Solutions to Exercise 5D Foundation questions

1a $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (2 \times 1) + (1 \times -1)$

$= 2 + 2 - 1$

$= 3$

1b $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 1^2$

$= 6$

$|\underset{\sim}{a}| = \sqrt{6}$

$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-1)^2$

$= 6$

$|\underset{\sim}{b}| = \sqrt{6}$

1c $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3$

$|\underset{\sim}{a}| = \sqrt{6}$

$|\underset{\sim}{b}| = \sqrt{6}$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$3 = \sqrt{6} \times \sqrt{6} \times \cos\theta$

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$$\theta = \cos^{-1}\frac{1}{2}$$

$$\theta = 60°$$

$$\theta = \frac{\pi}{3}$$

2a $\quad \underset{\sim}{a} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$$|\underset{\sim}{a}|^2 = 2^2 + 0^2 + 0^2$$

$$|\underset{\sim}{a}|^2 = 4$$

$$|\underset{\sim}{a}| = 2$$

$$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-2)^2$$

$$|\underset{\sim}{b}|^2 = 9$$

$$|\underset{\sim}{b}| = 3$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2 \times 2) + (0 \times 1) + (0 \times -2)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 4$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{4}{2 \times 3}$$

$$= \frac{2}{3}$$

2b $\quad \underset{\sim}{a} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$$|\underset{\sim}{a}|^2 = 1^2 + (-1)^2 + (-1)^2$$

$$|\underset{\sim}{a}|^2 = 3$$

$$|\underset{\sim}{a}| = \sqrt{3}$$

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$$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-1)^2$$

$$|\underset{\sim}{b}|^2 = 6$$

$$|\underset{\sim}{b}| = \sqrt{6}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (-1 \times 1) + (-1 \times -1)$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 - 1 + 1$$

$$= 2$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{2}{\sqrt{3} \times \sqrt{6}}$$

$$= \frac{2}{\sqrt{18}}$$

$$= \frac{2}{3\sqrt{2}}$$

$$= \frac{\sqrt{2}}{3}$$

3 $$\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$|\underset{\sim}{a}|^2 = 3^2 + (-2)^2 + (-3)^2$$

$$|\underset{\sim}{a}|^2 = 9 + 4 + 9$$

$$|\underset{\sim}{a}|^2 = 22$$

$$|\underset{\sim}{a}| = \sqrt{22}$$

$$\underset{\sim}{b} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$$

$$|\underset{\sim}{b}|^2 = (-1)^2 + 3^2 + (-4)^2$$

$$|\underset{\sim}{b}|^2 = 1 + 9 + 16$$

$$|\underset{\sim}{b}|^2 = 26$$

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$|\underset{\sim}{b}| = \sqrt{26}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3 \times -1) + (-2 \times 3) + (-3 \times -4)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3 - 6 + 12$

$= 3$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

So for

$\cos\theta = \dfrac{3}{2\sqrt{143}}$

$\text{LHS} = \cos\theta$

$= \dfrac{3}{\sqrt{22} \times \sqrt{26}}$

$= \dfrac{3}{2\sqrt{143}}$

$= \text{RHS}$

4a $\underset{\sim}{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$|\underset{\sim}{v_1}|^2 = 3^2 + 2^2 + 1^2$

$|\underset{\sim}{v_1}|^2 = 14$

$|\underset{\sim}{v_1}| = \sqrt{14}$

$\underset{\sim}{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$|\underset{\sim}{v_2}|^2 = 1^2 + 2^2 + 3^2$

$|\underset{\sim}{v_2}|^2 = 14$

$|\underset{\sim}{v_2}| = \sqrt{14}$

$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = (3 \times 1) + (2 \times 2) + (1 \times 3)$

$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = 3 + 4 + 3$

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$= 10$

$v_1 \cdot v_2 = |v_1||v_2| \cos\theta$

$\cos\theta = \dfrac{v_1 \cdot v_2}{|v_1||v_2|}$

$\cos\theta = \dfrac{10}{\sqrt{14} \times \sqrt{14}}$

$\cos\theta = \dfrac{5}{7}$

$\theta = \cos^{-1}\dfrac{5}{7}$

$\theta = 44.42 \ldots °$

$\theta \div 44°$

4b $v_1 = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

$|v_1|^2 = 5^2 + 3^2 + (-1)^2$

$|v_1|^2 = 25 + 9 + 1$

$|v_1|^2 = 35$

$|v_1| = \sqrt{35}$

$v_2 = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$

$|v_2|^2 = (-2)^2 + 2^2 + (-6)^2$

$|v_2|^2 = 4 + 4 + 36$

$|v_2| = \sqrt{44}$

$v_1 \cdot v_2 = (5 \times -2) + (3 \times 2) + (-1 \times -6)$

$v_1 \cdot v_2 = -10 + 6 + 6$

$= 2$

$v_1 \cdot v_2 = |v_1||v_2| \cos\theta$

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$$\cos\theta = \frac{\underset{\sim}{v_1} \cdot \underset{\sim}{v_2}}{|\underset{\sim}{v_1}||\underset{\sim}{v_2}|}$$

$$\cos\theta = \frac{2}{\sqrt{35} \times \sqrt{44}}$$

$$\cos\theta = \frac{1}{\sqrt{35} \times \sqrt{11}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{385}}$$

$$\theta = 87.08\ldots°$$

$$\theta \div 87°$$

5 $\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$

$$\hat{\underset{\sim}{\imath}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{\underset{\sim}{\jmath}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\hat{\underset{\sim}{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$proj_{\underset{\sim}{\imath}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a} \cdot \hat{\underset{\sim}{\imath}}}{\hat{\underset{\sim}{\imath}} \cdot \hat{\underset{\sim}{\imath}}}\right)\hat{\underset{\sim}{\imath}}$$

So for $proj_{\underset{\sim}{\imath}}\underset{\sim}{a} = 3\underset{\sim}{\imath}$

LHS $= proj_{\underset{\sim}{\imath}}\underset{\sim}{a}$

$$= \left(\frac{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 3\underset{\sim}{\imath}$$

$$= \text{RHS}$$

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$$proj_{\underset{\sim}{j}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a}\cdot\hat{\underset{\sim}{j}}}{\hat{\underset{\sim}{j}}\cdot\hat{\underset{\sim}{j}}}\right)\hat{\underset{\sim}{j}}$$

So for $proj_{\underset{\sim}{j}}\underset{\sim}{a} = -2\underset{\sim}{j}$

$$\text{LHS} = proj_{\underset{\sim}{j}}\underset{\sim}{a}$$

$$= \left(\frac{\begin{bmatrix}3\\-2\\5\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}{\begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}}\right)\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$= -2\underset{\sim}{j}$$

$$= \text{RHS}$$

$$proj_{\underset{\sim}{k}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a}\cdot\hat{\underset{\sim}{k}}}{\hat{\underset{\sim}{k}}\cdot\hat{\underset{\sim}{k}}}\right)\hat{\underset{\sim}{k}}$$

So for $proj_{\hat{\underset{\sim}{k}}}\underset{\sim}{a} = 5\underset{\sim}{k}$

$$\text{LHS} = proj_{\hat{\underset{\sim}{k}}}\underset{\sim}{a}$$

$$= \left(\frac{\begin{bmatrix}3\\-2\\5\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}{\begin{bmatrix}0\\0\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}}\right)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$= 5\underset{\sim}{k}$$

$$= \text{RHS}$$

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Solutions to Exercise 5D Development questions

6a $\underset{\sim}{a} = \underset{\sim}{\imath} + \underset{\sim}{\jmath} - \underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} - \underset{\sim}{k}$

$proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$= \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$

$= \dfrac{2 - 2 + 1}{4 + 4 + 1}\underset{\sim}{b}$

$= \dfrac{1}{9}\underset{\sim}{b}$

$= \dfrac{2}{9}\underset{\sim}{\imath} - \dfrac{2}{9}\underset{\sim}{\jmath} - \dfrac{1}{9}\underset{\sim}{k}$

6b $a = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

$proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$= \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$

$= \dfrac{12 + 2 - 2}{16 + 1 + 1}\underset{\sim}{b}$

$= \dfrac{2}{3}\underset{\sim}{b}$

$= \begin{bmatrix} \frac{8}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\underset{\sim}{k}$

Uncorrected proofs

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Chapter 5 worked solutions – Vectors

7a $\underset{\sim}{a} = 2\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} - 2\underset{\sim}{k}$

$\underset{\sim}{b} = 4\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 5\underset{\sim}{k}$

$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = |\underset{\sim}{a}| \cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$

$|\underset{\sim}{b}|^2 = 16 + 4 + 25 = 45$

$|\underset{\sim}{b}| = \sqrt{45}$

$= 3\sqrt{5}$

So

$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = \dfrac{8 - 6 - 10}{3\sqrt{5}}$

$= -\dfrac{8}{3\sqrt{5}}$

So the length of projection is $\dfrac{8}{3\sqrt{5}}$.

7b $a = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ 1 \end{bmatrix}$

$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = |\underset{\sim}{a}| \cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$

$|\underset{\sim}{b}|^2 = 64 + 16 + 1 = 81$

$|\underset{\sim}{b}| = \sqrt{81}$

$= 9$

Uncorrected proofs

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So

$$|proj_{\underset{\sim}{b}}\underset{\sim}{a}| = \frac{8+4+3}{9}$$

$$= \frac{15}{9}$$

$$= \frac{5}{3}$$

8a $A = (2, 7, -12)$

$B = (-1, 5, -5)$

$C = (4, 1, -4)$

$\overrightarrow{BA} = (2-(-1))\underset{\sim}{i} + (7-5)\underset{\sim}{j} + (-12-(-5))\underset{\sim}{k}$

$= 3\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$

$\overrightarrow{BC} = (4-(-1))\underset{\sim}{i} + (1-5)\underset{\sim}{j} + (-4-(-5))\underset{\sim}{k}$

$= 5\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

8b $\overrightarrow{BA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$

$\overrightarrow{BC} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

$\angle ABC = \cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}\right)$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = 0$

So:

$\angle ABC = \cos^{-1} 0$

$= 90°$

8c $\overrightarrow{AC} = \overrightarrow{BC} - \overrightarrow{BA}$

$= 2\underset{\sim}{i} - 6\underset{\sim}{j} + 8\underset{\sim}{k}$

$|\overrightarrow{BA}| = \sqrt{9+4+49}$

$= \sqrt{62}$

Uncorrected proofs

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$$\left|\overrightarrow{BC}\right| = \sqrt{25 + 16 + 1}$$
$$= \sqrt{42}$$
$$\left|\overrightarrow{AC}\right| = \sqrt{4 + 36 + 64}$$
$$= \sqrt{104}$$
$$= 2\sqrt{26}$$

Pythagoras' theorem: $\left|\overrightarrow{BA}\right|^2 + \left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{AC}\right|^2$

$$\text{LHS} = \left|\overrightarrow{BA}\right|^2 + \left|\overrightarrow{BC}\right|^2$$
$$= \left(\sqrt{62}\right)^2 + \left(\sqrt{42}\right)^2$$
$$= 62 + 42$$
$$= 104$$
$$= \left|\overrightarrow{AC}\right|^2$$
$$= \text{RHS}$$

So the equation is satisfied.

9a $A = (3, -3, 1)$

$B = (-2, 1, 2)$

$C = (4, 0, -1)$

$$\overrightarrow{AB} = \begin{bmatrix} -2 - 3 \\ 1 - (-3) \\ 2 - 1 \end{bmatrix}$$
$$= \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$
$$\overrightarrow{AC} = \begin{bmatrix} 4 - 3 \\ 0 - (-3) \\ -1 - 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Uncorrected proofs

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9b $\overrightarrow{BA} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$

$\overrightarrow{BC} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$\angle ABC = \cos^{-1}\left(\dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}\right)$

So:

$\angle ABC$

$= \cos^{-1}\left(\dfrac{-5 + 12 - 2}{\sqrt{25 + 16 + 1} \times \sqrt{1 + 9 + 4}}\right)$

$= \cos^{-1}\left(\dfrac{5}{\sqrt{42} \times \sqrt{14}}\right)$

$= \cos^{-1}\left(\dfrac{5}{14\sqrt{3}}\right)$

$= 78.100 \ldots °$

$\doteqdot 78°$

10 $P = (-4, -1, 6)$

$Q = (-5, 3, 4)$

$R = (-3, 4, -7)$

$\overrightarrow{QP} = \underset{\sim}{\imath} - 4\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\overrightarrow{QR} = 2\underset{\sim}{\imath} + \underset{\sim}{\jmath} - 11\underset{\sim}{k}$

$\overrightarrow{QP} \cdot \overrightarrow{QR} = |\overrightarrow{QP}||\overrightarrow{QR}| \cos\theta$

$\angle PQR = \cos^{-1}\left(\dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|}\right)$

$\overrightarrow{QP} \cdot \overrightarrow{QR} = 2 - 4 - 22$

$= -24$

$|\overrightarrow{QP}|^2 = 1^2 + (-4)^2 + (2)^2$

$= 21$

Uncorrected proofs

$|\overrightarrow{QP}| = \sqrt{21}$

$|\overrightarrow{QR}|^2 = 2^2 + 1^2 + (-11)^2$

$= 126$

$|\overrightarrow{QR}| = \sqrt{126}$

$= 3\sqrt{14}$

So

$$\angle PQR = \cos^{-1}\left(\frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{|\overrightarrow{QP}||\overrightarrow{QR}|}\right)$$

$$= \cos^{-1}\left(\frac{-24}{\sqrt{21} \times 3\sqrt{14}}\right)$$

$$= \cos^{-1}\left(\frac{-24}{21\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{-8}{7\sqrt{6}}\right)$$

$= 117.811 \ldots °$

$\div 117°49'$

11a $A = (1, 0, -1)$

$B = (1, 1, 1)$

$C = (0, 1, -1)$

$\overrightarrow{CB} = \underset{\sim}{\imath} + 2\underset{\sim}{k}$

$\overrightarrow{CA} = \underset{\sim}{\imath} - \underset{\sim}{\jmath}$

$\overrightarrow{CB} \cdot \overrightarrow{CA} = 1 + 0 + 0 = 1$

$|\overrightarrow{CB}|^2 = 1^2 + 0^2 + 2^2 = 5$

$|\overrightarrow{CB}| = \sqrt{5}$

$|\overrightarrow{CA}|^2 = 1^2 + (-1)^2 + 0^2 = 2$

$|\overrightarrow{CA}| = \sqrt{2}$

Uncorrected proofs

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$$\cos \angle ACB = \frac{\overrightarrow{CB} \cdot \overrightarrow{CA}}{|\overrightarrow{CB}||\overrightarrow{CA}|}$$

$$= \frac{1}{\sqrt{5} \times \sqrt{2}}$$

$$= \frac{1}{\sqrt{10}}$$

11b

$$\text{Area} = \frac{1}{2} ab \cos C$$

For ΔABC,

$$\text{Area} = \frac{1}{2} |\overrightarrow{CA}||\overrightarrow{CB}| \cos \angle ACB$$

Since $\cos \angle ACB = \frac{1}{\sqrt{10}}$, $|\overrightarrow{CB}| = \sqrt{5}$ and $|\overrightarrow{CA}| = \sqrt{2}$

$$\text{Area} = \frac{1}{2} \times \sqrt{2} \times \sqrt{5} \times \frac{1}{\sqrt{10}}$$

$$= \frac{1}{2} \text{ square unit}$$

12a $P = (-4, 3, -1)$

$A = (3, 2, 1)$

$B = (0, -4, 1)$

$$\overrightarrow{AP} = \begin{bmatrix} -4-3 \\ 3-2 \\ -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0-3 \\ -4-2 \\ 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

Uncorrected proofs

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12b $\overrightarrow{AP} = \underset{\sim}{p}$

$$= \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$\overrightarrow{AB} = \underset{\sim}{b}$

$$= \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\ \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{(-3 \times -7) + (-6 \times 1) + (0 \times -2)}{(-3 \times -3) + (-6 \times -6) + (0 \times 0)} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{21 - 6}{9 + 36} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{15}{45} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

12c $d = \left| proj_{\underset{\sim}{b}}\ \underset{\sim}{p} - \underset{\sim}{p} \right|$

$$proj_{\underset{\sim}{b}}\ \underset{\sim}{p} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{p} = \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\ \underset{\sim}{p} - \underset{\sim}{p} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$$

Uncorrected proofs

$$d^2 = 6^2 + (-3)^2 + 2^2$$
$$= 36 + 9 + 4$$
$$= 49$$
$$d = 7 \text{ units}$$

13a $P = (3, -2, 1)$

$A = (1, -11, -4)$

$B = (9, 3, 8)$

$\overrightarrow{AP} = \underset{\sim}{p}$

$$= \begin{bmatrix} 3-1 \\ -2-(-11) \\ 1-(-4) \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix}$$

$\overrightarrow{AB} = \underset{\sim}{b}$

$$= \begin{bmatrix} 9-1 \\ 3-(-11) \\ 8-(-4) \end{bmatrix}$$
$$= \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$

$$proj_{\underset{\sim}{b}} \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$
$$= \frac{(8 \times 2) + (14 \times 9) + (12 \times 5)}{(8 \times 8) + (14 \times 14) + (12 \times 12)} \times \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$
$$= \frac{16 + 126 + 60}{64 + 196 + 144} \times \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$
$$= \frac{202}{404} \times 2 \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

Uncorrected proofs

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$$d = \left|proj_{\underset{\sim}{b}}\ \underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$proj_{\underset{\sim}{b}}\ \underset{\sim}{p} - \underset{\sim}{p} = \begin{bmatrix}4\\7\\6\end{bmatrix} - \begin{bmatrix}2\\9\\5\end{bmatrix}$$

$$= \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

$$d^2 = (2^2 + (-1)^2 + 1^2)$$

$$= 9$$

$$d = 3 \text{ units}$$

13b $P = (0, 0, 3)$

$A = (1, 2, 1)$

$B = (4, 0, 0)$

$$\overrightarrow{AP} = \underset{\sim}{p}$$

$$= \begin{bmatrix}0-1\\0-2\\3-1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\-2\\2\end{bmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}$$

$$= \begin{bmatrix}4-1\\0-2\\0-1\end{bmatrix}$$

$$= \begin{bmatrix}3\\-2\\-1\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\ \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$$

$$= \frac{(3 \times -1) + (-2 \times -2) + (-1 \times 2)}{(3 \times 3) + (-2 \times -2) + (-1 \times -1)} \times \begin{bmatrix}3\\-2\\-1\end{bmatrix}$$

$$= \frac{-3 + 4 - 2}{9 + 4 + 1} \times \begin{bmatrix}3\\-2\\-1\end{bmatrix}$$

Uncorrected proofs

$$= -\frac{1}{14} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$d = \left| proj_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} \right|$

$$proj_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} = -\frac{1}{14} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 11 \\ 30 \\ -27 \end{bmatrix}$$

$$d^2 = \left(\frac{1}{14}\right)^2 (11^2 + 30^2 + (-27)^2)$$

$$= \frac{1}{196}(121 + 900 + 729)$$

$$= \frac{1750}{196}$$

$$= \frac{125}{14}$$

$$d = \sqrt{\frac{125}{14}}$$

$$= \frac{5\sqrt{70}}{14} \text{ units}$$

14 $\overrightarrow{AG} = (0 - a)\underset{\sim}{i} + (a - 0)\underset{\sim}{j} + (a - 0)\underset{\sim}{k}$

$$= -a\underset{\sim}{i} + a\underset{\sim}{j} + a\underset{\sim}{k}$$

$\overrightarrow{CE} = (a - 0)\underset{\sim}{i} + (0 - a)\underset{\sim}{j} + (a - 0)\underset{\sim}{k}$

$$= a\underset{\sim}{i} - a\underset{\sim}{j} + a\underset{\sim}{k}$$

To find the acute angle we use $\overrightarrow{EC}$.

$\overrightarrow{EC} = -\overrightarrow{CE}$

$$= -a\underset{\sim}{i} + a\underset{\sim}{j} - a\underset{\sim}{k}$$

$\overrightarrow{AG} \cdot \overrightarrow{EC} = a^2 + a^2 - a^2$

$$= a^2$$

Uncorrected proofs

$$\left|\overrightarrow{AG}\right|^2 = a^2 + (-a)^2 + a^2$$
$$= 3a^2$$
$$\left|\overrightarrow{AG}\right| = \sqrt{3a^2}$$
$$= \sqrt{3}a$$
$$\left|\overrightarrow{EC}\right|^2 = (-a)^2 + a^2 + (-a)^2$$
$$= 3a^2$$
$$\left|\overrightarrow{EC}\right| = \sqrt{3a^2}$$
$$= \sqrt{3}a$$
$$\cos\theta = \frac{\overrightarrow{AG} \cdot \overrightarrow{EC}}{\left|\overrightarrow{AG}\right|\left|\overrightarrow{EC}\right|}$$
$$= \frac{a^2}{\sqrt{3}a \times \sqrt{3}a}$$
$$= \frac{a^2}{3a^2}$$
So $\theta = \arccos\left(\frac{1}{3}\right)$

15 $O = (0, 0, 0)$

$B = (1, 2, 0)$

$D = (0, 0, 3)$

$F = (1, 2, 3)$

For $\overrightarrow{OF}$ and $\overrightarrow{BD}$,

$$\overrightarrow{OF} = (1-0)\underset{\sim}{i} + (2-0)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$$
$$= \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\overrightarrow{BD} = (0-1)\underset{\sim}{i} + (0-2)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$$
$$= -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$$
$$\overrightarrow{OF} \cdot \overrightarrow{BD} = -1 - 4 + 9$$
$$= 4$$

Uncorrected proofs

$$\left|\overrightarrow{OF}\right|^2 = 1^2 + 2^2 + 3^2$$
$$= 14$$
$$\left|\overrightarrow{OF}\right| = \sqrt{14}$$
$$\left|\overrightarrow{BD}\right|^2 = (-1)^2 + (-2)^2 + 3^2$$
$$= 14$$
$$\left|\overrightarrow{BD}\right| = \sqrt{14}$$
$$\cos\theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{BD}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{BD}\right|}$$
$$= \frac{4}{\sqrt{14} \times \sqrt{14}}$$
$$= \frac{2}{7}$$

For $\overrightarrow{OF}$ and $\overrightarrow{CE}$,

$$\overrightarrow{OF} = (1-0)\underset{\sim}{\imath} + (2-0)\underset{\sim}{\jmath} + (3-0)\underset{\sim}{k}$$
$$= \underset{\sim}{\imath} + 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$$
$$\overrightarrow{CE} = (1-0)\underset{\sim}{\imath} + (0-2)\underset{\sim}{\jmath} + (3-0)\underset{\sim}{k}$$
$$= \underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$$
$$\overrightarrow{OF} \cdot \overrightarrow{CE} = 1 - 4 + 9$$
$$= 6$$
$$\left|\overrightarrow{OF}\right|^2 = 1^2 + 2^2 + 3^2$$
$$= 14$$
$$\left|\overrightarrow{OF}\right| = \sqrt{14}$$
$$\left|\overrightarrow{CE}\right|^2 = (1)^2 + (-2)^2 + 3^2$$
$$= 14$$
$$\left|\overrightarrow{CE}\right| = \sqrt{14}$$
$$\cos\theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{CE}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{CE}\right|}$$

Uncorrected proofs

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$$= \frac{6}{\sqrt{14} \times \sqrt{14}}$$

$$= \frac{3}{7}$$

For $\overrightarrow{OF}$ and $\overrightarrow{AG}$,

$\overrightarrow{OF} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{AG} = (0-1)\underset{\sim}{i} + (2-0)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$

$= -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OF} \cdot \overrightarrow{AG} = -1 + 4 + 9$

$= 12$

$\left|\overrightarrow{OF}\right| = \sqrt{14}$

$\left|\overrightarrow{AG}\right|^2 = (-1)^2 + 2^2 + 3^2$

$= 14$

$\left|\overrightarrow{AG}\right| = \sqrt{14}$

$$\cos\theta = \frac{\overrightarrow{OE} \cdot \overrightarrow{AD}}{\left|\overrightarrow{OE}\right|\left|\overrightarrow{AD}\right|}$$

$$= \frac{8}{\sqrt{14} \times \sqrt{14}}$$

$$= \frac{6}{7}$$

16 Say

$a = |AD|$

$= |DC|$

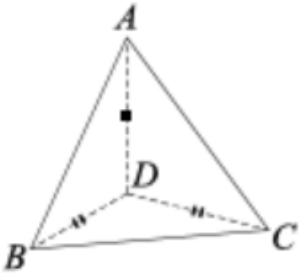
$= |BD|$

$A = (0, 0, a)$

$B = (0, a, 0)$

$C = (a, 0, 0)$

$D = (0, 0, 0)$



Uncorrected proofs

M is the midpoint between B and C.

$$M = \left(\frac{0+a}{2}, \frac{a+0}{2}, \frac{0+0}{2}\right)$$

$$= \left(\frac{a}{2}, \frac{a}{2}, 0\right)$$

So we want to find the angle made by $\overrightarrow{MD}$ and $\overrightarrow{MA}$.

$$\overrightarrow{MD} = \left(0 - \frac{a}{2}\right)\underset{\sim}{i} + \left(2 - \frac{a}{2}\right)\underset{\sim}{j} + (0-0)\underset{\sim}{k}$$

$$= -\frac{a}{2}\underset{\sim}{i} - \frac{a}{2}\underset{\sim}{j}$$

$$\overrightarrow{MA} = (0, 0, a) - \left(\frac{a}{2}, \frac{a}{2}, 0\right)$$

$$= -\frac{a}{2}\underset{\sim}{i} - \frac{a}{2}\underset{\sim}{j} + a\underset{\sim}{k}$$

$$\overrightarrow{MD} \cdot \overrightarrow{MA} = \left(-\frac{a}{2} \times -\frac{a}{2}\right) + \left(-\frac{a}{2} \times -\frac{a}{2}\right) + (a \times 0)$$

$$= \frac{2a^2}{4}$$

$$= \frac{a^2}{2}$$

$$\left|\overrightarrow{MD}\right|^2 = \left(-\frac{a}{2}\right)^2 + \left(-\frac{a}{2}\right)^2$$

$$= \frac{a^2}{4} + \frac{a^2}{4}$$

$$= \frac{a^2}{2}$$

$$\left|\overrightarrow{MD}\right| = \frac{a}{\sqrt{2}}$$

$$\left|\overrightarrow{MA}\right|^2 = \left(-\frac{a}{2}\right)^2 + \left(-\frac{a}{2}\right)^2 + a^2$$

$$= \frac{3a^2}{2}$$

$$\left|\overrightarrow{MA}\right| = \sqrt{\frac{3}{2}}a$$

Uncorrected proofs

$$\cos \angle AMD = \frac{\overrightarrow{MD} \cdot \overrightarrow{MA}}{\left|\overrightarrow{MD}\right|\left|\overrightarrow{MA}\right|}$$

$$= \frac{\left(\frac{a^2}{2}\right)}{\frac{a}{\sqrt{2}} \times \sqrt{\frac{3}{2}}a}$$

$$= \frac{\sqrt{2}}{2} \times \sqrt{\frac{2}{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\angle AMD = \cos^{-1}\frac{1}{\sqrt{3}}$$

17a $\overrightarrow{OA} = -5\underset{\sim}{\imath} + 22\underset{\sim}{\jmath} + 5\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{\imath} + 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\overrightarrow{OD} = -\underset{\sim}{\imath} + 2\underset{\sim}{\jmath} - 3\underset{\sim}{k}$

$\overrightarrow{BA} = (-5 - 1)\underset{\sim}{\imath} + (22 - 2)\underset{\sim}{\jmath} + (5 - 3)\underset{\sim}{k}$

$= -6\underset{\sim}{\imath} + 20\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\overrightarrow{BD} = (-1 - 1)\underset{\sim}{\imath} + (2 - 2)\underset{\sim}{\jmath} + (-3 - 3)\underset{\sim}{k}$

$= -2\underset{\sim}{\imath} - 6\underset{\sim}{k}$

$\overrightarrow{BA} \cdot \overrightarrow{BD} = (-6 \times -2) + (20 \times 0) + (2 \times -6)$

$= 12 + 0 - 12$

$= 0$

$$\cos \angle CBD = \frac{\overrightarrow{BA} \cdot \overrightarrow{BD}}{\left|\overrightarrow{BD}\right|\left|\overrightarrow{BA}\right|}$$

So

$\angle CBD = \cos^{-1} 0$

$\angle CBD = 90°$

Uncorrected proofs

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17b $\overrightarrow{OA} = -5\underset{\sim}{\imath} + 22\underset{\sim}{\jmath} + 5\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{\imath} + 2\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{\imath} + 3\underset{\sim}{\jmath} + 2\underset{\sim}{k}$

$\overrightarrow{OD} = -\underset{\sim}{\imath} + 2\underset{\sim}{\jmath} - 3\underset{\sim}{k}$

$\overrightarrow{AB} = (1 - (-5))\underset{\sim}{\imath} + (2 - 22)\underset{\sim}{\jmath} + (3 - 5)\underset{\sim}{k}$

$= 6\underset{\sim}{\imath} - 20\underset{\sim}{\jmath} - 2\underset{\sim}{k}$

$\overrightarrow{BC} = (4 - 1)\underset{\sim}{\imath} + (3 - 2)\underset{\sim}{\jmath} + (2 - 3)\underset{\sim}{k}$

$= 3\underset{\sim}{\imath} + \underset{\sim}{\jmath} - \underset{\sim}{k}$

$\overrightarrow{BD} = (-1 - 1)\underset{\sim}{\imath} + (2 - 2)\underset{\sim}{\jmath} + (-3 - 3)\underset{\sim}{k}$

$= -2\underset{\sim}{\imath} - 6\underset{\sim}{k}$

For $\overrightarrow{AB}$ to be perpendicular to $\overrightarrow{BC}$ and $\overrightarrow{BD}$,

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$ and $\overrightarrow{AB} \cdot \overrightarrow{BD} = 0$.

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (6 \times 3) + (-20 \times 1) + (-2 \times -1)$

$= 18 - 20 + 2$

$= 0$

So $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular.

$\overrightarrow{AB} \cdot \overrightarrow{BD} = (6 \times -2) + (-20 \times 0) + (-2 \times -6)$

$= -12 + 0 + 12$

$= 0$

So $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are perpendicular.

17c The volume of a tetrahedron is:

$V = \frac{1}{3} \times X \times Z$

$X =$ Area of the base of the pyramid

$Z =$ Height of the pyramid

We know that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are perpendicular.

Uncorrected proofs

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So the lengths $|\overrightarrow{AB}|$, $|\overrightarrow{BD}|$ and $|\overrightarrow{DA}|$ form a right-angled triangle.

$\overrightarrow{AB} = 6\underset{\sim}{i} - 20\underset{\sim}{j} - 2\underset{\sim}{k}$

$|\overrightarrow{AB}|^2 = 6^2 + (-20)^2 + (-2)^2$

$= 36 + 400 + 4$

$= 440$

$|\overrightarrow{AB}| = \sqrt{440}$

$= 2\sqrt{110}$

$\overrightarrow{BD} = -2\underset{\sim}{i} - 6\underset{\sim}{k}$

$|\overrightarrow{BD}|^2 = (-2)^2 + (-6)^2$

$= 40$

$|\overrightarrow{BD}| = \sqrt{40}$

$= 2\sqrt{10}$

So the area of the base will be

$X = \frac{1}{2} \times 2\sqrt{110} \times 2\sqrt{10}$

$= 20\sqrt{11}$ square units

Since $\angle CBD = 90°$, and ΔABD is perpendicular to $\overrightarrow{BC}$, the height Z will be $|\overrightarrow{BC}|$.

$\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

So $\overrightarrow{BC} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$|\overrightarrow{BC}|^2 = 3^2 + 1^2 + (-1)^2$

$= 9 + 1 + 1$

$= 11$

$|\overrightarrow{BC}| = Z = \sqrt{11}$

So the volume of the tetrahedron is

Uncorrected proofs

$$V = \frac{1}{3} \times X \times Z$$

$$V = \frac{1}{3} \times 20\sqrt{11} \times \sqrt{11}$$

$$V = \frac{20}{3} \times 11$$

$$V = \frac{220}{3} \text{ cubic units}$$

18a $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OB} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{AB} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$

We want to show that:

$\overrightarrow{OP} = (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}$

So:

$\overrightarrow{OP} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$

$= (2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) + \lambda(4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k})$

$= (2 + 4\lambda)\underset{\sim}{i} + \underset{\sim}{j} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}$

18b For $\overrightarrow{OP}$ and $\overrightarrow{AB}$ to be perpendicular:

$\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$

$\overrightarrow{AB} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$

$\overrightarrow{OP} = (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}$

$\overrightarrow{OP} \cdot \overrightarrow{AB} = (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k} \cdot 4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$

$0 = 4 \times (2 + 4\lambda) + (-4) \times (1 - 4\lambda) + 4 \times (4\lambda - 2)$

$0 = 16\lambda + 8 + 16\lambda - 4 + 16\lambda - 8$

$0 = 48\lambda - 4$

$48\lambda = 4$

Uncorrected proofs

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$$\lambda = \frac{1}{12}$$

18c $\angle AOP = \angle BOP$

So

$$\cos^{-1}\left(\frac{\overrightarrow{OA}\cdot\overrightarrow{OP}}{|\overrightarrow{OA}||\overrightarrow{OP}|}\right) = \cos^{-1}\left(\frac{\overrightarrow{OB}\cdot\overrightarrow{OP}}{|\overrightarrow{OB}||\overrightarrow{OP}|}\right)$$

which becomes

$$\frac{\overrightarrow{OA}\cdot\overrightarrow{OP}}{|\overrightarrow{OA}|} = \frac{\overrightarrow{OB}\cdot\overrightarrow{OP}}{|\overrightarrow{OB}|}$$

$$\overrightarrow{OA}\cdot\overrightarrow{OP} = \left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right)\cdot\left((2+4\lambda)\underset{\sim}{i} + (1-4\lambda)\underset{\sim}{j} + (4\lambda-2)\underset{\sim}{k}\right)$$

$$= 2(2+4\lambda) + (1-4\lambda) - 2(4\lambda-2)$$

$$= 4 + 8\lambda + 1 - 4\lambda - 8\lambda + 4$$

$$= -4\lambda + 9$$

$$\overrightarrow{OB}\cdot\overrightarrow{OP} = \left(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)\cdot\left((2+4\lambda)\underset{\sim}{i} + (1-4\lambda)\underset{\sim}{j} + (4\lambda-2)\underset{\sim}{k}\right)$$

$$= 6(2+4\lambda) - 3(1-4\lambda) + 2(4\lambda-2)$$

$$= 12 + 24\lambda - 3 + 12\lambda + 8\lambda - 4$$

$$= 44\lambda + 5$$

$$|\overrightarrow{OA}|^2 = 2^2 + 1^2 + (-2)^2 = 9$$

$$|\overrightarrow{OA}| = 3$$

$$|\overrightarrow{OB}|^2 = 6^2 + (-3)^2 + 2^2 = 49$$

$$|\overrightarrow{OB}| = 7$$

$$\frac{\overrightarrow{OA}\cdot\overrightarrow{OP}}{|\overrightarrow{OA}|} = \frac{\overrightarrow{OB}\cdot\overrightarrow{OP}}{|\overrightarrow{OB}|}$$

Substituting gives:

$$\frac{-4\lambda + 9}{3} = \frac{44\lambda + 5}{7}$$

$$-28\lambda + 63 = 132\lambda + 15$$

Uncorrected proofs

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$$-160\lambda = -48$$

$$-160\lambda = -48$$

$$\lambda = \frac{3}{10}$$

19 For θ

$$\theta = \cos^{-1}\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$\theta = \cos^{-1}\frac{4}{21}$$

So

$$\cos^{-1}\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \cos^{-1}\frac{4}{21}$$

$$\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{4}{21}$$

$$\underset{\sim}{a} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ \lambda \end{bmatrix}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ \lambda \end{bmatrix}\cdot\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

$$= (-12) + (8) + (3\lambda)$$

$$= 3\lambda - 4$$

$$|\underset{\sim}{a}|^2 = 6^2 + (-2)^2 + 3^2$$

$$= 36 + 4 + 9$$

$$= 49$$

$$|\underset{\sim}{a}| = 7$$

$$|\underset{\sim}{b}|^2 = (-2)^2 + (-4)^2 + \lambda^2$$

$$= 4 + 16 + \lambda^2$$

$$= 20 + \lambda^2$$

Uncorrected proofs

$|\underset{\sim}{b}| = \sqrt{20 + \lambda^2}$

So

$$\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{4}{21}$$

Solve for λ

$$\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|} = \frac{4}{21}$$

$$|\underset{\sim}{b}| = \frac{21 \times \underset{\sim}{a} \cdot \underset{\sim}{b}}{4 \times |\underset{\sim}{a}|}$$

$$|\underset{\sim}{b}| = \frac{21 \times (3\lambda - 4)}{4 \times 7}$$

$$|\underset{\sim}{b}| = \frac{3(3\lambda - 4)}{4}$$

$$\sqrt{20 + \lambda^2} = \frac{9\lambda - 12}{4}$$

$$\sqrt{20 + \lambda^2} = \frac{9}{4}\lambda - 3$$

$$20 + \lambda^2 = \left(\frac{9}{4}\lambda - 3\right)^2$$

$$20 + \lambda^2 = \frac{81}{16}\lambda^2 - \frac{54}{4}\lambda + 9$$

$$0 = \frac{81}{16}\lambda^2 - \lambda^2 - \frac{54}{4}\lambda - 11$$

$$0 = \frac{65}{16}\lambda^2 - \frac{54}{4}\lambda - 11$$

$$0 = \frac{65}{16}\lambda^2 - \frac{27}{2}\lambda - 11$$

$$0 = 65\lambda^2 - 216\lambda - 176$$

$$0 = (65\lambda + 44)(\lambda - 4)$$

$$\lambda = -\frac{44}{65} \text{ or } 4$$

Alternatively, using the quadratic formula:

$$\lambda = \frac{216 \pm \sqrt{(-216)^2 - 4 \times 65 \times -176}}{2 \times 65}$$

Uncorrected proofs

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$$\lambda = \frac{216 \pm \sqrt{92\,416}}{130}$$

$$\lambda = \frac{216 \pm 304}{130}$$

$$\lambda = \frac{520}{130} \text{ or } -\frac{88}{130}$$

$$\lambda = 4 \text{ or } -\frac{44}{65}$$

20a For A, B and P to be collinear, $\underset{\sim}{a}, \underset{\sim}{b}$ and $\underset{\sim}{p}$ all lie on one line.

If $\underset{\sim}{a}$ $\underset{\sim}{b}$ and $\underset{\sim}{p}$ all lie on a straight line then

$\overrightarrow{AB} + \overrightarrow{BP} = \overrightarrow{AP}$

So:

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{BP} = \underset{\sim}{p} - \underset{\sim}{b}$

$\overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{a}$

LHS $= \overrightarrow{AB} + \overrightarrow{BP}$

$= \underset{\sim}{b} - \underset{\sim}{a} + \underset{\sim}{p} - \underset{\sim}{b}$

$= \underset{\sim}{p} - \underset{\sim}{a}$

$= \overrightarrow{AP}$

$=$ RHS

So A, B and P are collinear.

20b $\underset{\sim}{a} = \underset{\sim}{\imath} + \underset{\sim}{\jmath}$

$\underset{\sim}{b} = 4\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 6\underset{\sim}{k}$

$\underset{\sim}{p} = \lambda\underset{\sim}{a} + (1 - \lambda)\underset{\sim}{b}$

$\overrightarrow{OA} = \underset{\sim}{\imath} + \underset{\sim}{\jmath}$

$\overrightarrow{OP} = \lambda\underset{\sim}{a} + (1 - \lambda)\underset{\sim}{b}$

Uncorrected proofs

$\overrightarrow{OP} = (\underset{\sim}{\imath} + \underset{\sim}{\jmath})\lambda + (1-\lambda)(4\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 6\underset{\sim}{k})$

$\overrightarrow{OP} = \lambda\underset{\sim}{\imath} + \lambda\underset{\sim}{\jmath} + 4\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + 6\underset{\sim}{k} - 4\lambda\underset{\sim}{\imath} + 2\lambda\underset{\sim}{\jmath} - 6\lambda\underset{\sim}{k}$

$\overrightarrow{OP} = (4-3\lambda)\underset{\sim}{\imath} + (3\lambda-2)\underset{\sim}{\jmath} + 6(1-\lambda)\underset{\sim}{k}$

$\left|\overrightarrow{OA}\right|^2 = 1 + 1 = 2$

$\left|\overrightarrow{OA}\right| = \sqrt{2}$

$|\overrightarrow{OP}|^2 = (4-3\lambda)^2 + (3\lambda-2)^2 + 36(1-\lambda)^2$

$|\overrightarrow{OP}|^2 = 16 - 24\lambda + 9\lambda^2 + 9\lambda^2 - 12\lambda + 4 + 36 - 72\lambda + 36\lambda^2$

$|\overrightarrow{OP}|^2 = 54\lambda^2 - 108\lambda + 56$

$|\overrightarrow{OP}| = \sqrt{2} \times \sqrt{(27\lambda^2 - 54\lambda + 28)}$

$\overrightarrow{OA} \cdot \overrightarrow{OP} = 4 - 3\lambda + 3\lambda - 2$

$\overrightarrow{OA} \cdot \overrightarrow{OP} = 2$

$\cos^{-1}\left(\dfrac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OP}\right|}\right) = \theta$

$\angle AOP = 60°$

$\dfrac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{OP}\right|} = \cos 60°$

$\dfrac{2}{\sqrt{2} \times \sqrt{2} \times \sqrt{(27\lambda^2 - 54\lambda + 28)}} = \dfrac{1}{2}$

$\sqrt{(27\lambda^2 - 54\lambda + 28)} = 2$

$27\lambda^2 - 54\lambda + 28 = 4$

$27\lambda^2 - 54\lambda + 24 = 0$

$9\lambda^2 - 18\lambda + 8 = 0$

$(3\lambda - 2)(3\lambda - 4) = 0$

$\lambda = \dfrac{2}{3}$ or $\lambda = \dfrac{4}{3}$

Uncorrected proofs

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5D

(21)(a) $\vec{CB} = -2\underset{\sim}{i} - 6\underset{\sim}{j} + 6\underset{\sim}{k}$

$\vec{CA} = 4\underset{\sim}{i} + 12\underset{\sim}{j}$

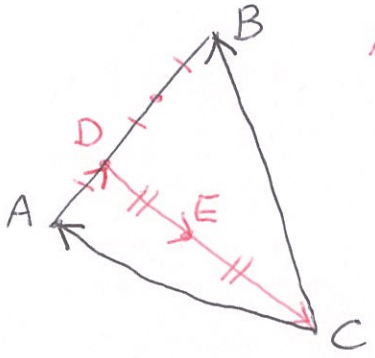
so $\vec{CB} \cdot \vec{CA} = -80$.

$\cos \angle ACB = \dfrac{\vec{CB} \cdot \vec{CA}}{|CB||CA|}$

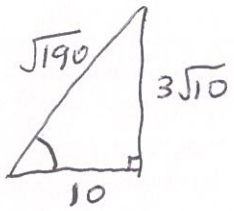
$= \dfrac{-80}{2\sqrt{19} \cdot 4\sqrt{10}}$

$= \dfrac{-80}{8\sqrt{190}}$

$= \dfrac{-10}{\sqrt{190}}$ (so $\angle ACB$ is obtuse)



$AD:AB = 1:3$



so $\sin \angle ACB = \dfrac{3\sqrt{10}}{\sqrt{190}} = \dfrac{3}{\sqrt{19}}$.

So area of $\Delta ABC = \frac{1}{2}|CB||CA| \sin \angle ACB$

$= \frac{1}{2} \cdot 2\sqrt{19} \cdot 4\sqrt{10} \cdot \dfrac{3}{\sqrt{19}}$

$= 12\sqrt{10}\ u^2$.

(b) $\vec{AB} = -6\underset{\sim}{i} - 18\underset{\sim}{j} + 6\underset{\sim}{k}$

so $\vec{AD} = -2\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}$

so $\vec{OD} = \vec{OA} + \vec{AD}$

$= (9\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}) + (-2\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k})$

$= 7\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

$\vec{DC} = -2\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}$

so $\vec{DE} = -\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

so $\vec{OE} = \vec{OD} + \vec{DE}$

$= 6\underset{\sim}{i} - 2\underset{\sim}{j}$

5D

(21)(a) [Alternative solution that avoids fractions]

Area of $\Delta ABC = \frac{1}{2}\left|\vec{CA}\right|\left|\vec{CB}\right| \sin \angle ACB$

$= \frac{1}{2}\left|\vec{CA}\right|\left|\vec{CB}\right| \sqrt{1-\cos^2 \angle ACB}$

$= \frac{1}{2}\sqrt{\left|\vec{CA}\right|^2\left|\vec{CB}\right|^2 - \left(\left|\vec{CA}\right|\left|\vec{CB}\right|\cos \angle ACB\right)^2}$

$= \frac{1}{2}\sqrt{\left(\vec{CA}\cdot\vec{CA}\right)\left(\vec{CB}\cdot\vec{CB}\right) - \left(\vec{CA}\cdot\vec{CB}\right)^2}$

$= \frac{1}{2}\sqrt{160 \times 76 - 80^2}$

$= \frac{1}{2}\sqrt{5760}$

$= 12\sqrt{10}\ u^2$, as before.

5D

(21)(c) $\overrightarrow{OE} \cdot \overrightarrow{AB} = (6\underset{\sim}{i} - 2\underset{\sim}{j}) \cdot (-6\underset{\sim}{i} - 18\underset{\sim}{j} + 6\underset{\sim}{k})$

$= -36 + 36$

$= 0$

and $\overrightarrow{OE} \cdot \overrightarrow{AC} = (6\underset{\sim}{i} - 2\underset{\sim}{j}) \cdot (-4\underset{\sim}{i} - 12\underset{\sim}{j})$

$= -24 + 24$

$= 0$

so $\overrightarrow{OE}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$,
so $\overrightarrow{OE}$ " " " the plane ABC.

(d) $V = \frac{1}{3} \times$ area of base $\times \perp$ height

$= \frac{1}{3} \times$ area of $\Delta ABC \times |\overrightarrow{OE}|$

$= \frac{1}{3} \times 12\sqrt{10} \times 2\sqrt{10}$

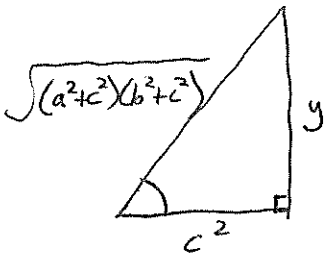
$= 80 \text{ u}^3$

(22) $|AOB| = \frac{ab}{2}$, $|AOC| = \frac{ac}{2}$, $|BOC| = \frac{bc}{2}$.

$\cos \angle ACB = \dfrac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}||\overrightarrow{CB}|}$

$= \dfrac{\begin{pmatrix} a \\ 0 \\ -c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -c \end{pmatrix}}{\sqrt{a^2+c^2} \cdot \sqrt{b^2+c^2}}$

$= \dfrac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$



$y^2 + c^4 = a^2b^2 + a^2c^2 + b^2c^2 + c^4$

so $y = \sqrt{a^2b^2 + a^2c^2 + b^2c^2}$.

$|ABC| = \frac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}| \sin \angle ACB$

$= \frac{1}{2}\sqrt{a^2+c^2} \cdot \sqrt{b^2+c^2} \cdot \dfrac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{\sqrt{(a^2+c^2)(b^2+c^2)}}$

so $|ABC| = \frac{1}{2}\sqrt{a^2b^2+a^2c^2+b^2c^2}$.

So $|AOB|^2 + |BOC|^2 + |COA|^2 = \frac{1}{4}a^2b^2 + \frac{1}{4}a^2c^2 + \frac{1}{4}b^2c^2$

$= \frac{1}{4}(a^2b^2 + a^2c^2 + b^2c^2)$

$= |ABC|^2$.