 = 5\lambda$

$2 = \lambda$

As $\lambda = 2$ for both equations, the point $(2, -8)$ lies on the line $\underset{\sim}{r}$.

5b $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(-13, 17)$

$\begin{bmatrix} -13 \\ 17 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$-13 = -4 + 3\lambda$

$-9 = 3\lambda$

$-3 = \lambda$

For y-equation:

$17 = 2 - 5\lambda$

$-15 = 5\lambda$

$-3 = \lambda$

As $\lambda = -3$ for both equations, the point $(-13, 17)$ lies on the line $\underset{\sim}{r}$.

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5c $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(8, -20)$

$\begin{bmatrix} 8 \\ -20 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$8 = -4 + 3\lambda$

$12 = 3\lambda$

$4 = \lambda$

For y-equation:

$-20 = 2 - 5\lambda$

$22 = 5\lambda$

$\frac{22}{5} = \lambda$

As there are different values for λ, the point $(8,20)$ does not lie on the line $\underset{\sim}{r}$.

6a $P(7, 0, -5)$

$\underset{\sim}{a} = 7\underset{\sim}{i} - 5\underset{\sim}{k}$

$\underset{\sim}{b} = -4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = 7\underset{\sim}{i} - 5\underset{\sim}{k} + \lambda(-4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k})$

6b $P(3, 4, 5)$

$\underset{\sim}{a} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$

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7a $P(3, -2, -4)$

$\underset{\sim}{r} = 2\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + \underset{\sim}{k} + \lambda(5\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} - \underset{\sim}{k})$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = 3\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} - 4\underset{\sim}{k}$

$\underset{\sim}{b} = 5\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} - \underset{\sim}{k}$

So

$\underset{\sim}{r} = 3\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} - 4\underset{\sim}{k} + \lambda(5\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} - \underset{\sim}{k})$

7b $P(-1, -1, 2)$

$\underset{\sim}{r} = \frac{1}{3}\underset{\sim}{\imath} - \frac{1}{3}\underset{\sim}{\jmath} - \underset{\sim}{k} + \lambda\left(\frac{1}{6}\underset{\sim}{\imath} + \frac{1}{3}\underset{\sim}{\jmath} + \frac{1}{2}\underset{\sim}{k}\right)$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = -1\underset{\sim}{\imath} - 1\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\underset{\sim}{b} = \frac{1}{6}\underset{\sim}{\imath} + \frac{1}{3}\underset{\sim}{\jmath} + \frac{1}{2}\underset{\sim}{k}$

We multiply the directional vector by a number without altering the direction it represents so:

$\underset{\sim}{b} = \frac{6}{6}\underset{\sim}{\imath} + \frac{6}{3}\underset{\sim}{\jmath} + \frac{6}{2}\underset{\sim}{k}$

$\underset{\sim}{b} = \underset{\sim}{\imath} + 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

So

$\underset{\sim}{r} = -\underset{\sim}{\imath} - \underset{\sim}{\jmath} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{\imath} + 2\underset{\sim}{\jmath} + 3\underset{\sim}{k})$

8a $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

$P = (8, -13, 11)$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

For x-equation:

$8 = 4 - 2\lambda$
$-4 = 2\lambda$

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$\lambda = -2$

For y-equation:

$-13 = -7 + 3\lambda$

$-6 = 3\lambda$

$\lambda = -2$

For z-equation:

$11 = -1 - 6\lambda$

$12 = -6\lambda$

$\lambda = -2$

As $\lambda = -2$ for all three equations, the point P lies on the line $\underset{\sim}{r}$.

8b $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

$P = (-4, 5, -25)$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

For x-equation:

$-4 = 4 - 2\lambda$
$8 = 2\lambda$
$\lambda = 4$

For y-equation:

$5 = -7 + 3\lambda$

$12 = 3\lambda$

$\lambda = 4$

For z-equation:

$-25 = -1 - 6\lambda$

$24 = 6\lambda$

$\lambda = 4$

As $\lambda = 4$ for all three equations, the point P lies on the line $\underset{\sim}{r}$.

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Solutions to Exercise 5F Development questions

9a i $x + 2y - 4 = 0$

$y = 2 - \frac{1}{2}x$

For $x = 0, y = 2$

$(0, 2)$

For $x = 2, y = 1$

$(2, 1)$

The directional vector, $\underset{\sim}{a}$ for this line is:

$\underset{\sim}{a} = \begin{bmatrix} 2 - 0 \\ 1 - 2 \end{bmatrix}$

$\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

9a ii $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

The vector, $\underset{\sim}{n}$, perpendicular to $\underset{\sim}{a}$, is:

$\underset{\sim}{n} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9a iii The vector equation for the perpendicular line through $(2, -3)$ is

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9b $x - y + 3 = 0$

$y = x + 3$

For $x = 0, y = 3$

$(0, 3)$

For $x = 1, y = 4$

$(1, 4)$

So the directional vector for this equation is:

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$$\underset{\sim}{a} = \begin{bmatrix} 1-0 \\ 4-3 \end{bmatrix}$$

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The vector, $\underset{\sim}{n}$, perpendicular to $\underset{\sim}{a}$, is:

$$\underset{\sim}{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The vector equation for the perpendicular line through $(1, -2)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

10 $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ m \end{bmatrix}$$

This gives:

$$x = x_1 + \lambda$$

$$x - x_1 = \lambda$$

$$\frac{y - y_1}{m} = \lambda$$

$$\frac{y - y_1}{m} = x - x_1$$

So

$$y - y_1 = m(x - x_1)$$

11a $A(4, 3)$

$B(6, 0)$

$$\overrightarrow{AB} = \begin{bmatrix} 6-4 \\ 0-3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$$\underset{\sim}{r} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda(2\underset{\sim}{i} - 3\underset{\sim}{j})$$

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11b $A(-7, 5)$

$B(-13, -8)$

$\overrightarrow{AB} = \begin{bmatrix} -13-(-7) \\ -8-5 \end{bmatrix}$

$= \begin{bmatrix} -6 \\ -13 \end{bmatrix}$

$\underset{\sim}{r} = -7\underset{\sim}{\imath} + 5\underset{\sim}{\jmath} - \lambda(6\underset{\sim}{\imath} + 13\underset{\sim}{\jmath})$

which is equivalent to:

$\underset{\sim}{r} = -7\underset{\sim}{\imath} + 5\underset{\sim}{\jmath} + \lambda(6\underset{\sim}{\imath} + 13\underset{\sim}{\jmath})$

12a $P(-1, 3, 1)$

$Q(2, 4, 5)$

$\overrightarrow{PQ} = \begin{bmatrix} 2-(-1) \\ 4-3 \\ 5-1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

$\underset{\sim}{r} = -\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} + \underset{\sim}{k} + \lambda(3\underset{\sim}{\imath} + \underset{\sim}{\jmath} + 4\underset{\sim}{k})$

12b $P(7, -11, 14)$

$Q(17, 9, -16)$

$\overrightarrow{PQ} = \begin{bmatrix} 17-7 \\ 9-(-11) \\ -16-14 \end{bmatrix}$

$= \begin{bmatrix} 10 \\ 20 \\ -30 \end{bmatrix}$

$\underset{\sim}{r} = 7\underset{\sim}{\imath} - 11\underset{\sim}{\jmath} + 14\underset{\sim}{k} + \lambda(10\underset{\sim}{\imath} + 20\underset{\sim}{\jmath} - 30\underset{\sim}{k})$

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13a $A(1, -2)$

$B(5, 4)$

$\overrightarrow{AB} = \begin{bmatrix} 5-1 \\ 4-(-2) \end{bmatrix}$

$= \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 6 \end{bmatrix}, 0 \leq \lambda \leq 1$

13b $A(-1, 1, -2)$

$B(2, 3, -1)$

$\overrightarrow{AB} = \begin{bmatrix} 2-(-1) \\ 3-1 \\ -1-(-2) \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, 0 \leq \lambda \leq 1$

14 $\underset{\sim}{r}_1 = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$

If $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are parallel then there is a value a, that is the ratio between the directional vectors of $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ so

$\begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix} = a \begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$

$6 = -4a$

$a = -\frac{3}{2}$

So for

Uncorrected proofs

$$\text{LHS} = \begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix}$$

$$= -\frac{3}{2}\begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$$

$$= \text{RHS}$$

Since LHS = RHS, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are parallel.

15a $\underset{\sim}{r}_1 = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

Equating the first two components and solving for λ:

$4 + \lambda = 7 + 6\mu$

$\lambda = 3 + 6\mu$

Equating the second two components and substituting for λ:

$8 + 2\lambda = 6 + 4\mu$

$2\lambda = -2 + 4\mu$

$2(3 + 6\mu) = -2 + 4\mu$

$6 + 12\mu = -2 + 4\mu$

$8\mu = -8$

$\mu = -1$

So the point of intersection is:

$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + (-1)\begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = (1, 2, 0)$

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15b $\underset{\sim}{r}_1 = \begin{bmatrix} 7 \\ -3 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 7 \\ -3 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}$

Equating the second two components and solving for λ:

$-3 - \lambda = 1 - 3\mu$

$\lambda = 3\mu - 4$

Equating the first two components and substituting for λ:

$7 + 4\lambda = -2 + 5\mu$

$4\lambda = -9 + 5\mu$

$4(3\mu - 4) = -9 + 5\mu$

$12\mu - 16 = -9 + 5\mu$

$7\mu = 7$

$\mu = 1$

So the point of intersection is:

$\begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix} = (3, -2, 6)$

16 $\underset{\sim}{r}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 3 \\ -3 \end{bmatrix}$

Equating the first two components and solving for λ:

$1 + 2\lambda = 1 - 4\mu$

$\lambda = -2\mu$

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Equating the second two components and substituting for λ:

$-\lambda = 1 + 3\mu$

$2\mu = 1 + 3\mu$

$2\mu = 1 + 3\mu$

$\mu = \frac{1}{5}$

Hence

$\lambda = -\frac{2}{5}$

Equating the third two components and substituting for λ and μ:

$-1 + \lambda = -3\mu$

LHS $= -1 - \frac{2}{5} = -\frac{7}{5}$ and

RHS $= -\frac{3}{5}$

LHS $\neq$ RHS

As the simultaneous equations are inconsistent the lines do not intersect.

These lines are not parallel as there is no value for a at which $v_1 = a\underset{\sim}{v}_2$

where v_1 and v_2 is the directional vectors for $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ respectively and a is a real number.

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are skewed.

17a $v_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

Equating the first components and solving for μ:

$3 + 2\lambda = -2 + \mu$

$\mu = 2\lambda + 5$

Uncorrected proofs

Equating the second components and substituting for μ:

$-2 - \lambda = -2 + 2\mu$

$-2 - \lambda = -2 + 2(2\lambda + 5)$

$-\lambda = 4\lambda + 10$

$-5\lambda = 10$

$\lambda = -2$

Hence

$\mu = 2(-2) + 5$

$\mu = 1$

Using the third components and substituting for λ and μ:

$3 + \lambda = 4 - 3\mu$

LHS $= 3 - 2 = 1$ and

RHS $= 4 - 3 = 1$

So v_1 and v_2 intersect.

$$v_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So v_1 intersects v_2 at $(-1, 0, 1)$ or $-\underset{\sim}{i} + \underset{\sim}{k}$.

17b $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$\underset{\sim}{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Using the first two components and solving for μ:

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$3 + 2\lambda = 2 - \mu$

$\mu = -1 - 2\lambda$

Using the second two components and substituting for μ:

$1 + \lambda = -1 + 2\mu$

$1 + \lambda = -1 + 2(-1 - 2\lambda)$

$4 = -5\lambda$

$\lambda = -\frac{4}{5}$

Hence

$\mu = -1 - 2\left(-\frac{4}{5}\right)$

$\mu = \frac{19}{5}$

Using the third two components and substituting for λ and μ:

$4 - \lambda = 1 + 3\mu$

LHS $= 4 - \left(-\frac{4}{5}\right) = \frac{16}{5}$ and

RHS $= 1 + 3\left(\frac{19}{5}\right) = \frac{39}{5}$

LHS $\neq$ RHS

As the simultaneous equations are inconsistent, the lines do not intersect.

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are skewed.

18a For the points $(2, 0, 1)$ and $(-1, 3, 4)$

the directional vector $\underset{\sim}{v}_1$ will be:

$$\underset{\sim}{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

$$\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

So a vector for the line will be:

Uncorrected proofs

$$\underset{\sim}{r}_1 = \begin{bmatrix}2\\0\\1\end{bmatrix} + \lambda\begin{bmatrix}3\\-3\\-3\end{bmatrix}$$

For the points $(-1, 3, 0)$ and $(4, -2, 5)$,

the directional vector $\underset{\sim}{v}_1$ will be:

$$\underset{\sim}{v}_2 = \begin{bmatrix}-1\\3\\0\end{bmatrix} - \begin{bmatrix}4\\-2\\5\end{bmatrix}$$

$$\underset{\sim}{v}_2 = \begin{bmatrix}-5\\5\\-5\end{bmatrix}$$

So a vector for the line will be:

$$\underset{\sim}{r}_2 = \begin{bmatrix}-1\\3\\0\end{bmatrix} + \mu\begin{bmatrix}-5\\5\\-5\end{bmatrix}$$

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ will intersect for $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$.

$$\begin{bmatrix}2\\0\\1\end{bmatrix} + \lambda\begin{bmatrix}3\\-3\\-3\end{bmatrix} = \begin{bmatrix}-1\\3\\0\end{bmatrix} + \mu\begin{bmatrix}-5\\5\\-5\end{bmatrix}$$

Using the first two components and solving for μ:

$$2 + 3\lambda = -1 - 5\mu$$

$$\mu = -\frac{3}{5} - \frac{3}{5}\lambda$$

Using the third two components and substituting for μ:

$$1 - 3\lambda = -5\mu$$

$$1 - 3\lambda = -5\left(-\frac{3}{5} - \frac{3}{5}\lambda\right)$$

$$1 - 3\lambda = 3 + 3\lambda$$

$$-2 = 6\lambda$$

$$\lambda = -\frac{1}{3}$$

Hence

$$\mu = -\frac{3}{5} - \frac{3}{5}\lambda$$

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$$\mu = -\frac{3}{5} - \frac{3}{5}\left(-\frac{1}{3}\right)$$

$$\mu = -\frac{2}{5}$$

So substituting λ in $\underset{\sim}{r}_1$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ intersect at $(1, 1, 2)$.

18b In order to find the angle between the two vectors, we can use the directional vectors of $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.

$$\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

$$\underset{\sim}{v}_2 = \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

$$\theta = \cos^{-1}\frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||v_2|}$$

$$v_1 \cdot \underset{\sim}{v}_2 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

$$= (3 \times -5) + (-3 \times 5) + (-3 \times -5)$$

$$= -15$$

$$|v_1|^2 = (3)^2 + (-3)^2 + (-3)^2 = 27$$

$$|v_1| = \sqrt{27} = 3\sqrt{3}$$

$$|v_2|^2 = (-5)^2 + 5^2 + (-5)^2 = 75$$

$$|v_2| = \sqrt{75} = 5\sqrt{3}$$

So

$$\theta = \cos^{-1}\frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||v_2|}$$

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$$\theta = \cos^{-1}\left(-\frac{15}{3\sqrt{3} \times 5\sqrt{3}}\right)$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

For the acute angle,

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\theta \div 70.5°$$

19

$$\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\underset{\sim}{r}_2 = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

So if $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ intersect then, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$.

$$\begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

Equating the first two components and solving for a:

$$2 + \lambda = a - \mu$$

$$a = 2 + \lambda + \mu$$

Equating the third two components and solving for λ:

$$13 + 3\lambda = -2 - 3\mu$$

$$3\lambda = -15 - 3\mu$$

$$\lambda = -5 - \mu$$

Equating the second two components and substituting for λ:

$$9 + 2\lambda = 7 + 2\mu$$

$$9 + 2(-5 - \mu) = 7 + 2\mu$$

$$-10 - 2\mu = -2 + 2\mu$$

$$-4\mu = 8$$

$$\mu = -2$$

Uncorrected proofs

Hence

$\lambda = -5 - \mu$

$\lambda = -5 - (-2)$

$\lambda = -3$

So

$a = 2 + \lambda + \mu$

$a = 2 - 3 - 2$

$a = -3$

20a $\underset{\sim}{r} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x = \lambda$

$y = -4 + 2\lambda$

$y = -4 + 2x$

So, for $x = 1$ and $x = 2$ respectively, two points on the line will be

$A = (1, -2)$

$B = (2, 0)$

20b Using A and B from part a:

$P = (-2, 3)$

$A = (1, -2)$

$B = (2, 0)$

$\overrightarrow{AP} = \begin{bmatrix} -2 - 1 \\ 3 - (-2) \end{bmatrix}$

$\overrightarrow{AP} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 2 - 1 \\ 0 - (-2) \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Uncorrected proofs

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20c $\overrightarrow{AB} = \underset{\sim}{b}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AP} = \underset{\sim}{p}$

$\underset{\sim}{p} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{(1 \times -3) + (2 \times 5)}{1^2 + 2^2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{7}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{1}{5}\begin{bmatrix} 7 \\ 14 \end{bmatrix}$

20d The perpendicular distance d from the point P to the line is:

$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{1}{5}\begin{bmatrix} 7 \\ 14 \end{bmatrix}$

$\underset{\sim}{p} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

So

$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$

$d = \left|\dfrac{1}{5}\begin{bmatrix} 7 \\ 14 \end{bmatrix} - \begin{bmatrix} -3 \\ 5 \end{bmatrix}\right|$

$d = \left\|\begin{bmatrix} \dfrac{22}{5} \\ -\dfrac{11}{5} \end{bmatrix}\right\|$

$d^2 = \left(\dfrac{22}{5}\right)^2 + \left(-\dfrac{11}{5}\right)^2$

$d^2 = \dfrac{484}{25} + \dfrac{121}{25}$

Uncorrected proofs

$$d = \sqrt{\frac{121}{5}}$$

$$d = \frac{11\sqrt{5}}{5} \text{ units}$$

21a

$$\underset{\sim}{r} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$x = -1 + \lambda$$

$$y = 1$$

$$z = 2\,\lambda$$

So

$$x = -1 + \frac{z}{2}$$

So, for $y = 1,\ z = 0$ and $y = 1,\ z = 2$ respectively, two points on the line will be

$$A = (-1, 1, 0)$$

$$B = (0, 1, 2)$$

21b Using A and B from part a:

$$P = (1, -1, 1)$$

$$A = (-1, 1, 0)$$

$$B = (0, 1, 2)$$

$$\overrightarrow{AP} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{AP} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Uncorrected proofs

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$$\overrightarrow{AB} = \begin{bmatrix}0\\1\\2\end{bmatrix} - \begin{bmatrix}-1\\1\\0\end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

21c $\overrightarrow{AB} = \underset{\sim}{b}$

$$\underset{\sim}{b} = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$\overrightarrow{AP} = \underset{\sim}{p}$$

$$\underset{\sim}{p} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{(1 \times 2) + (0 \times -2) + (2 \times 1)}{1^2 + 0^2 + 2^2}\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{4}{5}\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix}$$

21d The perpendicular distance d from the point P to the line is:

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix}$$

$$\underset{\sim}{p} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

So

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

Uncorrected proofs

$$d = \left|\frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix} - \begin{bmatrix}2\\-2\\1\end{bmatrix}\right|$$

$$d = \left\|\begin{bmatrix}-\frac{6}{5}\\2\\\frac{3}{5}\end{bmatrix}\right\|$$

$$d^2 = \left(-\frac{6}{5}\right)^2 + 2^2 + \left(\frac{3}{5}\right)^2$$

$$d^2 = \frac{36}{25} + 4 + \frac{9}{25}$$

$$d = \sqrt{\frac{145}{25}}$$

$$d = \frac{\sqrt{145}}{5} \text{ units}$$

22a Two lines are parallel if there is value a at which $\underset{\sim}{n}_1 = a\underset{\sim}{n}_2$

where $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ are the directional vectors for $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ respectively and a is a real number.

In this case the directional vectors $\underset{\sim}{n}_1 = \underset{\sim}{n}_2$ so the lines must be parallel.

22b For the line $\underset{\sim}{v}_2$

where $\underset{\sim}{v}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix} + \mu\begin{bmatrix}1\\-2\\3\end{bmatrix}$

A point on the line $\underset{\sim}{v}_2$ is:

$P = (1, -2, 1)$

The position vector for P, $\underset{\sim}{c}$ will be

$\underset{\sim}{c} = \begin{bmatrix}1\\-2\\1\end{bmatrix}$ or $\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + \underset{\sim}{k}$

Uncorrected proofs

22c The perpendicular distance d from the point P to the line is $\underset{\sim}{v}_2$:

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$$

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

Finding two points on $\underset{\sim}{v}_2$

$$\underset{\sim}{v}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix} + \mu\begin{bmatrix}1\\-2\\3\end{bmatrix}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\-2\\1\end{bmatrix} + \mu\begin{bmatrix}1\\-2\\3\end{bmatrix}$$

$$x = 1 + \mu$$

$$\mu = x - 1$$

$$y = -2 - 2\mu$$

$$y = -2 - 2(x - 1)$$

$$y = -2x \qquad (1)$$

$$z = 1 + 3\mu$$

$$z = 3x - 2 \qquad (2)$$

Using (1) and (2), solve for $x = 0$ and $x = 2$ for points A and B respectively.

$$A = (0, 0, -2)$$

$$B = (2, -4, 4)$$

$$\overrightarrow{AB} = \begin{bmatrix}2\\-4\\4\end{bmatrix} - \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix}2\\-4\\6\end{bmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}$$

$$\underset{\sim}{c} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$$

$$\overrightarrow{AP} = \begin{bmatrix}2\\-4\\4\end{bmatrix} - \begin{bmatrix}2\\1\\-2\end{bmatrix}$$

Uncorrected proofs

$\overrightarrow{AP} = \begin{bmatrix} 0 \\ -5 \\ 6 \end{bmatrix}$

$\overrightarrow{AP} = \underset{\sim}{p}$

$|\underset{\sim}{b}|^2 = 2^2 + (-4)^2 + 6^2$

$|\underset{\sim}{b}|^2 = 4 + 16 + 36$

$|\underset{\sim}{b}| = 56$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \dfrac{0 + 20 + 36}{56}\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$

$d = \left|\begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \\ 6 \end{bmatrix}\right|$

$d = \left|\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right|$

$d^2 = 2^2 + 1^2 + 0^2$

$d^2 = 5$

$d = \sqrt{5}$ units

23a If $ABCD$ is a rhombus then all sides will be equal and opposing sides will be parallel.

$\underset{\sim}{a} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\overrightarrow{AB} = (\underset{\sim}{a} + \underset{\sim}{b}) - (\underset{\sim}{a})$

Uncorrected proofs

$\overrightarrow{AB} = \underset{\sim}{b}$

$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\overrightarrow{BC} = (\underset{\sim}{a} + \underset{\sim}{b} +) - (\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{BC} = \underset{\sim}{c}$

$\overrightarrow{BC} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\overrightarrow{CD} = \underset{\sim}{b}$

$\overrightarrow{CD} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\overrightarrow{CD} = (\underset{\sim}{a} + \underset{\sim}{c}) - (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$

$\overrightarrow{DA} = (\underset{\sim}{a}) - (\underset{\sim}{a} + \underset{\sim}{c})$

$\overrightarrow{DA} = -\underset{\sim}{c}$

$\overrightarrow{DA} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

So in order for the opposing sides to be parallel, for real numbers a and b,

$\overrightarrow{AB} = a\overrightarrow{CD}$ and $\overrightarrow{BC} = b\overrightarrow{DA}$

These hold for $a = 1$ and $b = -1$

If all sides are equal in length, then:

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2 = \left|\overrightarrow{DA}\right|^2$

We already know that

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{CD}\right|^2$ and $\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{DA}\right|^2$

So we need to see if $\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2$

$\left|\overrightarrow{BC}\right|^2 = 2^2 + 3^2$

$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2$

So as $ABCD$ has all sides equal in length and opposite sides are parallel,

$ABCD$ is a rhombus.

Uncorrected proofs

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23b The bisector of angle $\angle ABC$ is BD.

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{OB} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\overrightarrow{OD} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{OD} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda(\overrightarrow{OD} - \overrightarrow{OB})$

$\underset{\sim}{r}_{BD} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

$\underset{\sim}{r}_{BD} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

As the bisector of angle $\angle ABC$ is perpendicular to $\angle BAD$,

the vector equation for AC will be

$\underset{\sim}{r}_{AC} = \overrightarrow{OA} + \lambda \begin{bmatrix} -1 \\ (-1) \times 1 \end{bmatrix}$

$\underset{\sim}{r}_{AC} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

24a For $\overrightarrow{OM}$

$\overrightarrow{OA} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{OC} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$

$\overrightarrow{OM} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$

$\overrightarrow{OM} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}\right)$

$\overrightarrow{OM} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$

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24b For a vector equation for the line BD

$$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$$

$$\overrightarrow{OB} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{OD} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$$

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right)$$

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

24c Solving for λ:

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$z = 2\lambda$$

$$\lambda = \frac{z}{2}$$

$$M = \left(2, \frac{1}{2}, \frac{3}{2}\right)$$

So

$$\lambda = \frac{3}{2} \times \frac{1}{2}$$

$$\lambda = \frac{3}{4}$$

Substituting this back into $\underset{\sim}{r}_{BD}$ should be equal to M if M is on the line:

LHS $= \underset{\sim}{r}_{BD}$

Uncorrected proofs

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$= M$$

$$= \text{RHS}$$

So M lies on the line $\underset{\sim}{r}_{BD}$.

24d $BM : MD$

$$\overrightarrow{OB} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{OD} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{OM} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$\overrightarrow{BM} = \left(\begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right)$$

$$\overrightarrow{BM} = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

Uncorrected proofs

$$\overrightarrow{MD} = \left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \right)$$

$$\overrightarrow{MD} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$BM : MD = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} : \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= 3 : 1$$

25a The interval of the line segment AB.

25b The ray with endpoint B in the direction of $\underset{\sim}{b} - \underset{\sim}{a}$.

25b The ray with endpoint A in the direction of $\underset{\sim}{a} - \underset{\sim}{b}$.

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5F

(26)(a) P is the variable point $(-2+\lambda, 1, 2-\lambda)$,

so $\vec{AP} = \begin{pmatrix} -3+\lambda \\ 0 \\ 1-\lambda \end{pmatrix}$

so $|\vec{AP}|^2 = (\lambda-3)^2 + (1-\lambda)^2$

$= 2\lambda^2 - 8\lambda + 10$

so $|\vec{AP}| = \sqrt{2\lambda^2 - 8\lambda + 10}$.

(b) $|\vec{AP}| = \sqrt{2(\lambda^2 - 4\lambda + 4) + 10 - 8}$

$= \sqrt{2(\lambda-2)^2 + 2}$

so the minimum distance from A to l is $\sqrt{2}$ units, and this occurs when $\lambda = 2$.

(c) When $\lambda = 2$, $\vec{AP} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$.

The direction of l is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1 + 0 + 1$

$= 0$,

so $\vec{AP} \perp$ the direction vector of l,

so the minimum distance is the perpendicular distance.

(d) The dot product of $\vec{AP}$ and the direction vector of l is $\begin{pmatrix} -3+\lambda \\ 0 \\ 1-\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -3 + \lambda - 1 + \lambda$

$= 2\lambda - 4$.

For the minimum (i.e. perpendicular) distance,

$2\lambda - 4 = 0$

$\lambda = 2$.

When $\lambda = 2$, $\vec{AP} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and $|\vec{AP}| = \sqrt{(-1)^2 + 0^2 + 1^2}$

$= \sqrt{2}$.

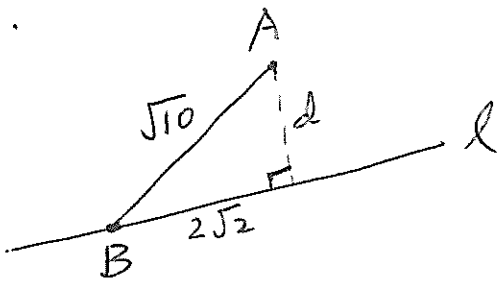
5F

(26)(e) Suppose we let B be the point on ℓ corresponding to $\lambda = 0$.

So B is the point $(-2, 1, 2)$.

Then $\overrightarrow{BA} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$,

and so $|\overrightarrow{BA}| = \sqrt{10}$.



Now, $|\text{proj}_{\ell} \overrightarrow{BA}| = \dfrac{\left|\overrightarrow{BA} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right|}{\left|\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right|}$

$$= \frac{3+1}{\sqrt{2}}$$

$$= 2\sqrt{2}.$$

Finally, by Pythagoras,

$d^2 = (\sqrt{10})^2 - (2\sqrt{2})^2$

$= 2$

so d, the minimum distance from A to ℓ, is $\sqrt{2}$.