Solutions to Exercise 5A Foundation questions

1 yz-plane: $x = 0$

xz-plane: $y = 0$

xy-plane: $z = 0$

2a $(-2,\ 3, 1)$ lies in the second octant.

2b $(2,\ 3, -1)$ lies in the fifth octant.

2c $(2, -3, 1)$ lies in the fourth octant.

2d $(-2,\ 3, -1)$ lies in the sixth octant.

2e $(2, -3, -1)$ lies in the eighth octant.

2f $(-2, -3, 1)$ lies in the third octant.

3a $(3, 2, 5 - 6) = (3, 2, -1)$

3b $(3 - 8, 2, 5) = (-5, 2, 5)$

3c $(3, 2 + 10, 5) = (3, 12, 5)$

3d $(3 + 5\ , 2, 5 + 7) = (8, 2, 12)$

3e $(3, 2 - 3, 5 - 4) = (3, -1, 1)$

3f $(3, 2, 5 \times -1) = (3, 2, -5)$

© Cambridge University Press 2019

3g $(3 \times -1, 2, 5) = (-3, 2, 5)$

3h $(3, 2 \times -1, 5) = (3, -2, 5)$

3i $(3, 2 \times -1, 5 \times -1) = (3, -2, -5)$

4a

$A = (2, 0, 0)$

$B = (2, 2, 0)$

$C = (2, 2, 2)$

$D = (2, 0, 2)$

$O = (0, 0, 0)$

$P = (0, 2, 0)$

$Q = (0, 2, 2)$

$R = (0, 0, 2)$

4b For a right-angled triangle, $a^2 + b^2 = c^2$

So for any diagonal on a face:

$2^2 + 2^2 = c^2$

$c = \sqrt{8}$

$= 2\sqrt{2}$

4c For a right-angled triangle, $a^2 + b^2 = c^2$

In the case of the space diagonal, $a = 2$ and $b = 2\sqrt{2}$

So

$2^2 + 2\sqrt{2}^2 = c^2$

$c = \sqrt{12}$

$= 2\sqrt{3}$

© Cambridge University Press 2019

4d The plane $OABP$ has the equation: $z = 0$

The plane $RDCQ$ has the equation: $z = 2$

The plane $ABCD$ has the equation: $x = 2$

The plane $OPQR$ has the equation: $x = 0$

The plane $OADR$ has the equation: $y = 0$

The plane $PBCQ$ has the equation: $y = 2$

5a $C = (2, 4, 3)$

$A = (2, 0, 0)$

$B = (2, 4, 0)$

$D = (2, 0, 3)$

$P = (0, 4, 0)$

$Q = (0, 4, 3)$

$R = (0, 0, 3)$

5b For a right-angled triangle, $a^2 + b^2 = c^2$

So for any diagonal on a face:

$2^2 + 4^2 = c^2$

$c = \sqrt{20}$

$= 2\sqrt{5}$

5c For a right-angled triangle $a^2 + b^2 = c^2$

In the case of the space diagonal, $a = 3$ and $b = 2\sqrt{5}$

So

$3^2 + \left(2\sqrt{5}\right)^2 = c^2$

$c = \sqrt{29}$

© Cambridge University Press 2019

5d The plane $OABP$ has the equation: $z = 0$

The plane $RDCQ$ has the equation: $z = 3$

The plane $ABCD$ has the equation: $x = 2$

The plane $OPQR$ has the equation: $x = 0$

The plane $OADR$ has the equation: $y = 0$

The plane $PBCQ$ has the equation: $y = 4$

6a

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\text{Area} = \frac{1}{2} \times 3 \times 4$$

$$= 6 \text{ square units}$$

6b

$$\text{Volume} = \frac{1}{3} \times \text{base} \times \text{height}$$

$$= \frac{1}{3} \times 6 \times 5$$

$$= 10 \text{ cubic units}$$

© Cambridge University Press 2019

Solutions to Exercise 5A Development questions

7a $AB^2 = (B_x - A_x)^2 + (B_y - A_y)^2 + (B_z - A_z)^2$

The three points are:

$O = (0, 0, 0)$

$A = (2, 6, 3)$

$B = (-3, 5, -8)$

$|\overrightarrow{OA}|^2| = (2-0)^2 + (6-0)^2 + (3-0)^2$

$|\overrightarrow{OA}|^2 = 4 + 36 + 9 = 49$

So $|\overrightarrow{OA}| = \sqrt{49} = 7$

$|\overrightarrow{OB}|^2 = (-3-0)^2 + (5-0)^2 + (-8-0)^2$

$|\overrightarrow{OB}|^2 = 9 + 25 + 64 = 98$

So $|\overrightarrow{OB}| = \sqrt{98} = 7\sqrt{2}$

$|\overrightarrow{AB}|^2 = (-3-2)^2 + (5-6)^2 + (-8-3)^2$

$|\overrightarrow{AB}|^2 = 25 + 1 + 121 = 147$

So $\overrightarrow{AB} = \sqrt{147} = 7\sqrt{3}$

7b $|\overrightarrow{OA}| = 7, |\overrightarrow{OA}|^2 = 49$

$|\overrightarrow{OB}| = 7\sqrt{2}, |\overrightarrow{OB}|^2 = 98$

$|\overrightarrow{AB}| = 7\sqrt{3}, |\overrightarrow{AB}|^2 = 147$

For a right-angled triangle, Pythagoras's theorem should hold; that is:

$|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 = |\overrightarrow{AB}|^2$

LHS $= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2$

$= 49 + 98$

$= 147$

$= AB^2$

Uncorrected proofs

© Cambridge University Press 2019

$= \text{RHS}$

Pythagoras's theorem is satisfied so the angle opposite the hypotenuse is right-angled. That is, $\angle AOB = 90°$.

8 $A = (3, -1, -3)$

$B = (1, -5, 7)$

$C = (-1, 3, 3)$

$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Midpoint of the line AB:

$M = \left(\frac{3 + 1}{2}, \frac{-1 - 5}{2}, \frac{-3 + 7}{2}\right) = (2, -3, 2)$

Midpoint of the line AC:

$N = \left(\frac{3 - 1}{2}, \frac{-1 + 3}{2}, \frac{-3 + 3}{2}\right) = (1, 1, 0)$

$M = (2, -3, 2)$

$N = (1, 1, 0)$

For MN:

$\left|\overrightarrow{MN}\right|^2 = (N_x - M_x)^2 + \left(N_y - M_y\right)^2 + (N_z - M_z)^2$

$\left|\overrightarrow{MN}\right|^2 = (1 - 2)^2 + (1 - (-3))^2 + (0 - 2)^2$

$\left|\overrightarrow{MN}\right|^2 = 1 + 16 + 4 = 21$

So $\left|\overrightarrow{MN}\right| = \sqrt{21}$

For BC:

$\left|\overrightarrow{BC}\right|^2 = (-1 - 1)^2 + (3 - (-5))^2 + (3 - 7)^2$

$\left|\overrightarrow{BC}\right|^2 = 4 + 64 + 16 = 68$

So $\left|\overrightarrow{BC}\right| = \sqrt{84} = 2\sqrt{21}$

$\overrightarrow{BC} = 2\overrightarrow{MN}$

LHS $= \overrightarrow{BC}$

$= 2\sqrt{21}$

Uncorrected proofs

© Cambridge University Press 2019

$= 2\overrightarrow{MN}$

$=$ RHS

9a $P = (-6, -8, 14)$

$Q = (-10, 20, 22)$

$M = \left(\frac{-16-6}{2}, \frac{20-8}{2}, \frac{22+14}{2}\right)$

$= \left(\frac{-16}{2}, \frac{12}{2}, \frac{36}{2}\right)$

$M = (-8, 6, 18)$

9b

$X = \frac{1}{2}(OP + OM)$

$= \frac{1}{2}(-14, -2, 32)$

$= (-7, -1, 16)$

$Y = \frac{1}{2}(OQ + OM)$

$= \frac{1}{2}(-18, 26, 40)$

$= \frac{1}{2}(-9, 13, 20)$

10 $P(1, 0, 0)$

$Q(-3, -1, 1)$

$R(-2, 3, 4)$

$C(-1, 1, 2)$

If the lengths of CP, CQ and CR are equal, the points P, Q and R are shown to be on the surface of a sphere centred at C:

$\left|\overrightarrow{CP}\right|^2 = \left(1 - (-1)\right)^2 + (0-1)^2 + (0-2)^2$

$\left|\overrightarrow{CP}\right| = \sqrt{4+1+4} = 3$

Uncorrected proofs

© Cambridge University Press 2019

$|\overrightarrow{CQ}|^2 = (-3-(-1))^2 + (-1-1)^2 + (1-2)^2$

$|\overrightarrow{CQ}| = \sqrt{4+4+1} = 3$

$|\overrightarrow{CR}|^2 = (-2-(-1))^2 + (3-1)^2 + (4-2)^2$

$|\overrightarrow{CR}| = \sqrt{4+1+4} = 3$

So the lengths $|\overrightarrow{CP}| = |\overrightarrow{CQ}| = |\overrightarrow{CR}|$ and the conditions are satisfied.

11 So the length between $(x, x+5, x-2)$ and $(1, 0, -1)$ will be:

$\sqrt{(x-1)^2 + (x+5)^2 + (x-1)^2}$

$= \sqrt{x^2 - 2x + 1 + x^2 + 10x + 25 + x^2 - 2x + 1}$

$= \sqrt{3x^2 + 6x + 27}$

We know

$\sqrt{3x^2 + 6x + 27} = 2\sqrt{6}$

$3x^2 + 6x + 27 = 24$

$3x^2 + 6x + 3 = 0$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$

12a $A = (4, 2, 6)$

$B = (-2, 0, 2)$

$C = (10, -2, 4)$

In order for the triangle to be an isosceles triangle, two sides must be of equal length:

$|\overrightarrow{AB}|^2 = (-6)^2 + (-2)^2 + (-4)^2$

$= 36 + 4 + 16$

$= 56$

$|\overrightarrow{AB}| = \sqrt{56} = 2\sqrt{14}$

Uncorrected proofs

$$\left|\overrightarrow{AC}\right|^2 = 6^2 + (-4)^2 + (-2)^2$$
$$= 36 + 16 + 4$$
$$= 56$$
$$\left|\overrightarrow{AC}\right| = \sqrt{56} = 2\sqrt{14}$$
$$\left|\overrightarrow{BC}\right|^2 = 12^2 + (-2)^2 + 2^2$$
$$= 144 + 4 + 4$$
$$= 152$$
$$\left|\overrightarrow{BC}\right| = \sqrt{152} = 2\sqrt{38}$$
$$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right| = 2\sqrt{14}$$

So as exactly two sides of the triangle have the same length,

ΔABC is an isosceles triangle.

12b The area of the triangle is:

$$\frac{1}{2}\text{base} \times \text{height}$$

The base of the triangle is:

$$\left|\overrightarrow{BC}\right| = 2\sqrt{38}$$

The height of the triangle can be found as the length from A to the middle point between BC, M:

$B(-2, 0, 2)$

$C(10, -2, 4)$

$$M = \left(\frac{10-2}{2}, \frac{-2+0}{2}, \frac{4+2}{2}\right)$$
$$= \left(-\frac{8}{2}, -\frac{2}{2}, \frac{6}{2}\right)$$

$M = (4, -1, 3)$

$AM^2 = ((4-4)^2, (2-(-1))^2, (3-6)^2)$

$AM = (0, 3, -3)$

$AM^2 = 0^2 + (-3)^2 + (-3)^2$

Uncorrected proofs

© Cambridge University Press 2019

$AM = \sqrt{18} = 3\sqrt{2}$

So:

$$\text{Area} = \frac{1}{2}\left(2\sqrt{38}\right) \times 3\sqrt{2}$$

$$= \sqrt{38} \times 3\sqrt{2}$$

$$= 6\sqrt{19} \text{ units}^2$$

13a For the plane:

$3x + 4y + 6z = 12$

The intersecting plane on the xy-plane will occur when $z = 0$.

So $3x + 4y = 12$

13b For the plane:

$3x + 4y + 6z = 12$

The intersecting plane on the xz-plane will occur when $y = 0$.

So $3x + 6z = 12$

The intersecting plane on the yz-plane will occur when $x = 0$.

So $4y + 6z = 12$

13c A line, as they fit the form: $y = mx + c$

13d The intersection of two non-parallel planes will be a line.

Uncorrected proofs

© Cambridge University Press 2019

5A

(14) In ΔPRO,

$PO^2 = a^2+b^2+c^2$, $PR^2 = b^2+c^2$ and $OR^2 = a^2$

so $PO^2 = PR^2 + OR^2$

so $\angle PRO = 90^\circ$ (converse of Pythagoras).

The proofs that PSO and PTO are right-angles are very similar.

(15) (a) z can take any real value.

(b) The horizontal plane $z=k$, for any real value of k, intersects the cylinder $x^2+y^2=4$ in the horizontal circle $x^2+y^2=4$. So the intersection is the circle with centre $(0,0,k)$ and radius 2 in the plane $z=k$.

(c) It is the curved surface of a cylinder of infinite height with radius 2 units. The z-axis is its axis of symmetry.

(d) The surface is graphed in the 'Answers to Chapter Five' in the text book.

(16) (a) $z = x^2+y^2 \geq 0$ for all $x, y \in \mathbb{R}$, since a square is never negative.

(b) The circle $x^2+y^2=k$, where $k>0$, with centre $(0,0,k)$ and radius $\sqrt{k}$.

If $k=0$, then the intersection is the origin.

If $k<0$, then there is no intersection.

(c) The xz-plane has equation $y=0$.

So the intersection is the parabola defined by $z=x^2$ and $y=0$.

(d) It is graphed in the 'Answers to Chapter Five' section of the text book.