Chapter 5 worked solutions – Vectors

Solutions to Exercise 5E Foundation questions

1 For a quadrilateral $OABC$, where the diagonals OB and AC bisect each other

Let M be the point where the diagonals intersect one another.

Since M bisects both diagonals:

$$\overrightarrow{OM} = \frac{\overrightarrow{OB}}{2} = \overrightarrow{MB}$$

$$\overrightarrow{AM} = \frac{\overrightarrow{AC}}{2} = \overrightarrow{MC}$$

Now we consider two opposite sides of the quadrilateral:

$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OM} + \overrightarrow{MA} \\ &= \overrightarrow{OM} - \overrightarrow{AM} \\ &= \frac{\overrightarrow{OB}}{2} - \frac{\overrightarrow{AC}}{2}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{CM} + \overrightarrow{MB} \\ &= -\overrightarrow{MC} + \overrightarrow{MB} \\ &= -\frac{\overrightarrow{AC}}{2} + -\frac{\overrightarrow{OB}}{2} \\ &= \overrightarrow{OA}\end{aligned}$$

Therefore sides OA and CB are parallel and equal. This is sufficient to prove that $OABC$ is a parallelogram.

2a $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ as they are perpendicular.

2b $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

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Chapter 5 worked solutions – Vectors

2c Since the diagonals are perpendicular, $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

Therefore

$(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$

Expanding:

$\underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$

From part a:

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$

Therefore

$\underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$

$|\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = 0$

$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2$

$|\underset{\sim}{c}| = |\underset{\sim}{a}|$

Therefore $OABC$ is a rectangle whose sides are equal, making it a square.

3a $\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

$|\overrightarrow{OA}| = |\overrightarrow{OB}|$ as both $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$ are the radius of the circle.

For $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= |\overrightarrow{OA}|^2$

$= |\overrightarrow{OB}|^2$

$= \underset{\sim}{b} \cdot \underset{\sim}{b}$

$=$ RHS

Thus $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

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3b $\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

Therefore:

$\overrightarrow{AM} = \underset{\sim}{m} - \underset{\sim}{a}$

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = \left|\overrightarrow{AM}\right|^2$

Similarly:

$\overrightarrow{BM} = \underset{\sim}{m} - \underset{\sim}{b}$

$(\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b}) = \left|\overrightarrow{BM}\right|^2$

Because M bisects AB, we know that $\left|\overrightarrow{AM}\right| = \left|\overrightarrow{BM}\right|$.

Therefore $\left|\overrightarrow{AM}\right|^2 = \left|\overrightarrow{BM}\right|^2$

Therefore $(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$.

3c For $\overrightarrow{OM} \perp \overrightarrow{AB}$, it is sufficient to prove that $\overrightarrow{OM} \cdot \overrightarrow{AB} = 0$

From part b:

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

$\underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$-2\underset{\sim}{m} \cdot \underset{\sim}{a} = -2\underset{\sim}{m} \cdot \underset{\sim}{b}$ (using the fact that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$)

$\underset{\sim}{m} \cdot \underset{\sim}{a} = \underset{\sim}{m} \cdot \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OM} \cdot \overrightarrow{AB}$

$= \underset{\sim}{m} \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{m} \cdot \underset{\sim}{b} - \underset{\sim}{m} \cdot \underset{\sim}{a}$

$= 0$

Therefore $\overrightarrow{OM} \perp \overrightarrow{AB}$.

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4 The converse result is that if M lies on the chord AB and $\overrightarrow{OM} \perp \overrightarrow{AB}$, then M bisects AB.

We have:

$\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$

and since $\overrightarrow{OM} \perp \overrightarrow{AB}$, we have

$\overrightarrow{OM} \cdot \overrightarrow{AB} = 0$

Therefore

$\underset{\sim}{m} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$

$\underset{\sim}{m} \cdot \underset{\sim}{b} = \underset{\sim}{m} \cdot \underset{\sim}{a}$

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a})$

$= \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$ (using $\underset{\sim}{m} \cdot \underset{\sim}{b} = \underset{\sim}{m} \cdot \underset{\sim}{a}$ and $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$)

$= (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

Therefore

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

$|(\underset{\sim}{m} - \underset{\sim}{a})|^2 = |(\underset{\sim}{m} - \underset{\sim}{b})|^2$

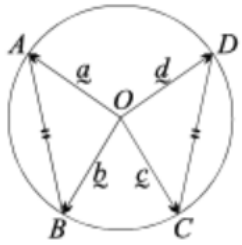
$\left|\overrightarrow{AM}\right| = \left|\overrightarrow{BM}\right|$

Therefore M bisects AB.

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Solutions to Exercise 5E Development questions

5a $|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{d} - \underset{\sim}{c}|^2$



$|\overrightarrow{AB}| = |\overrightarrow{DC}|$

$\underset{\sim}{b} - \underset{\sim}{a} = \overrightarrow{AB}$

$\underset{\sim}{d} - \underset{\sim}{c} = \overrightarrow{DC}$

LHS $= |\underset{\sim}{b} - \underset{\sim}{a}|^2$

$= |\overrightarrow{AB}|^2$

$= |\overrightarrow{DC}|^2$

$= |\underset{\sim}{d} - \underset{\sim}{c}|^2$

$=$ RHS

So the equation is satisfied.

5b For $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{d} \cdot \underset{\sim}{c}$

$|\overrightarrow{AB}| = |\overrightarrow{DC}|$

As they are radial:

$|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$= |\underset{\sim}{c}|$

$= |\underset{\sim}{d}|$

$|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{d} - \underset{\sim}{c}|^2$

$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{d}|^2 + |\underset{\sim}{c}|^2 - 2\underset{\sim}{d} \cdot \underset{\sim}{c}$

$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{d}|^2 - |\underset{\sim}{c}|^2 + 2\underset{\sim}{d} \cdot \underset{\sim}{c}$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= \frac{1}{2}|\underset{\sim}{a}|^2 + \frac{1}{2}|\underset{\sim}{b}|^2 - \frac{1}{2}|\underset{\sim}{d}|^2 - \frac{1}{2}|\underset{\sim}{c}|^2 + \underset{\sim}{d} \cdot \underset{\sim}{c}$

$= \frac{1}{2}|\underset{\sim}{a}|^2 + \frac{1}{2}|\underset{\sim}{a}|^2 - \frac{1}{2}|\underset{\sim}{a}|^2 - \frac{1}{2}|\underset{\sim}{a}|^2 + \underset{\sim}{d} \cdot \underset{\sim}{c}$

$= \underset{\sim}{d} \cdot \underset{\sim}{c}$

Uncorrected proofs

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$= \text{RHS}$

So the equation is satisfied.

5c For $\angle AOB = \angle COD$

$\text{LHS} = \angle AOB$

$= \cos^{-1}\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \cos^{-1}\dfrac{\underset{\sim}{c}\cdot\underset{\sim}{d}}{|\underset{\sim}{c}||\underset{\sim}{d}|}$

$= \angle COD$

$= \text{RHS}$

So the equation is satisfied.

6a $\overrightarrow{AB} = \underset{\sim}{b} + \underset{\sim}{a}$

$\overrightarrow{ON} = \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$

$\overrightarrow{OM} = \dfrac{1}{2}\underset{\sim}{c}$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

For $\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c})$

$\text{LHS} = \overrightarrow{MN}$

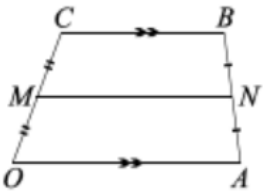
$= \overrightarrow{ON} - \overrightarrow{OM}$

$= \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a}) - \dfrac{1}{2}\underset{\sim}{c}$

$= \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a} - \underset{\sim}{c})$

$= \text{RHS}$

So the equation is satisfied.



Uncorrected proofs

6b For $\underset{\sim}{b} - \underset{\sim}{c} = k\underset{\sim}{a}$

$\overrightarrow{CB} || \overrightarrow{OA}$ where $\underset{\sim}{a} = \overrightarrow{OA}$

So as $\overrightarrow{CB}$ and $\overrightarrow{OA}$ are parallel there must exist a constant value k whereby

$\overrightarrow{CB} = k\underset{\sim}{a}$

LHS $= \underset{\sim}{b} - \underset{\sim}{c}$

$= \overrightarrow{CB}$

$= k\underset{\sim}{a}$

$=$ RHS

6c $\underset{\sim}{b} - \underset{\sim}{c} = k\underset{\sim}{a}$

$\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c})$

$\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a} + k\underset{\sim}{a})$

$\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{a}(1 + k)$

and

$\overrightarrow{OA} || \overrightarrow{CB}$

Thus $\overrightarrow{MN} || \overrightarrow{OA}$ and $\overrightarrow{MN} || \overrightarrow{CB}$.

7 $O = (0\ ,0\ ,0)$

$A = (a_1, a_2, a_3)$

$B = (b_1, b_2, b_3)$

$C = (c_1, c_2, c_3)$

So the mid points are:

$M = \frac{1}{2}(a_1, a_2, a_3)$

$N = \frac{1}{2}(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$P = \frac{1}{2}(b_1 + c_1, b_2 + c_2, b_3 + c_3)$

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$R = \frac{1}{2}(c_1, c_2, c_3)$

In order for $MNPR$ to be a parallelogram, both pairs of opposite sides are parallel, in which:

$|\overrightarrow{MN}| = \lambda|\overrightarrow{PR}|$ and $|\overrightarrow{NP}| = \mu|\overrightarrow{RM}|$, where λ and μ are real numbers.

$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \frac{1}{2}\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{b}$

$\overrightarrow{PR} = \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) = -\frac{1}{2}\underset{\sim}{b}$

So

$|\overrightarrow{MN}| = \lambda|\overrightarrow{PR}|$

LHS $= |\overrightarrow{MN}|$

$= \frac{1}{2}\underset{\sim}{b}$

$= -|\overrightarrow{PR}|$

$=$ RHS

$\overrightarrow{NP} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) - \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{NP} = \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$

$\overrightarrow{RM} = \frac{1}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}$

So

$|\overrightarrow{NP}| = \mu|\overrightarrow{RM}|$

LHS $= |\overrightarrow{NP}|$

Uncorrected proofs

$$= \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$$

$$= -\left|\overrightarrow{RM}\right|$$

$$= \text{RHS}$$

Hence both pairs of opposite sides are parallel (and equal in length).

Thus $MNPR$ is a parallelogram.

8a We want to find $\angle FOD$.

$$\angle FOD = \cos^{-1}\frac{\overrightarrow{OF}\cdot\overrightarrow{OD}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{OD}\right|}$$

$$\overrightarrow{OD} = \begin{bmatrix}0\\a\\a\end{bmatrix}$$

$$\overrightarrow{OF} = \begin{bmatrix}a\\0\\a\end{bmatrix}$$

$$\left|\overrightarrow{OD}\right|^2 = a^2 + a^2$$

$$\left|\overrightarrow{OD}\right| = \sqrt{2}a$$

$$\left|\overrightarrow{OF}\right|^2 = a^2 + a^2$$

$$\left|\overrightarrow{OF}\right| = \sqrt{2}a$$

$$\overrightarrow{OF}\cdot\overrightarrow{OD} = (0\times a + a\times 0 + a\times a)$$

$$\overrightarrow{OF}\cdot\overrightarrow{OD} = a^2$$

So

$$\angle FOD = \cos^{-1}\left(\frac{a^2}{\sqrt{2}a\times\sqrt{2}a}\right)$$

$$= \cos^{-1}\frac{1}{2}$$

$$= 60°$$

8b The triangle $\angle FOD$ is an equilateral triangle as all interior angles of the triangle are 60°.

Uncorrected proofs

8c The triangular pyramid $OBDF$ is a regular tetrahedron, as each face of the tetrahedron is a equilateral triangle.

8d

$$\overrightarrow{OX} = \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

$$\angle FXD = \cos^{-1}\frac{\overrightarrow{XF}\cdot\overrightarrow{XD}}{\left|\overrightarrow{XF}\right|\left|\overrightarrow{XD}\right|}$$

$$\overrightarrow{XF} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} - \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

$$\overrightarrow{XF} = \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

$$\overrightarrow{XD} = \begin{bmatrix} -\frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

$$\left|\overrightarrow{XD}\right|^2 = \frac{1}{4}a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2$$

$$\left|\overrightarrow{XD}\right|^2 = \frac{3}{4}a^2$$

$$\left|\overrightarrow{XD}\right| = \frac{\sqrt{3}}{2}a$$

$$\left|\overrightarrow{XF}\right|^2 = \frac{1}{4}a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2$$

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$$\left|\overrightarrow{XF}\right|^2 = \frac{3}{4}a^2$$

$$\left|\overrightarrow{XF}\right| = \frac{\sqrt{3}}{2}a$$

$$\overrightarrow{XF} \cdot \overrightarrow{XD} = -\frac{1}{2}a \times \frac{1}{2}a - \frac{1}{2}a \times \frac{1}{2}a + \frac{1}{2}a \times \frac{1}{2}a$$

$$\overrightarrow{OF} \cdot \overrightarrow{OD} = -\frac{1}{4}a^2$$

So

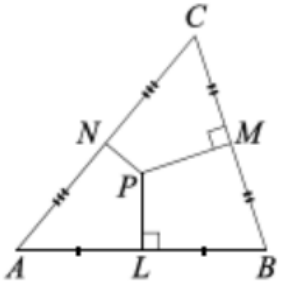
$$\angle FXD = \cos^{-1}\left(\frac{-\frac{1}{4}a^2}{\frac{\sqrt{3}}{2}a \times \frac{\sqrt{3}}{2}a}\right)$$

$$= \cos^{-1}\left(-\frac{1}{3}\right)$$

$$= 109.471\ldots°$$

$$\doteqdot 109° 28'$$

9a $\overrightarrow{PN} = \overrightarrow{BN} - \overrightarrow{BP}$



$\overrightarrow{AB} = \underset{\sim}{u}$

$\overrightarrow{BC} = \underset{\sim}{v}$

$\overrightarrow{PL} = \underset{\sim}{w}$

$$\overrightarrow{BN} = \frac{1}{2}\left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$

$$\overrightarrow{BN} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$$

$$\overrightarrow{BP} = -\overrightarrow{LB} - \overrightarrow{PL}$$

$$\overrightarrow{PB} = -\frac{1}{2}\underset{\sim}{u} - \underset{\sim}{w}$$

$$\overrightarrow{PN} = \overrightarrow{BN} - \overrightarrow{BP}$$

$$\overrightarrow{PN} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u}) - \left(-\frac{1}{2}\underset{\sim}{u} - \underset{\sim}{w}\right)$$

$$\overrightarrow{PN} = \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{u} + \underset{\sim}{w}$$

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$$\overrightarrow{PN} = \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}$$

9b As $\overrightarrow{AB} \perp \overrightarrow{PL}$

$$\overrightarrow{AB} \cdot \overrightarrow{PL} = 0$$

$$\underset{\sim}{u} \cdot \underset{\sim}{w} = 0$$

As $\overrightarrow{BC} \perp \overrightarrow{MP}$

$$\overrightarrow{BC} \cdot \overrightarrow{MP} = 0$$

$$\overrightarrow{MP} = \overrightarrow{ML} - \overrightarrow{PL}$$

$$\overrightarrow{MP} = \left(-\frac{1}{2}\underset{\sim}{v} + \left(-\frac{1}{2}\underset{\sim}{u}\right)\right) - \underset{\sim}{w}$$

$$\overrightarrow{MP} = -\frac{1}{2}\underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v} - \underset{\sim}{w}$$

So

$$\underset{\sim}{v} \cdot \left(-\frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v}) - \underset{\sim}{w}\right) = 0$$

$$0 = -\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} - \underset{\sim}{v} \cdot \underset{\sim}{w}$$

$$\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} = -\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{w}$$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{AC} = \underset{\sim}{u} + \underset{\sim}{v}$$

So for the triangle ΔABC to be concurrent, $\overrightarrow{PN} \perp \overrightarrow{AC}$ and $\overrightarrow{PN} \cdot \overrightarrow{AC} = 0$

$$\text{LHS} = \overrightarrow{PN} \cdot \overrightarrow{AC}$$

$$= \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w} \cdot (\underset{\sim}{u} + \underset{\sim}{v})$$

$$= \left(\frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}\right) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$$

$$= \left(\frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}\right) \cdot (\underset{\sim}{u} + \underset{\sim}{v})$$

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$$= \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{w} \cdot \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \underset{\sim}{w} \cdot \underset{\sim}{v}$$

$$= \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{w} \cdot \underset{\sim}{u} + \left(\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{w}\right) + \underset{\sim}{w} \cdot \underset{\sim}{v}$$

$$= \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u}$$

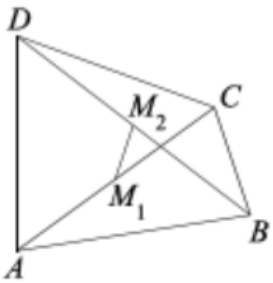
$$= 0$$

$$= \text{RHS}$$

So $\overrightarrow{PN} \perp \overrightarrow{AC}$.

Thus, the perpendicular bisectors of the sides of ΔABC are concurrent.

10 $\overrightarrow{AB} = \underset{\sim}{b}$



$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$

$\overrightarrow{CD} = \underset{\sim}{d} - \underset{\sim}{c}$

$\overrightarrow{DA} = -\underset{\sim}{d}$

$\overrightarrow{AC} = \underset{\sim}{c}$

$\left|\overrightarrow{AC}\right|^2 = \underset{\sim}{c} \cdot \underset{\sim}{c}$

$\overrightarrow{BD} = \underset{\sim}{d} - \underset{\sim}{b}$

$\left|\overrightarrow{BD}\right|^2 = |\underset{\sim}{d} - \underset{\sim}{b}|^2$

$\left|\overrightarrow{BD}\right|^2 = \underset{\sim}{d} \cdot \underset{\sim}{d} - 2\underset{\sim}{d} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

For M_1 and M_2 where M_1 is the centre of AC and M_2 is the centre of BD,

$\overrightarrow{AM_1} = \frac{1}{2}\underset{\sim}{c}$

$\overrightarrow{AM_2} = \overrightarrow{AB} + \overrightarrow{BM_2}$

$= \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{d} - \underset{\sim}{b})$

$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b})$

$\overrightarrow{M_1M_2} = \overrightarrow{AM_2} - \overrightarrow{AM_1}$

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$$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b}) - \frac{1}{2}\underset{\sim}{c}$$

$$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c})$$

$$\left|\overrightarrow{M_1M_2}\right|^2 = \frac{1}{4}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c})^2$$

$$\left|\overrightarrow{M_1M_2}\right|^2 = \frac{1}{4}(\underset{\sim}{d}\cdot\underset{\sim}{d} + \underset{\sim}{d}\cdot\underset{\sim}{b} - \underset{\sim}{d}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{c}\cdot\underset{\sim}{d} - \underset{\sim}{c}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c})$$

$$\left|\overrightarrow{M_1M_2}\right|^2 = \frac{1}{4}(\underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} - 2\underset{\sim}{d}\cdot\underset{\sim}{c})$$

$$4\left|\overrightarrow{M_1M_2}\right|^2 = \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} - 2\underset{\sim}{d}\cdot\underset{\sim}{c}$$

$$4\left|\overrightarrow{M_1M_2}\right|^2 = \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{c}\cdot(\underset{\sim}{b} + \underset{\sim}{d})$$

So

$$2\underset{\sim}{c}\cdot(\underset{\sim}{b} + \underset{\sim}{d}) = \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - 4\left|\overrightarrow{M_1M_2}\right|^2$$

Prove:

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DA}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + 4\left|\overrightarrow{M_1M_2}\right|^2$$

LHS

$$= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DA}\right|^2$$

$$= |\underset{\sim}{b}|^2 + |\underset{\sim}{c} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{c}|^2 + |-\underset{\sim}{d}|^2$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} + (\underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{c}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}) + (\underset{\sim}{d}\cdot\underset{\sim}{d} - 2\underset{\sim}{d}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{c}) + \underset{\sim}{d}\cdot\underset{\sim}{d}$$

$$= 2\underset{\sim}{c}\cdot\underset{\sim}{c} + 2\underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{c}\cdot(\underset{\sim}{b} + \underset{\sim}{d})$$

$$= 2\underset{\sim}{c}\cdot\underset{\sim}{c} + 2\underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{b}\cdot\underset{\sim}{b} - \left(\underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} + 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} - 4\left|\overrightarrow{M_1M_2}\right|^2\right)$$

$$= \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} + 4\left|\overrightarrow{M_1M_2}\right|^2$$

$$= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + 4\left|\overrightarrow{M_1M_2}\right|^2$$

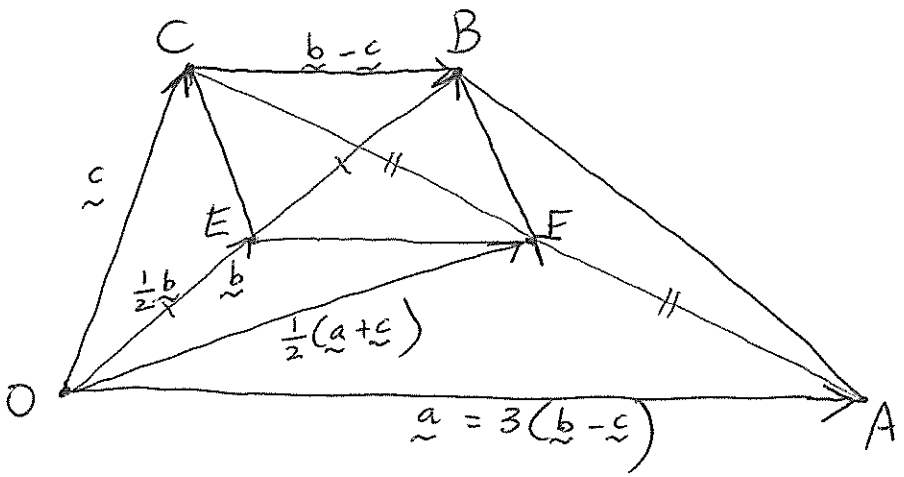
= RHS

So the equation is satisfied.

Uncorrected proofs

5E

(11)



Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$.

Then $\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$ and so $\overrightarrow{OA} = \underset{\sim}{a} = 3(\underset{\sim}{b} - \underset{\sim}{c})$.

Also, $\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\underset{\sim}{b}$

and $\overrightarrow{OF} = \frac{1}{2}\overrightarrow{AC}$

$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$.

So $\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE}$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c} - \underset{\sim}{b})$

$= \frac{1}{2}(3(\underset{\sim}{b} - \underset{\sim}{c}) + \underset{\sim}{c} - \underset{\sim}{b})$

$= \frac{1}{2}(2\underset{\sim}{b} - 2\underset{\sim}{c})$

$= \underset{\sim}{b} - \underset{\sim}{c}$

$= \overrightarrow{CB}$.

So EFBC is a parallelogram (opposite sides CB and EF are parallel and equal).

5E

(12) Let OABC be a quadrilateral with vertices $O(0,0,0)$, $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$ and $C(c_1,c_2,c_3)$.

$$\vec{OA} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \vec{AB} = \begin{pmatrix} b_1-a_1 \\ b_2-a_2 \\ b_3-a_3 \end{pmatrix}, \quad \vec{BC} = \begin{pmatrix} c_1-b_1 \\ c_2-b_2 \\ c_3-b_3 \end{pmatrix}, \quad \vec{CO} = \begin{pmatrix} -c_1 \\ -c_2 \\ -c_3 \end{pmatrix}$$

So the sum of the squares of the sides is

$$|\vec{OA}|^2 + |\vec{AB}|^2 + |\vec{BC}|^2 + |\vec{CO}|^2$$

$$= a_1^2 + a_2^2 + a_3^2 + (b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2$$
$$+ (c_1-b_1)^2 + (c_2-b_2)^2 + (c_3-b_3)^2 + c_1^2 + c_2^2 + c_3^2$$

$$= 2\left(a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2\right)$$
$$- 2\left(a_1b_1 + a_2b_2 + a_3b_3 + b_1c_1 + b_2c_2 + b_3c_3\right). \quad (1)$$

The diagonals are OB and AC,

where $\vec{OB} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} c_1-a_1 \\ c_2-a_2 \\ c_3-a_3 \end{pmatrix}$.

So the sum of the squares of the diagonals is

$$|\vec{OB}|^2 + |\vec{AC}|^2 = b_1^2 + b_2^2 + b_3^2 + (c_1-a_1)^2 + (c_2-a_2)^2 + (c_3-a_3)^2$$

$$= \left(a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2\right)$$
$$- 2\left(a_1c_1 + a_2c_2 + a_3c_3\right). \quad (2)$$

Let the midpoints of OB and AC be M and N respectively.

Then $\vec{OM} = \frac{1}{2}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and $\vec{ON} = \vec{OA} + \frac{1}{2}\vec{AC} = \frac{1}{2}\begin{pmatrix} c_1+a_1 \\ c_2+a_2 \\ c_3+a_3 \end{pmatrix}$

so $\vec{MN} = \frac{1}{2}\begin{pmatrix} a_1-b_1+c_1 \\ a_2-b_2+c_2 \\ a_3-b_3+c_3 \end{pmatrix}$

5E

(12) [continued]

so 4 times the square of the distance between the midpoints of the diagonals is

$$4\left|\overrightarrow{MN}\right|^2 = 4 \cdot \frac{1}{4}\left((a_1-b_1+c_1)^2+(a_2-b_2+c_2)^2+(a_3-b_3+c_3)^2\right)$$

$$= \left(a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2+c_1^2+c_2^2+c_3^2\right)$$

$$-2\left(a_1b_1-a_1c_1+b_1c_1+a_2b_2-a_2c_2+b_2c_2+a_3b_3-a_3c_3+b_3c_3\right)$$

(3)

Now to prove the theorem:

It is clear that

equation (1) = equation (2) + equation (3),

so the result is proven.

5E [Alternatively, questions 10 and 12 are essentially the same and can be done as follows:]

Using the diagram in question 10, translate the figure so that A coincides with the origin.

Then let $\overrightarrow{AB} = \overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{AC} = \overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{AD} = \overrightarrow{OD} = \underset{\sim}{d}$.

So $\overrightarrow{AM_1} = \overrightarrow{OM_1} = \frac{1}{2}\underset{\sim}{c}$ and $\overrightarrow{AM_2} = \overrightarrow{OM_2} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{d})$

and thus $\overrightarrow{M_1M_2} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{d} - \underset{\sim}{c})$.

$$\text{So LHS} = |\underset{\sim}{b}|^2 + |\underset{\sim}{c} - \underset{\sim}{b}|^2 + |\underset{\sim}{d} - \underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} + (\underset{\sim}{c} - \underset{\sim}{b})\cdot(\underset{\sim}{c} - \underset{\sim}{b}) + (\underset{\sim}{d} - \underset{\sim}{c})\cdot(\underset{\sim}{d} - \underset{\sim}{c}) + \underset{\sim}{d}\cdot\underset{\sim}{d}$$

$$= \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\underset{\sim}{c}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d}$$

$$= 2\left(\underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} - \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$

$$\text{and RHS} = |\underset{\sim}{c}|^2 + |\underset{\sim}{d} - \underset{\sim}{b}|^2 + 4\cdot\left|\frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{d} - \underset{\sim}{c})\right|^2$$

$$= \underset{\sim}{c}\cdot\underset{\sim}{c} + (\underset{\sim}{d} - \underset{\sim}{b})\cdot(\underset{\sim}{d} - \underset{\sim}{b}) + 4\cdot\frac{1}{4}(\underset{\sim}{b} + \underset{\sim}{d} - \underset{\sim}{c})\cdot(\underset{\sim}{b} + \underset{\sim}{d} - \underset{\sim}{c})$$

$$= \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} - 2\underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{d} - \underset{\sim}{b}\cdot\underset{\sim}{c}$$
$$+ \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{d}\cdot\underset{\sim}{d} - \underset{\sim}{c}\cdot\underset{\sim}{d}$$
$$- \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{c}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{c}$$

$$= 2\left(\underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d} - \underset{\sim}{b}\cdot\underset{\sim}{c} - \underset{\sim}{c}\cdot\underset{\sim}{d}\right)$$

$$= \text{LHS}.$$