Solutions to Exercise 5B Foundation questions

1a i $P(2, -3, 5)$

$$\overrightarrow{OP} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

1a ii $P(2, -3, 5)$

$$\overrightarrow{OP} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$$

1b i $P(-4, 0, 13)$

$$\overrightarrow{OP} = \begin{bmatrix} -4 \\ 0 \\ 13 \end{bmatrix}$$

1b ii $P(-4, 0, 13)$

$$\overrightarrow{OP} = -4\underset{\sim}{i} + 13\underset{\sim}{k}$$

1c i $P(a, -2a, -3a)$

$$\overrightarrow{OP} = \begin{bmatrix} a \\ -2a \\ -3a \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

1c ii $P(a, -2a, -3a)$

$$\overrightarrow{OP} = a\underset{\sim}{i} - 2a\underset{\sim}{j} - 3a\underset{\sim}{k}$$

2a $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{k}$

$$|\underset{\sim}{a}|^2 = 4^2 + (-3)^2$$

$$|\underset{\sim}{a}|^2 = 16 + 9$$

$$|\underset{\sim}{a}|^2 = 25$$

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$|\underset{\sim}{a}| = 5$

$\hat{\underset{\sim}{a}} = \frac{1}{|\underset{\sim}{a}|}\underset{\sim}{a}$

So

$\hat{\underset{\sim}{a}} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$\hat{\underset{\sim}{a}} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$

2b $\underset{\sim}{a} = \underset{\sim}{\imath} + 2\underset{\sim}{\jmath} - 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + (-2)^2$

$|\underset{\sim}{a}|^2 = 1 + 4 + 4$

$|\underset{\sim}{a}|^2 = 9$

$|\underset{\sim}{a}| = 3$

$\hat{\underset{\sim}{a}} = \frac{1}{|\underset{\sim}{a}|}\underset{\sim}{a}$

So

$\hat{\underset{\sim}{a}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

3a $\underset{\sim}{v} = -\underset{\sim}{\imath} - 4\underset{\sim}{\jmath} + \underset{\sim}{k}$

$|\underset{\sim}{v}|^2 = (-1)^2 + (-4)^2 + 1^2$

$|\underset{\sim}{v}|^2 = 1 + 16 + 1$

$|\underset{\sim}{v}|^2 = 18$

$|\underset{\sim}{v}| = \sqrt{18}$

$|\underset{\sim}{v}| = 3\sqrt{2}$

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$$\hat{\underset{\sim}{v}} = \frac{1}{|\underset{\sim}{v}|}\underset{\sim}{v}$$

So

$$\hat{\underset{\sim}{v}} = \frac{1}{3\sqrt{2}}\begin{bmatrix}-1\\-4\\1\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{\sqrt{2}}{6}\begin{bmatrix}-1\\-4\\1\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{1}{6}\begin{bmatrix}-\sqrt{2}\\-4\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

3b $\underset{\sim}{v} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

$|\underset{\sim}{v}|^2 = 5^2 + 3^2 + (-4)^2$

$|\underset{\sim}{v}|^2 = 25 + 9 + 16$

$|\underset{\sim}{v}|^2 = 50$

$|\underset{\sim}{v}| = \sqrt{50}$

$|\underset{\sim}{v}| = 5\sqrt{2}$

$$\hat{\underset{\sim}{v}} = \frac{1}{|\underset{\sim}{v}|}\underset{\sim}{v}$$

So

$$\hat{\underset{\sim}{v}} = \frac{1}{5\sqrt{2}}\begin{bmatrix}5\\3\\-4\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{\sqrt{2}}{10}\begin{bmatrix}5\\3\\-4\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{1}{10}\begin{bmatrix}5\sqrt{2}\\3\sqrt{2}\\-4\sqrt{2}\end{bmatrix}$$

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4a $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$2\underset{\sim}{p} + \underset{\sim}{q} = 2\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 8 \\ -4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ -10 \\ 23 \end{bmatrix}$

4b $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$\left|2\underset{\sim}{p} + \underset{\sim}{q}\right| = \left|2\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right|$

$= \left|\begin{bmatrix} 8 \\ -4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right|$

$= \left|\begin{bmatrix} 5 \\ -10 \\ 23 \end{bmatrix}\right|$

$= \sqrt{5^2 + (-10)^2 + 23^2}$

$= \sqrt{25 + 100 + 529}$

$= \sqrt{654}$

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4c $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$\underset{\sim}{p} - 5\underset{\sim}{q} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$= \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 15 \\ 30 \\ -45 \end{bmatrix}$

$= \begin{bmatrix} 19 \\ 28 \\ -38 \end{bmatrix}$

4d $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$\left|\underset{\sim}{p} - 5\underset{\sim}{q}\right| = \left|\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right|$

$= \left|\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 15 \\ 30 \\ -45 \end{bmatrix}\right|$

$= \left|\begin{bmatrix} 19 \\ 28 \\ -38 \end{bmatrix}\right|$

$= \sqrt{19^2 + 28^2 + (-38)^2}$

$= \sqrt{361 + 784 + 1444}$

$= \sqrt{2589}$

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5a $\underset{\sim}{p} = 2\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} - \underset{\sim}{k}$

$\underset{\sim}{q} = 5\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$

$= (5\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 3\underset{\sim}{k}) - (2\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} - \underset{\sim}{k})$

$= 3\underset{\sim}{\imath} - 12\underset{\sim}{\jmath} + 4\underset{\sim}{k}$

5b $\underset{\sim}{p} = 2\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} - \underset{\sim}{k}$

$\underset{\sim}{q} = 5\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 3\underset{\sim}{k}$

$\overrightarrow{QP} = \underset{\sim}{p} - \underset{\sim}{q}$

$= (2\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} - \underset{\sim}{k}) - (5\underset{\sim}{\imath} - 5\underset{\sim}{\jmath} + 3\underset{\sim}{k})$

$= -3\underset{\sim}{\imath} + 12\underset{\sim}{\jmath} - 4\underset{\sim}{k}$

5c We know from question 5a that $\overrightarrow{PQ} = 3\underset{\sim}{\imath} - 12\underset{\sim}{\jmath} + 4\underset{\sim}{k}$.

$\left|\overrightarrow{PQ}\right|^2 = 3^2 + (-12)^2 + 4^2$

$\left|\overrightarrow{PQ}\right|^2 = 9 + 144 + 16$

$\left|\overrightarrow{PQ}\right|^2 = 169$

$\left|\overrightarrow{PQ}\right| = 13$

6a $\overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$

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$$= \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$$

6b $\overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}$

6c We know from question 6b that $\overrightarrow{AB} = \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}$.

$\left|\overrightarrow{AB}\right|^2 = (-8)^2 + (-3)^2 + 2^2$

$\left|\overrightarrow{AB}\right|^2 = 64 + 9 + 4$

$\left|\overrightarrow{AB}\right|^2 = 77$

$\left|\overrightarrow{AB}\right|^2 = \sqrt{77}$

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Solutions to Exercise 5B Development questions

7 $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix}$

$\lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{b} = \begin{bmatrix} 14 \\ 26 \\ -18 \end{bmatrix}$

$\lambda_1 \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 26 \\ -18 \end{bmatrix}$

$3\lambda_1 - 2\lambda_2 = 14 \qquad (1)$

$-\lambda_1 + 4\lambda_2 = -18$

$\lambda_1 = 4\lambda_2 + 18 \qquad (2)$

Substituting (2) back into equation (1) then solving for λ_2 we get:

$3(4\lambda_2 + 18) - 2\lambda_2 = 14$

$12\lambda_2 + 54 - 2\lambda_2 = 14$

Hence

$10\lambda_2 = -40$

$\lambda_2 = -4$

$\lambda_1 = 4\lambda_2 + 18$

$\lambda_1 = -16 + 18 = 2$

8 $\underset{\sim}{a} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$

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$$\lambda_1 \underset{\sim}{a} + \lambda_2 \underset{\sim}{b} + \lambda_3 \underset{\sim}{c} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$$

$-\lambda_1 + 4\lambda_3 = -7$

$\lambda_1 = 7 + 4\lambda_3$ (1)

$2\lambda_1 - 2\lambda_2 + 3\lambda_3 = -14$ (2)

$\lambda_2 - 2\lambda_3 = 7$

$\Rightarrow \lambda_2 = 7 + 2\lambda_3$ (3)

Substituting (1) and (3) back into equation (2) then solving for λ_3 we get:

$\Rightarrow 2(7 + 4\lambda_3) - 2(7 + 2\lambda_3) + 3\lambda_3 = -14$

$\Rightarrow 14 + 8\lambda_3 - 14 - 4\lambda_3 + 3\lambda_3 = -14$

$\Rightarrow 7\lambda_3 = -14$

$\Rightarrow \lambda_3 = -2$

Solving for λ_1

$\lambda_1 = 7 - 4(-2)$

$= -1$

Solving for λ_2

$\lambda_2 = 7 + 2(-2)$

$= 3$

9a $A = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

$C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

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$$D = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0-(-1) \\ 2-4 \\ 1-(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\overrightarrow{CD} = \begin{bmatrix} 0-3 \\ 8-2 \\ -7-5 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$$

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, there exists a value whereby $\overrightarrow{AB} = a\overrightarrow{CD}$.

$$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = a\begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$$

This holds true if $a = -3$, thus $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

9b $A = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$

$$B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 3-0 \\ 2-2 \\ 5-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$\overrightarrow{AD} = \begin{bmatrix} 0-(-1) \\ 8-4 \\ -7-(-3) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix}$$

If $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel, there exists a value whereby $\overrightarrow{BC} = a\overrightarrow{AD}$.

$$\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = a\begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix}$$

There is no value for a to satisfy this equation, thus $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are not parallel

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10 $A(-2,-1,0)$

$B(0,5,-2)$

$C(4,17,-6)$

$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$

$\overrightarrow{BC} = \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$

For the points A, B and C to be colinear the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ must be parallel.
If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel, there exists a value whereby $\overrightarrow{AB} = a\overrightarrow{BC}$.

So $\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = a\begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$, which is satisfied for $a = \frac{1}{2}$.

$\overrightarrow{BC} = 2\overrightarrow{AB}$

Thus points A, B and C are collinear.

11 $A(5,4,7)$

$B(7,-1,-4)$

$C(-1,-3,-5)$

$D(-3,2,6)$

In order for $ABCD$ to be a parallelogram the vectors $\overrightarrow{AB}, \overrightarrow{CD}$ and $\overrightarrow{BC}, \overrightarrow{AD}$ will be parallel and the diagonals bisect one another.

$\overrightarrow{AB} = \begin{bmatrix} 2 \\ -5 \\ -11 \end{bmatrix}$

$\overrightarrow{CD} = \begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, there exists a value, a whereby $\overrightarrow{AB} = a\overrightarrow{CD}$.

$\begin{bmatrix} 2 \\ -5 \\ -11 \end{bmatrix} = a\begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$, which is satisfied for $a = -1$.

So $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel (and equal in length since $a = -1$).

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$\overrightarrow{BC} = \begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$

$\overrightarrow{AD} = \begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$

If $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel, there exists a value b whereby $\overrightarrow{BC} = b\overrightarrow{AD}$.

$\begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix} = b\begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$, which is satisfied for $b = 1$.

So $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel (and equal in length as $b = 1$).

We can show that the diagonals bisect one another if the midpoints of AC and BD are the same.

$$M_{AC} = \begin{bmatrix} \frac{5-1}{2} \\ \frac{4-3}{2} \\ \frac{7-5}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$M_{BD} = \begin{bmatrix} \frac{7-3}{2} \\ \frac{-1+2}{2} \\ \frac{-4+6}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$M_{AC} = M_{BD}$

Since the opposite sides are parallel and the diagonals bisect each other, the points $ABCD$ form a parallelogram.

12 $\hat{A} = 3\underset{\sim}{\imath} - 8\underset{\sim}{\jmath} - 2\underset{\sim}{k}$

$\hat{B} = 2\underset{\sim}{\imath} + 4\underset{\sim}{\jmath} + 5\underset{\sim}{k}$

$\hat{C} = -2\underset{\sim}{\imath} - 2\underset{\sim}{\jmath} + \underset{\sim}{k}$

$\hat{D} = \hat{A} + \overrightarrow{AD}$

As it is a parallelogram, $\overrightarrow{AD}, \overrightarrow{BC}$ are parallel and so, $\overrightarrow{AD} = \overrightarrow{BC}$.

$\hat{D} = \hat{A} + \overrightarrow{BC}$

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$\widehat{D} = \hat{A} + (\hat{C} - \hat{B})$

$\widehat{D} = 3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k} + (-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} - (2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}))$

$\widehat{D} = 3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k} - 4\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$

$\widehat{D} = -\underset{\sim}{i} - 14\underset{\sim}{j} - 6\underset{\sim}{k}$

13 $A(2, 1, 3)$

Find the magnitude of $\overrightarrow{OA}$.

$\left|\overrightarrow{OA}\right|^2 = 2^2 + 1^2 + 3^2 = 14$

$\left|\overrightarrow{OA}\right| = \sqrt{14}$

Let B, C and D be the values of A on the x-, y- and z-axis respectively so:

$B(2, 0, 0)$

$C(0, 1, 0)$

$D(0, 0, 3)$

For the angle between A and the x-axis:

$\overrightarrow{OB} = 2$

$\angle AOB = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.69° \doteqdot 58°$

For the angle between A and the y-axis:

$\overrightarrow{OC} = 1$

$\angle AOC = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.5° \doteqdot 74°$

For the angle between A and the z-axis:

$\overrightarrow{OD} = 3$

$\angle AOD = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 36.7° \doteqdot 37°$

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14a $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 5 \\ 5 \\ -8 \end{bmatrix}$

$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$

$k = 1$

$\ell = 2$

$\underset{\sim}{p} = \frac{1}{1+2}\left(2\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \\ -8 \end{bmatrix}\right)$

$= \frac{1}{3}\begin{bmatrix} 9 \\ 3 \\ -12 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}$

14b If P divides $\overrightarrow{AB}$ externally $k = -1$ or $\ell = -2$

So using $k = -1, \ell = 2$:

$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$

$\underset{\sim}{p} = \frac{1}{-1+2}\left(2\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ 5 \\ -8 \end{bmatrix}\right)$

$= \begin{bmatrix} -1 \\ -7 \\ 4 \end{bmatrix}$

15a $\underset{\sim}{a} = -4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k}$

$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$

$k = 2$

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$\ell = 3$

$$\underset{\sim}{p} = \frac{1}{2+3}\left(3(-4\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} + 5\underset{\sim}{k}) + 2(6\underset{\sim}{\imath} - 8\underset{\sim}{\jmath} + 10\underset{\sim}{k})\right)$$

$$= \frac{1}{5}\left(-12\underset{\sim}{\imath} - 9\underset{\sim}{\jmath} + 15\underset{\sim}{k} + 12\underset{\sim}{\imath} - 16\underset{\sim}{\jmath} + 20\underset{\sim}{k}\right)$$

$$= \frac{1}{5}\left(-25\underset{\sim}{\jmath} + 35\underset{\sim}{k}\right)$$

$$= -5\underset{\sim}{\jmath} + 7\underset{\sim}{k}$$

15b If P divides $\overrightarrow{AB}$ externally, $k = -2$ or $\ell = -3$.

So using $k = -2$, $\ell = 3$:

$$\underset{\sim}{a} = -4\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} + 5\underset{\sim}{k}$$

$$\underset{\sim}{b} = 6\underset{\sim}{\imath} - 8\underset{\sim}{\jmath} + 10\underset{\sim}{k}$$

$$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$$

$$\underset{\sim}{p} = \frac{1}{-2+3}\left(3(-4\underset{\sim}{\imath} - 3\underset{\sim}{\jmath} + 5\underset{\sim}{k}) - 2(6\underset{\sim}{\imath} - 8\underset{\sim}{\jmath} + 10\underset{\sim}{k})\right)$$

$$= \left(-12\underset{\sim}{\imath} - 9\underset{\sim}{\jmath} + 15\underset{\sim}{k} - 12\underset{\sim}{\imath} + 16\underset{\sim}{\jmath} - 20\underset{\sim}{k}\right)$$

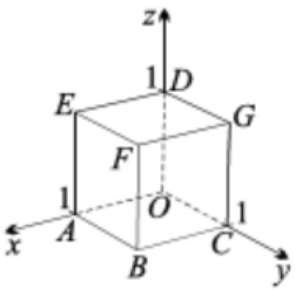
$$= -24\underset{\sim}{\imath} + 7\underset{\sim}{\jmath} - 5\underset{\sim}{k}$$

16a $A = (1, 0, 0)$

$G = (0, 1, 1)$

$$\overrightarrow{AG} = (0-1)\underset{\sim}{\imath} + (1-0)\underset{\sim}{\jmath} + (1-0)\underset{\sim}{k}$$

$$= -\underset{\sim}{\imath} + \underset{\sim}{\jmath} + \underset{\sim}{k}$$



16b $\overrightarrow{AG} = -\underset{\sim}{\imath} + \underset{\sim}{\jmath} + \underset{\sim}{k}$

$$\left|\overrightarrow{AG}\right|^2 = (-1)^2 + 1^2 + 1^2$$

$$\left|\overrightarrow{AG}\right| = \sqrt{3}$$

Uncorrected proofs

16c $\overrightarrow{OG} = \underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{OH} = \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{j} + \underset{\sim}{k})$

$\overrightarrow{OH} = \frac{1}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$

$|\overrightarrow{OH}|^2 = \left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2 = \frac{6}{4}$

$|\overrightarrow{OH}| = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}$

17a We want to show that $(\lambda_1 + \lambda_2)\underset{\sim}{a} = \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{a}$, where $\lambda_1, \lambda_2 \epsilon \mathbb{R}$

$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

LHS $= (\lambda_1 + \lambda_2)\underset{\sim}{a}$

$= (\lambda_1 + \lambda_2)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$= \lambda_1\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda_2\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$= \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{a}$

$=$ RHS

17b We want to show that $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\lambda \epsilon \mathbb{R}$

$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

LHS $= \lambda(\underset{\sim}{a} + \underset{\sim}{b})$

Uncorrected proofs

$$= \lambda\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right)$$

$$= \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \lambda \underset{\sim}{a} + \lambda \underset{\sim}{b}$$

$$= \text{RHS}$$

Uncorrected proofs

5B

(18)(a) The vector equation is

$$\lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix}+\lambda_2\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda_3\begin{pmatrix}0\\-1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

from which we obtain the simultaneous equations:

$$\lambda_1+\lambda_2=0 \quad (1)$$

$$\lambda_1+2\lambda_2-\lambda_3=0 \quad (2)$$

$$\lambda_1+\lambda_3=0 \quad (3)$$

If we let $\lambda_1=k$, it follows from (1) and (3) that $\lambda_2=\lambda_3=-k$.

Clearly k can be non-zero, so there are infinitely many non-trivial solutions for $\lambda_1,\lambda_2,\lambda_3$. (e.g. $1,-1,-1$)

In fact, any non-zero value of k provides us with the non-trivial solution $\lambda_1=k$, $\lambda_2=-k$, $\lambda_3=-k$.

So the set of vectors $\underset{\sim}{a},\underset{\sim}{b},\underset{\sim}{c}$ is linearly dependent, since $k\underset{\sim}{a}-k\underset{\sim}{b}-k\underset{\sim}{c}=\underset{\sim}{0}$ for all $k\in\mathbb{R}$.

(b) $\lambda_1+\lambda_2=0 \quad (1)$

$$\lambda_1+2\lambda_2-\lambda_3=0 \quad (2)$$

$$\lambda_1+2\lambda_3=0 \quad (3)$$

$2\times(1)-(2)$: $\lambda_1+\lambda_3=0 \quad (4)$

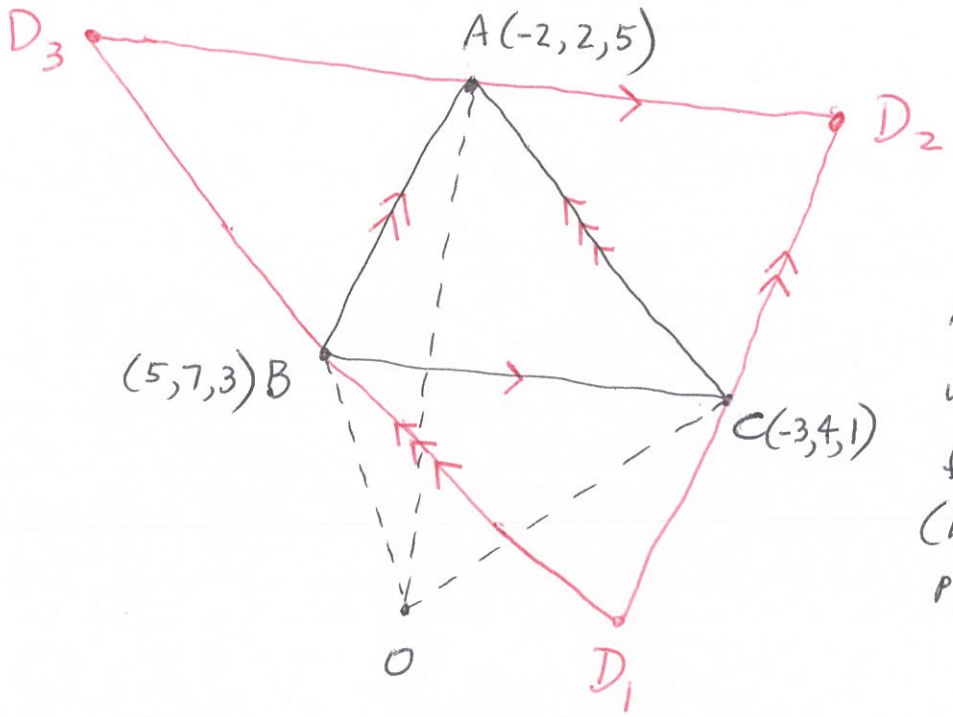
$(3)-(4)$: $\lambda_3=0$

so $\lambda_1=\lambda_2=\lambda_3=0$ is the only solution.

So the set of vectors is linearly independent.

5B

(19) In the plane containing A, B and C, there are three possible locations of D.
These are the points D_1, D_2 and D_3 shown below.
In the diagram a line is drawn through each vertex of ΔABC parallel to the opposite side.



Note that $\Delta ABC \parallel\parallel \Delta D_1 D_2 D_3$ with enlargement factor 2.
(Look at the parallelograms.)

Let $\vec{OA} = \underset{\sim}{a}$, $\vec{OB} = \underset{\sim}{b}$ and $\vec{OC} = \underset{\sim}{c}$.

Then $\vec{OD_1} = \vec{OB} + \vec{BD_1}$
$= \vec{OB} + \vec{AC}$
$= \underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}$
so $D_1 = (4, 9, -1)$

and $\vec{OD_2} = \vec{OC} + \vec{CD_2}$
$= \vec{OC} + \vec{BA}$
$= \underset{\sim}{c} + \underset{\sim}{a} - \underset{\sim}{b}$
so $D_2 = (-10, -1, 3)$

and $\vec{OD_3} = \vec{OA} + \vec{AD_3}$
$= \vec{OA} + \vec{CB}$
$= \underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$
so $D_3 = (6, 5, 7)$.

5B

(20) Given $\lambda \underset{\sim}{a} + \mu \underset{\sim}{b} = l \underset{\sim}{a} + m \underset{\sim}{b}$ (*)

with $\underset{\sim}{a}, \underset{\sim}{b}$ non-zero and non-parallel.

Re-arranging (*),

$$(\lambda - l)\underset{\sim}{a} + (\mu - m)\underset{\sim}{b} = \underset{\sim}{0} \quad (**)$$

Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel, it follows that $\underset{\sim}{b}$ is not a scalar multiple of $\underset{\sim}{a}$. So, from question 18, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent. So the only solution to equation (**) is the trivial solution $\lambda - l = \mu - m = 0$.

Thus, $\lambda = l$ and $\mu = m$.