Solutions to Exercise 5C Foundation questions

1a $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = 4$

$|\underset{\sim}{b}| = 6$

$\theta = 45°$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 4 \times 6 \times \cos 45°$

$= 24 \times \frac{\sqrt{2}}{2}$

$= 12\sqrt{2}$

1b $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = 5$

$|\underset{\sim}{b}| = 8$

$\theta = 120°$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 5 \times 8 \times \cos 120°$

$= 40 \times -\frac{1}{2}$

$= -20$

2a $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3 \times 2) + (-1 \times 6) + (5 \times 1)$

$= 6 - 6 + 5$

$= 5$

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Chapter 5 worked solutions – Vectors

2b $\underset{\sim}{a} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (x_1 \times x_2) + (y_1 \times y_2) + (z_1 \times z_2)$

$= x_1x_2 + y_1y_2 + z_1z_2$

2c $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

$\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

$= a_1b_1 + a_2b_2 + a_3b_3$

3 $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = a_1^2 + a_2^2 + a_3^2$

For $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= a_1^2 + a_2^2 + a_3^2$

$= |\underset{\sim}{a}|^2$

$=$ RHS

Thus $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

4a $\underset{\sim}{a} = 2\underset{\sim}{i} - 7\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{b} = -4\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2 \times -4) + (-7 \times 1) + (3 \times 5)$

$= -8 - 7 + 15$

$= 0$

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4b As $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ this means that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 90°, as $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$ and both $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$ are non-zero.

So we can conclude that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

5 $\underset{\sim}{a} = \begin{bmatrix} 13 \\ 23 \\ 7 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

If $\underset{\sim}{a} \perp \underset{\sim}{b}$ and $\underset{\sim}{a} \perp \underset{\sim}{c}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ and vice versa.

For $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= (13 \times 2) + (23 \times 1) + (7 \times -7)$

$= 26 + 23 - 49$

$= 0$

$=$ RHS

For $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{c}$

$= (13 \times 3) + (23 \times -2) + (7 \times 1)$

$= 39 - 46 + 7$

$= 0$

$=$ RHS

Thus $\underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$

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6a $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = (-3 \times -3) + (9 \times 9) + (6 \times 6)$

$= 9 + 81 + 36$

$= 126$

6b $\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$2\underset{\sim}{b} \cdot \underset{\sim}{b} = 2\big((8 \times 8) + (4 \times 4) + (-10 \times -10)\big)$

$= 2(64 + 16 + 100)$

$= 2 \times 180$

$= 360$

6c $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-3 \times 8) + (9 \times 4) + (6 \times -10)$

$= -24 + 36 - 60$

$= -48$

6d $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} \cdot \left(\begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix} \right)$

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$$= \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 13 \\ -4 \end{bmatrix}$$

$$= (-3 \times 5) + (9 \times 13) + (6 \times -4)$$

$$= -15 + 117 - 24$$

$$= 78$$

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Solutions to Exercise 5C Development questions

7a $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 2^2 = 5$

$|\underset{\sim}{a}| = \sqrt{9} = 3$

$|\underset{\sim}{b}|^2 = 2^2 + 6^2 + (-3)^2 = 49$

$|\underset{\sim}{b}| = \sqrt{49} = 7$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (2 \times 6) + (2 \times -3)$

$= 2 + 12 + -6$

$= 8$

Substituting the associated values into the inequation:

$-21 \leq 8 \leq 21$

So the Cauchy-Schwarz inequality, $-|\underset{\sim}{a}||\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$ is satisfied.

7b $\underset{\sim}{a} = -\underset{\sim}{\imath} + 3\underset{\sim}{\jmath}$

$\underset{\sim}{b} = -6j + 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 1^2 + 3^2 = 10$

$|\underset{\sim}{a}| = \sqrt{10}$

$|\underset{\sim}{b}|^2 = (-6)^2 + 2^2 = 40$

$|\underset{\sim}{b}| = \sqrt{40} = 2\sqrt{10}$

$|\underset{\sim}{a}||\underset{\sim}{b}| = 2\sqrt{10} \times \sqrt{10} = 20$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-1 \times 0) + (-6 \times 3) + (0 \times 2)$

$= -18$

Substituting the associated values into the inequation:

$-20 \leq -18 \leq 20$

So the Cauchy-Schwarz inequality, $-|\underset{\sim}{a}||\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$ is satisfied.

Uncorrected proofs

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8a $\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 2^2 = 9$

$|\underset{\sim}{a}| = \sqrt{9} = 3$

$|\underset{\sim}{b}|^2 = 2^2 + 6^2 + (-3)^2 = 49$

$|\underset{\sim}{b}| = \sqrt{49} = 7$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (1+2)^2 + (2+6)^2 + (2-3)^2$

$= 9 + 64 + 1$

$= 74$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{74}$

$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Substituting we get:

$4 \leq \sqrt{74} \leq 10$

So the triangle inequality holds.

8b $\underset{\sim}{a} = -\underset{\sim}{\imath} + 3\underset{\sim}{\jmath}$

$\underset{\sim}{b} = -6j + 2\underset{\sim}{k}$

$|\underset{\sim}{a}| = \sqrt{10}$

$|\underset{\sim}{b}| = 2\sqrt{10}$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (-1+0)^2 + (3-6)^2 + (0+2)^2$

$= 1 + 9 + 4$

$= 14$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{14}$

$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Substituting we get:

$\big|\sqrt{10} - 2\sqrt{10}\big| \leq \sqrt{14} \leq \sqrt{10} + 2\sqrt{10}$

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$\sqrt{10} \leq \sqrt{14} \leq 3\sqrt{10}$

So the triangle inequality holds.

9 $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} -2 \\ 9 \\ -5 \end{bmatrix}$

$\underset{\sim}{d} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 4-2 \\ 1-3 \\ 3-5 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$

$\overrightarrow{CD} = \begin{bmatrix} -3-(-2) \\ 1-9 \\ 2-(-5) \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

$\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

$= -2 + 16 - 14$

$= 0$

So as $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

10a $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \\ -5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ \lambda \\ -2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} 3 \\ \lambda \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ -5 \end{bmatrix}$

$= (2 \times 3) + (-2 \times \lambda) + (-5 \times -2)$

Uncorrected proofs

$= -2\lambda + 16$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

So

$-2\lambda + 16 = 0$

$-2\lambda = -16$

$\lambda = 8$

10b $\underset{\sim}{a} = \begin{bmatrix} -4 \\ \lambda + 3 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} \lambda \\ 5 \\ -\lambda^2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -4 \\ \lambda + 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 5 \\ -\lambda^2 \end{bmatrix}$

$= (-4 \times \lambda) + ((\lambda + 3) \times 5) + (2 \times -\lambda^2)$

$= -2\lambda^2 + \lambda + 15$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

So

$-2\lambda^2 + \lambda + 15 = 0$

$2\lambda^2 - \lambda - 15 = 0$

$(2\lambda + 5)(\lambda - 3) = 0$

$\lambda = -\frac{5}{2}$ or $\lambda = 3$

11 Let $\underset{\sim}{a} = \underset{\sim}{\imath} - \underset{\sim}{\jmath} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{\imath} + \underset{\sim}{\jmath} - 3\underset{\sim}{k}$

For a vector $\underset{\sim}{c}$ which is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\underset{\sim}{c} = \lambda_1\underset{\sim}{\imath} + \lambda_2\underset{\sim}{\jmath} + \lambda_3\underset{\sim}{k}$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 2\lambda_1 + \lambda_2 - 3\lambda_3$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \lambda_1 - \lambda_2 + 2\lambda_3$

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Rearranging $\underset{\sim}{b} \cdot \underset{\sim}{c}$ and $\underset{\sim}{a} \cdot \underset{\sim}{c}$ for λ_1:

$\lambda_1 - \lambda_2 + 2\lambda_3 = 0$

$\lambda_1 = \lambda_2 - 2\lambda_3$

$2\lambda_1 + \lambda_2 - 3\lambda_3 = 0$

$2\lambda_1 = 3\lambda_3 - \lambda_2$

Equating λ_1:

$2(\lambda_2 - 2\lambda_3) = 3\lambda_3 - \lambda_2$

$3\lambda_2 = 7\lambda_3 = \lambda_1$

So a value for $\underset{\sim}{c}$ will be a linear combination of of $\lambda_1\left(\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} + 3\underset{\sim}{k}\right)$ where $\lambda_1 \in R$

One such vector is $\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} + 3\underset{\sim}{k}$.

12 $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

For $\underset{\sim}{a} \cdot (\lambda\underset{\sim}{b}) = \lambda(\underset{\sim}{a} \cdot \underset{\sim}{b})$

LHS $= \underset{\sim}{a} \cdot (\lambda\underset{\sim}{b})$

$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \left(\lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right)$

$= \lambda a_1 b_1 + \lambda a_2 b_2 + \lambda a_3 b_3$

$= \lambda(a_1 b_1 + a_2 b_2 + a_3 b_3)$

$= \lambda(\underset{\sim}{a} \cdot \underset{\sim}{b})$

$=$ RHS

So the equation is satisfied.

13 $\underset{\sim}{a} = a_1\underset{\sim}{\imath} + a_2\underset{\sim}{\jmath} + a_3\underset{\sim}{k}$

$\underset{\sim}{b} = b_1\underset{\sim}{\imath} + b_2\underset{\sim}{\jmath} + b_3\underset{\sim}{k}$

$\underset{\sim}{c} = c_1\underset{\sim}{\imath} + c_2\underset{\sim}{\jmath} + c_3\underset{\sim}{k}$

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$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

LHS $= \underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c})$

$= (a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}) \cdot ((b_1 + c_1)\underset{\sim}{i} + (b_2 + c_2)\underset{\sim}{j} + (b_3 + c_3)\underset{\sim}{k})$

$= (a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3)$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

= RHS

So the equation is satisfied.

14a A vector is a unit vector if its magnitude is equal to 1

$\underset{\sim}{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$

$|\underset{\sim}{u}|^2 = \frac{1}{2}(1 + 1) = 1$

$|\underset{\sim}{u}| = 1$

So $\underset{\sim}{u}$ is a unit vector.

$\underset{\sim}{v} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$|\underset{\sim}{v}|^2 = \frac{1}{2}(1 + 1) = 1$

$|\underset{\sim}{v}| = 1$

So $\underset{\sim}{v}$ is a unit vector.

$\underset{\sim}{w} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$|\underset{\sim}{w}|^2 = 1$

$|\underset{\sim}{w}| = 1$

So $\underset{\sim}{w}$ is a unit vector.

$\underset{\sim}{u} \cdot \underset{\sim}{w} = \frac{1}{\sqrt{2}}(0 + 0 + 0) = 0$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \frac{1}{2}(-1 + 1 + 0) = 0$

Uncorrected proofs

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \frac{1}{\sqrt{2}}(0 + 0 + 0) = 0$

So $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular to each other.

Thus $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthonormal.

14b A vector is a unit vector if its magnitude is equal to 1.

$$\underset{\sim}{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$|\underset{\sim}{u}|^2 = \frac{1}{2}(1 + 1) = 1$$

$$|\underset{\sim}{u}| = 1$$

So $\underset{\sim}{u}$ is a unit vector.

$$\underset{\sim}{v} = \frac{1}{2\sqrt{2}}\begin{bmatrix}1\\\sqrt{6}\\-1\end{bmatrix}$$

$$|\underset{\sim}{v}|^2 = \frac{1}{8}(1 + 6 + 1) = 1$$

$$|\underset{\sim}{v}| = 1$$

So $\underset{\sim}{v}$ is a unit vector.

$$\underset{\sim}{w} = \frac{1}{2\sqrt{2}}\begin{bmatrix}-\sqrt{3}\\\sqrt{2}\\\sqrt{3}\end{bmatrix}$$

$$|\underset{\sim}{w}|^2 = \frac{1}{8}(3 + 2 + 3) = 1$$

$$|\underset{\sim}{w}| = 1$$

So $\underset{\sim}{w}$ is a unit vector.

$\underset{\sim}{u} \cdot \underset{\sim}{w} = \frac{1}{4}(-\sqrt{3} + 0 + \sqrt{3}) = 0$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \frac{1}{4}(1 + 0 - 1) = 0$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \frac{1}{8}\left(-\sqrt{3} + \sqrt{2}\sqrt{6} - \sqrt{3}\right) = \frac{1}{8}(-\sqrt{3} + 2\sqrt{3} - \sqrt{3}) = 0$

So $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular to each other.

Thus $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthonormal.

Uncorrected proofs

15a $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})$

Using the distributive law:

$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) - \underset{\sim}{a} \cdot \underset{\sim}{b}$

For $\underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = 0$

$\text{LHS} = \underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= 0$

$= \text{RHS}$

So the equation is satisfied.

15b $(\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c} = (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}$

$(\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c} = (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}$

If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c}$:

$\underset{\sim}{a} = \lambda\underset{\sim}{c}$, where $\lambda \epsilon \mathbb{R}$

$\text{LHS} = (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\lambda\underset{\sim}{c} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\underset{\sim}{c} \cdot \underset{\sim}{b})\lambda\underset{\sim}{c}$

$= (\underset{\sim}{c} \cdot \underset{\sim}{b})\underset{\sim}{a}$

$= \text{RHS}$

So the equation is satisfied.

If $\underset{\sim}{b}$ is perpendicular to a and $\underset{\sim}{c}$:

$\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\text{LHS} = (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{c}$

$= \text{RHS}$

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So the equation is satisfied.

16a $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ are perpendicular then:

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

$\left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}\right) = 0$

$(a_1 + b_1)(a_1 - b_1) + (a_2 + b_2)(a_2 - b_2) + (a_3 + b_3)(a_3 - b_3) = 0$

$(a_1^2 - b_1^2) + (a_2^2 - b_2^2) + (a_3^2 - b_3^2) = 0$

$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2$

$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2$

Hence $|\underset{\sim}{a}| = |\underset{\sim}{b}|$.

16b $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$|(\underset{\sim}{a} + \underset{\sim}{b})| = |(\underset{\sim}{a} - \underset{\sim}{b})|$

$\left\|\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}\right\|$

$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2} =$
$= \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$

Raising both sides to the power of 2 gives:

$(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$

$a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 + a_3^2 + 2a_3b_3 + b_3^2$

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$= a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2$

$4a_1b_1 + 4a_2b_2 + 4a_3b_3 = 0$

$a_1b_1 + a_2b_2 + a_3b_3 = 0$

For $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be perpendicular $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

LHS $= \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= a_1b_1 + a_2b_2 + a_3b_3$

$= 0$

$=$ RHS

So the equation is satisfied.

17 $\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Given:

$\overrightarrow{AB} \perp \overrightarrow{OC} \Rightarrow \overrightarrow{AB}.\overrightarrow{OC} = 0$

$\overrightarrow{BC} \perp \overrightarrow{OA} \Rightarrow \overrightarrow{BC}.\overrightarrow{OA} = 0$

$(b - a).c = 0$ and $(c - b).a = 0$

$bc = ca$ and $ab = ca$

$bc = ab$ (1)

$\overrightarrow{AC}.\overrightarrow{OB} = (c - a).b$

$= bc - ba$

$= bc - bc$ from (1)

$= 0$

Hence, $\overrightarrow{AC} \perp \overrightarrow{OB}$.

Uncorrected proofs

18 $|\underset{\sim}{a}| = 2$

$|\underset{\sim}{b}| = 3$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b}}$

$= \sqrt{2^2 + 3^2 + 2 \times 5}$

$= \sqrt{23}$

19 $|\underset{\sim}{u}| = 2\sqrt{2}$

$|\underset{\sim}{v}| = 2\sqrt{3}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = -4$

$|\underset{\sim}{u} - \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}$

$|\underset{\sim}{u} - \underset{\sim}{v}| = \sqrt{|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}}$

$= \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{3}\right)^2 - (2 \times -4)}$

$= \sqrt{8 + 12 + 8}$

$= \sqrt{28}$

$= 2\sqrt{7}$

20 $\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Squaring the middle term gives:

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2$

Rearranging for $-2|\underset{\sim}{a}||\underset{\sim}{b}|$ gives:

$-2|\underset{\sim}{a}||\underset{\sim}{b}| = -|\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

We know from the Cauchy-Schwarz inequality that:

$-|\underset{\sim}{a}||\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$

So

Uncorrected proofs

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$$-2|\underset{\sim}{a}||\underset{\sim}{b}| = -|\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

$$-|\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \leq 2|\underset{\sim}{a}||\underset{\sim}{b}|$$

$$|\underset{\sim}{a}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2 \leq |\underset{\sim}{a} + \underset{\sim}{b}|^2$$

$$(|\underset{\sim}{a}| - |\underset{\sim}{b}|)^2 \leq |\underset{\sim}{a} + \underset{\sim}{b}|^2$$

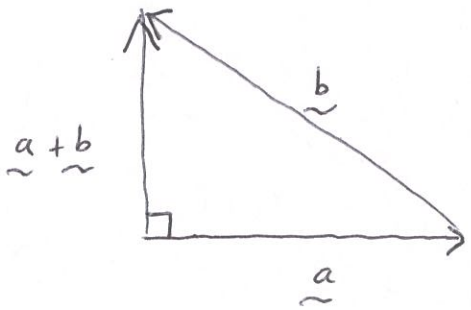
Hence

$$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}|$$

Uncorrected proofs

5C

(21)



Given $|\underset{\sim}{b}| = \sqrt{2}\,|\underset{\sim}{a}|$

so $|\underset{\sim}{b}|^2 = 2|\underset{\sim}{a}|^2$

so $\underset{\sim}{b}\cdot\underset{\sim}{b} = 2\,\underset{\sim}{a}\cdot\underset{\sim}{a}$. (*)

Since $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$,

$(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a} = 0$

so $\underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

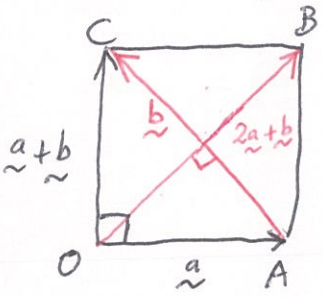
so $\frac{1}{2}\,\underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ (from (*))

so $2\,\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b} = 0$

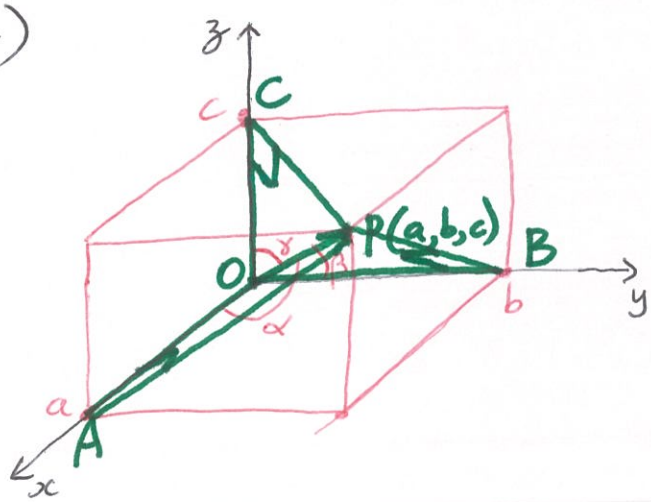
so $(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{b} = 0$

so $2\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

Note that $\underset{\sim}{a}$ and $\underset{\sim}{a}+\underset{\sim}{b}$ represent the sides of a square.



(22)



OP is the hypotenuse of each of the right-angled triangles OAP, OBP, OCP.

So, by basic trig,

$\cos\alpha = \dfrac{a}{|\overrightarrow{OP}|}$,

$\cos\beta = \dfrac{b}{|\overrightarrow{OP}|}$ and $\cos\gamma = \dfrac{c}{|\overrightarrow{OP}|}$

so $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{a^2+b^2+c^2}{|\overrightarrow{OP}|^2}$

$= \dfrac{|\overrightarrow{OP}|^2}{|\overrightarrow{OP}|^2}$

$= 1$.