Solutions to Exercise 4I Foundation questions

1a

$$\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$$

Let $u = 5 + x^3$

$du = 3x^2\,dx$

$x = 1, u = 6$

$x = -1, u = 4$

Hence

$$\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$$

$$= \frac{1}{3}\int_{4}^{6} \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\left[-\frac{1}{u}\right]_4^6$$

$$= \frac{1}{3}\left(-\frac{1}{6}+\frac{1}{4}\right)$$

$$= \frac{1}{36}$$

1b

$$\int_{0}^{\pi} x\sin x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \sin x$

$u' = 1, v = -\cos x$

Hence

$$\int_{0}^{\pi} x\sin x\,dx$$

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$= [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\, dx$

$= (\pi - 0) + [\sin x]_0^{\pi}$

$= \pi + (0 - 0)$

$= \pi$

1c

Let $\frac{2x+2}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$

$2x + 2 = A(x-1) + B(x+3)$

When $x = -3$,

$-4 = -4A$

$A = 1$

When $x = 1$,

$4 = 4B$

$B = 1$

Hence

$\frac{2x+2}{(x+3)(x-1)} = \frac{1}{x+3} + \frac{1}{x-1}$

$\int_2^3 \frac{2x+2}{(x+3)(x-1)} dx$

$= \int_2^3 \left(\frac{1}{x+3} + \frac{1}{x-1}\right) dx$

$= [\ln|x+3| + \ln|x-1|]_2^3$

$= (\ln 6 + \ln 2) - (\ln 5 + \ln 1)$

$= \ln\left(\frac{6 \times 2}{5 \times 1}\right)$

$= \ln\frac{12}{5}$

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1d

$$\int_0^2 \frac{x-1}{x+1}\,dx$$

$$= \int_0^2 \frac{x+1-2}{x+1}\,dx$$

$$= \int_0^2 \left(1-\frac{2}{x+1}\right)dx$$

$$= [x-2\ln|x+1|]_0^2$$

$$= (2-2\ln 3)-(0-0)$$

$$= 2-2\ln 3$$

1e

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$$

Let $u = \sin x$

$du = \cos x\,dx$

When $x = \frac{\pi}{2}, u = 1$

When $x = \frac{\pi}{4}, u = \frac{\sqrt{2}}{2}$

Hence

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{3}{u^4}\,du$$

$$= \left[-\frac{3}{3u^3}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \left[-\frac{1}{u^3}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

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$$= -1 + \frac{8}{2\sqrt{2}}$$

$$= -1 + \frac{4}{\sqrt{2}}$$

$$= -1 + \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= -1 + 2\sqrt{2}$$

$$= 2\sqrt{2} - 1$$

1f

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4 - 9x^2}} dx$$

$$= \frac{1}{3} \int_0^{\frac{1}{3}} \frac{3}{\sqrt{2^2 - (3x)^2}} dx$$

$$= \frac{1}{3} \left[\sin^{-1} \left(\frac{3x}{2} \right) \right]_0^{\frac{1}{3}}$$

$$= \frac{1}{3} \left(\frac{\pi}{6} - 0 \right)$$

$$= \frac{\pi}{18}$$

2a

$$\int \frac{x}{\sqrt{1 + x^2}} dx$$

Let $u = 1 + x^2$

$du = 2x\, dx$

Hence

$$\int \frac{x}{\sqrt{1 + x^2}} dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}} du$$

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$$= \frac{1}{2} \times 2\sqrt{u} + C$$

$$= \sqrt{u} + C$$

$$= \sqrt{1 + x^2} + C$$

2b

$$\int \frac{1 + x}{1 + x^2} dx$$

$$= \int \left(\frac{1}{1 + x^2} + \frac{x}{1 + x^2} \right) dx$$

$$= \int \frac{1}{1 + x^2} dx + \int \frac{x}{1 + x^2} dx$$

$$= \int \frac{1}{1 + x^2} dx + \frac{1}{2} \int \frac{2x}{1 + x^2} dx$$

$$= \tan^{-1} x + \frac{1}{2} \ln(1 + x^2) + C \qquad \text{(since } 1 + x^2 \text{ is positive)}$$

2c

$$\int \sin x \cos^4 x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int \sin x \cos^4 x \, dx$$

$$= -\int u^4 \, du$$

$$= -\frac{1}{5} u^5 + C$$

$$= -\frac{1}{5} \cos^5 x + C$$

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2d

$$\frac{1}{2x^2+3x+1}=\frac{1}{(x+1)(2x+1)}$$

Let $\frac{1}{(x+1)(2x+1)}=\frac{A}{x+1}+\frac{B}{2x+1}$

$1=A(2x+1)+B(x+1)$

When $x=-1$,

$1=-A$

$A=-1$

When $x=-\frac{1}{2}$

$1=\frac{1}{2}B$

$B=2$

Hence

$$\int\frac{1}{(x+1)(2x+1)}dx$$

$$=\int\left(\frac{-1}{x+1}+\frac{2}{2x+1}\right)dx$$

$$=-\ln|x+1|+\ln|2x+1|+C$$

$$=\ln\left|\frac{2x+1}{x+1}\right|+C$$

2e

$$\int x^3\ln x\,dx$$

$$\int uv'\,dx=uv-\int u'v\,dx$$

Therefore

$u=\ln x\,,v'=x^3$

$u'=\frac{1}{x},v=\frac{1}{4}x^4$

Hence

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$$\int x^3 \ln x \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

2f

$$\int \sin^3 2x \, dx$$

$$= \int \sin^2 2x \sin 2x \, dx$$

$$= \int (1 - \cos^2 2x) \sin 2x \, dx$$

Let $u = \cos 2x$

$du = -2 \sin 2x \, dx$

Hence

$$\int (1 - \cos^2 2x) \sin 2x \, dx$$

$$= -\frac{1}{2}\int (1 - u^2) \, du$$

$$= -\frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C$$

$$= -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + C$$

$$= \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$$

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2g

$$\int \frac{1}{x^2 + 6x + 25}\,dx$$

$$= \int \frac{1}{x^2 + 6x + 9 + 16}\,dx$$

$$= \int \frac{1}{(x+3)^2 + 4^2}\,dx$$

$$= \tan^{-1}\left(\frac{x+3}{4}\right) + C$$

2h

$$\int 3x \cos 3x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 3x, v' = \cos 3x$

$u' = 3, v = \frac{1}{3}\sin 3x$

Hence

$$\int 3x \cos 3x\,dx$$

$$= x \sin 3x - \int \sin 3x\,dx$$

$$= x \sin 3x + \frac{1}{3}\cos 3x + C$$

2i

$$\int \frac{x}{\sqrt{4+x}}\,dx$$

$$= \int \frac{4 + x - 4}{\sqrt{4+x}}\,dx$$

$$= \int \left(\frac{4+x}{\sqrt{4+x}} - \frac{4}{\sqrt{4+x}}\right)dx$$

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$$= \int \left(\sqrt{4+x} - \frac{4}{\sqrt{4+x}}\right) dx$$

Let $u = 4 + x$

$du = dx$

Hence

$$\int \left(\sqrt{4+x} - \frac{4}{\sqrt{4+x}}\right) dx$$

$$= \int \left(\sqrt{u} - \frac{4}{\sqrt{u}}\right) du$$

$$= \int \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right) du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}\left(\sqrt{4+x}\right)^3 - 8\sqrt{4+x} + C$$

$$= \sqrt{4+x}\left(\frac{2}{3}(4+x) - 8\right) + C$$

$$= \sqrt{4+x}\left(\frac{2}{3}(4+x) - \frac{2}{3}(12)\right) + C$$

$$= \frac{2}{3}(x-8)\sqrt{4+x} + C$$

3a

$$\int_0^1 x^2 e^{-x}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = x^2, v' = e^{-x}$

$u' = 2x, v = -e^{-x}$

Hence

$$\int_0^1 x^2 e^{-x}\, dx$$

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$$= [-x^2 e^{-x}]_0^1 + \int_0^1 2xe^{-x}\,dx$$

$$= -\frac{1}{e} + \int_0^1 2xe^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 2x, v' = e^{-x}$

$u' = 2, v = -e^{-x}$

Hence

$$-\frac{1}{e} + \int_0^1 2xe^{-x}\,dx$$

$$= -\frac{1}{e} - [2xe^{-x}]_0^1 + \int_0^1 2e^{-x}\,dx$$

$$= -\frac{1}{e} - \frac{2}{e} + [-2e^{-x}]_0^1$$

$$= -\frac{3}{e} - \frac{2}{e} + 2$$

$$= 2 - \frac{5}{e}$$

3b

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos^5 x \sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\cos^5 x \sin x\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

When $x = \frac{\pi}{2}, u = 0$

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When $x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^5 x \sin x \, dx$$

$$= -\int_1^0 (1 - u^2)u^5 \, du$$

$$= \int_0^1 (u^5 - u^7) \, du$$

$$= \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{1}{24}$$

3c

Let $\frac{x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$x = A(x^2 + 1) + (Bx + C)(x + 1)$

When $x = -1$,

$-1 = 2A$

$A = -\frac{1}{2}$

When $x = 0$,

$0 = A + C$

$C = -A$

$C = \frac{1}{2}$

When $x = 1$,

$1 = 2A + 2B + 2C$

$1 = -1 + 2B + 1$

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$B = \frac{1}{2}$

Hence

$$\int_0^1 \frac{x}{(x+1)(x^2+1)}dx$$

$$= \int_0^1 \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1}\right)dx$$

$$= \frac{1}{2}\int_0^1 \left(-\frac{1}{x+1} + \frac{x+1}{x^2+1}\right)dx$$

$$= \frac{1}{2}\int_0^1 \left(-\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{2}\left[-\ln|x+1| + \frac{1}{2}\ln(x^2+1) + \tan^{-1}x\right]_0^1 \qquad \text{(since } x^2+1 \text{ is positive)}$$

$$= \frac{1}{2}\left(-\ln 2 + \frac{1}{2}\ln 2 + \frac{\pi}{4}\right) - \frac{1}{2}(0+0+0)$$

$$= \frac{1}{8}(-4\ln 2 + 2\ln 2 + \pi)$$

$$= \frac{1}{8}(\pi - 2\ln 2)$$

3d

$$\int_0^{\frac{1}{2}} (1-x^2)^{-\frac{3}{2}}dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{\left(\sqrt{1-x^2}\right)^3}dx$$

Let $x = \sin u$

$dx = \cos u\, du$

When $x = \frac{1}{2}, u = \frac{\pi}{6}$

When $x = 0, u = 0$

Therefore

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$$\int_0^{\frac{1}{2}} \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\left(\sqrt{1-\sin^2 u}\right)^3}\,du$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\sqrt{\cos^6 u}}\,du \quad \text{(noting that in this interval } \cos u \text{ is the positive square root)}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\cos^3 u}\,du$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 u}\,du$$

$$= \int_0^{\frac{\pi}{6}} \sec^2 u\,du$$

$$= [\tan u]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{\sqrt{3}}$$

3e

$$\int_0^1 \frac{1-x^2}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x^2}{1+x^2}\right)dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{1+x^2-1}{1+x^2}\right)dx$$

$$= \left(\int_0^1 \frac{1}{1+x^2} - \frac{1+x^2}{1+x^2} + \frac{1}{1+x^2}\right)dx$$

$$= \int_0^1 \left(\frac{2}{1+x^2} - 1\right)dx$$

$$= [2\tan^{-1} x - x]_0^1$$

$$= \frac{\pi}{2} - 1$$

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3f

$$\int_2^4 \frac{x}{\sqrt{6x-8-x^2}}dx$$

$$= \int_2^4 \frac{x}{\sqrt{1-(x-3)^2}}dx$$

Let $u = x - 3$

$x = u + 3$

$du = dx$

When $x = 4, u = 1$

When $x = 2, u = -1$

Hence

$$\int_2^4 \frac{x}{\sqrt{1-(x-3)^2}}dx$$

$$= \int_{-1}^1 \frac{u+3}{\sqrt{1-u^2}}du$$

$$= \int_{-1}^1 \left(\frac{u}{\sqrt{1-u^2}} + \frac{3}{\sqrt{1-u^2}}\right)du$$

$$= \left[-\frac{1}{2}\sqrt{1-u^2} + 3\sin^{-1}u\right]_{-1}^1$$

$$= \left(0 + 3 \times \frac{\pi}{2}\right) - \left(0 + 3 \times -\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{2} + \frac{3\pi}{2}$$

$$= 3\pi$$

3g

$$\int_0^1 \frac{\sqrt{x}}{1+x}dx$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

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When $x = 1, u = 1$

When $x = 0, u = 0$

Hence

$$\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$$

$$= \int_0^1 \frac{u}{1+u^2}\,2u\,du$$

$$= 2\int_0^1 \frac{u^2}{1+u^2}\,du$$

$$= 2\int_0^1 \frac{1+u^2-1}{1+u^2}\,du$$

$$= 2\int_0^1 \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right)du$$

$$= 2\int_0^1 \left(1 - \frac{1}{1+u^2}\right)du$$

$$= 2[u - \tan^{-1} u]_0^1$$

$$= 2\left(1 - \frac{\pi}{4}\right) - 2(0-0)$$

$$= \frac{1}{2}(4-\pi)$$

3h

$$\int_0^{\sqrt{3}} \tan^{-1} x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \tan^{-1} x\,, v' = 1$

$$u' = \frac{1}{1+x^2}, v = x$$

Hence

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$$\int_0^{\sqrt{3}} \tan^{-1} x \, dx$$

$$= [x \tan^{-1} x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} dx$$

$$= \left(\frac{\sqrt{3}\pi}{3} - 0\right) - \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2} dx$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}[\ln|1+x^2|]_0^{\sqrt{3}}$$

$$= \frac{\pi}{\sqrt{3}} - \left(\frac{1}{2}\ln 4 - 0\right)$$

$$= \frac{\pi}{\sqrt{3}} - \ln 2$$

3i

$$\int_0^{\frac{\pi}{4}} \sin 2x \cos 3x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin(2x-3x) + \sin(2x+3x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (-\sin x + \sin 5x) \, dx$$

$$= \frac{1}{2}\left[\cos x - \frac{1}{5}\cos 5x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\left(\frac{\sqrt{2}}{2} - \frac{1}{5}\left(-\frac{\sqrt{2}}{2}\right)\right) - \left(1 - \frac{1}{5}\right)\right)$$

$$= \frac{1}{2}\left(\frac{3\sqrt{2}}{5} - \frac{4}{5}\right)$$

$$= \frac{1}{10}(3\sqrt{2} - 4)$$

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3j

$$\int_0^{\pi} e^{-x}\cos x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \cos x\,, v' = e^{-x}$

$u' = -\sin x\,, v = -e^{-x}$

Hence

$$\int_0^{\pi} e^{-x}\cos x\,dx$$

$$= [-e^{-x}\cos x]_0^{\pi} - \int_0^{\pi} e^{-x}\sin x\,dx$$

Consider $\int_0^{\pi} e^{-x}\sin x\,dx$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \sin x\,, v' = e^{-x}$

$u' = \cos x\,, v = -e^{-x}$

Hence

$$\int_0^{\pi} e^{-x}\sin x\,dx$$

$$= [-e^{-x}\sin x]_0^{\pi} + \int_0^{\pi} e^{-x}\cos x\,dx$$

Therefore

$$\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} - \int_0^{\pi} e^{-x}\sin x\,dx$$

$$\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} - [-e^{-x}\sin x]_0^{\pi} - \int_0^{\pi} e^{-x}\cos x\,dx$$

$$2\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} + [e^{-x}\sin x]_0^{\pi}$$

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$$2\int_0^{\pi} e^{-x}\cos x\,dx = (e^{-\pi}+1)+(0+0)$$

$$2\int_0^{\pi} e^{-x}\cos x\,dx = e^{-\pi}+1$$

$$\int_0^{\pi} e^{-x}\cos x\,dx = \frac{1}{2}(1+e^{-\pi})$$

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Solutions to Exercise 4I Development questions

4a

$$\frac{x-1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$x-1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$x-1 = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$x-1 = (A+B)x^2 + (B+C-A)x + A + C$$

Equating coefficients gives:

$$A+B=0$$

$$B+C-A=1$$

$$A+C=-1$$

$$B=-A$$

$$C=-1-A$$

$$(-A)+(-1-A)-A=1$$

$$-3A=2$$

$$\therefore A=-\frac{2}{3}$$

$$\therefore B=\frac{2}{3}$$

$$\therefore C=-\frac{1}{3}$$

4b

$$\int_0^1 \frac{x^3+x}{x^3+1}\,dx$$

$$= \int_0^1 \frac{x^3+1+x-1}{x^3+1}\,dx$$

$$= \int_0^1 \left(1+\frac{x-1}{x^3+1}\right)dx$$

Uncorrected proofs

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$$= \int_0^1 \left(1 - \frac{\frac{2}{3}}{x+1} + \frac{\frac{2}{3}x - \frac{1}{3}}{x^2 - x + 1}\right) dx$$

$$\int_0^1 \left(1 - \frac{2}{3} \times \frac{1}{x+1} + \frac{1}{3} \times \frac{2x-1}{x^2 - x + 1}\right) dx$$

$$= \left[x - \frac{2}{3}\ln|x+1| + \frac{1}{3}\ln(x^2 - x + 1)\right]_0^1$$

$$= 1 - \frac{2}{3}\ln 2$$

5

$$\int x^3 e^{-x^2}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$$u = x^2, v' = xe^{-x^2}$$

$$u' = 2x, v = -\frac{1}{2}e^{-x^2}$$

Hence

$$\int x^3 e^{-x^2}\, dx$$

$$= -\frac{1}{2}x^2 e^{-x^2} + \int xe^{-x^2}\, dx$$

$$= -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2}$$

$$= -\frac{1}{2}e^{-x^2}(1 + x^2) + C$$

6a

$$\int_0^{\frac{\pi}{3}} \sec^4 x\, dx$$

$$\int_0^{\frac{\pi}{3}} \sec^2 x \sec^2 x\, dx$$

Uncorrected proofs

$$\int_0^{\frac{\pi}{3}} (1 + \tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{3}} (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int_0^{\sqrt{3}} (1 + u^2) \, du$$

$$= \left[u + \frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \sqrt{3} + \sqrt{3}$$

$$= 2\sqrt{3}$$

6b

$$\int_0^{\frac{\pi}{3}} \sec^6 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x \sec^2 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x \, (1 + \tan^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x \, dx + \int_0^{\frac{\pi}{3}} \sec^4 x \tan^2 x \, dx$$

$$= 2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x \sec^2 x \tan^2 x \, dx$$

$$= 2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x \, (1 + \tan^2 x) \tan^2 x \, dx$$

Uncorrected proofs

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Let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 0$

Hence

$$2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x\,(1 + \tan^2 x)\tan^2 x\, dx$$

$$= 2\sqrt{3} + \int_0^{\sqrt{3}} (1 + u^2)u^2\, du$$

$$= 2\sqrt{3} + \int_0^{\sqrt{3}} (u^2 + u^4)\, du$$

$$= 2\sqrt{3} + \left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^{\sqrt{3}}$$

$$= 2\sqrt{3} + \sqrt{3} + \frac{9}{5}\sqrt{3}$$

$$= \frac{10}{5}\sqrt{3} + \frac{5}{5}\sqrt{3} + \frac{9}{5}\sqrt{3}$$

$$= \frac{24\sqrt{3}}{5}$$

7a

$$\int_0^{\frac{\pi}{2}} \frac{1}{3 + 5\cos x} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1 + t^2} dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

Uncorrected proofs

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$$

$$= \int_0^1 \frac{1}{3+5\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \int_0^1 \left(\frac{\frac{1}{4}}{2-t} + \frac{\frac{1}{4}}{2+t}\right)dt \quad \text{(using cover – up rule)}$$

$$= \frac{1}{4}\int_0^1 \left(\frac{1}{2-t} + \frac{1}{2+t}\right)dt$$

$$= \frac{1}{4}\left[-\ln|2-t| + \ln|2+t|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 + \ln 2 - \ln 2)$$

$$= \frac{1}{4}\ln 3$$

7b

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

Uncorrected proofs

$$\cos x = \frac{1 - t^2}{1 + t^2}$$

$$\sin x = \frac{2t}{1 + t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx$$

$$= \int_0^1 \frac{1}{\frac{1 - t^2}{1 + t^2} - 2 \times \frac{2t}{1 + t^2} + 3} \times \frac{2}{1 + t^2}\,dt$$

$$= \int_0^1 \frac{2}{1 - t^2 - 4t + 3 + 3t^2}\,dt$$

$$= \int_0^1 \frac{2}{2t^2 - 4t + 4}\,dt$$

$$= \int_0^1 \frac{1}{t^2 - 2t + 2}\,dt$$

$$= \int_0^1 \frac{1}{(t - 1)^2 + 1}\,dt$$

$$= [\tan^{-1}(t - 1)]_0^1$$

$$= \tan^{-1} 0 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4}$$

8a

$$\frac{4t}{(1 + t)^2(1 + t^2)} = \frac{A}{1 + t} + \frac{B}{(1 + t)^2} + \frac{Ct + D}{1 + t^2}$$

$$4t = A(1 + t)(1 + t^2) + B(1 + t^2) + (Ct + D)(1 + t)^2$$

$$4t = A(t^3 + t^2 + t + 1) + B + Bt^2 + (Ct + D)(1 + 2t + t^2)$$

$$4t = At^3 + At^2 + At + A + B + Bt^2 + Ct + 2Ct^2 + Ct^3 + D + 2Dt + Dt^2$$

$$4t = (A + C)t^3 + (A + B + 2C + D)t^2 + (A + C + 2D)t + A + B + D$$

Equating coefficients gives:

$$A + C = 0$$

Uncorrected proofs

$A + B + 2C + D = 0$

$A + C + 2D = 4$

$A + B + D = 0$

$(A + B + D) + 2C = 0$

$2C = 0$

$\therefore C = 0$

$\therefore A = 0$

$\therefore D = 2$

$\therefore B = -2$

8b

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \sin x} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1 + t^2} dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

$$\sin x = \frac{2t}{1 + t^2}$$

Hence

$$= \int_0^1 \frac{\sin x}{1 + \sin x} dx$$

$$= \int_0^1 \frac{\frac{2t}{1 + t^2}}{1 + \frac{2t}{1 + t^2}} \times \frac{2}{1 + t^2} dt$$

$$= \int_0^1 \frac{\frac{4t}{1 + t^2}}{1 + t^2 + 2t} dt$$

Uncorrected proofs

$$= \int_0^1 \frac{\frac{4t}{1+t^2}}{(1+t)^2}\,dt$$

$$= \int_0^1 \frac{4t}{(1+t)^2(1+t^2)}\,dt$$

$$= \int_0^1 \left(\frac{-2}{(1+t)^2} + \frac{2}{1+t^2}\right)dt$$

$$= \left[\frac{2}{1+t} + 2\tan^{-1}(t)\right]_0^1$$

$$= \frac{2}{1+1} + 2 \times \frac{\pi}{4} - 2$$

$$= 1 + \frac{\pi}{2} - 2$$

$$= \frac{\pi}{2} - 1$$

9

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$$

Let $u = \sqrt[6]{x}$

$x = u^6$

$dx = 6u^5\,du$

$x = 64, u = 2$

$x = 1, u = 1$

Hence

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\,dx$$

$$= \int_1^2 \frac{1}{\sqrt{u^6} + \sqrt[3]{u^6}} \times 6u^5\,du$$

$$= \int_1^2 \frac{1}{u^3 + u^2} \times 6u^5\,du$$

$$= 6\int_1^2 \frac{u^3}{u+1}\,du$$

Uncorrected proofs

$$= 6\int_1^2 \frac{u^3 + 1 - 1}{u + 1}\, du$$

$$= 6\int_1^2 \frac{(u + 1)(u^2 - u + 1) - 1}{u + 1}\, du$$

$$= 6\int_1^2 \left(u^2 - u + 1 - \frac{1}{u + 1}\right) du$$

$$= 6\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u + 1|\right]_1^2$$

$$= 6\left(\frac{8}{3} - 2 + 2 - \ln 3 - \frac{1}{3} + \frac{1}{2} - 1 + \ln 2\right)$$

$$= 6\left(\frac{7}{3} - \frac{1}{2} + \ln\frac{2}{3}\right)$$

$$= 14 - 3 + 6\ln\frac{2}{3}$$

$$= 11 - 6\ln\frac{3}{2}$$

10a

$$\int \sqrt{a^2 - x^2}\, dx$$

Let $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

Hence

$$\int \sqrt{a^2 - x^2}\, dx$$

$$= \int \sqrt{a^2 - (a\sin\theta)^2}\, a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1 - \sin^2\theta)}\, a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2\cos^2\theta}\, a\cos\theta\, d\theta$$

Uncorrected proofs

$$= \int a\cos\theta\, a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}\times\frac{\sqrt{a^2-x^2}}{a}\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{a^2}\right) + C$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

10b

$$\int \sqrt{a^2-x^2}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$$u = \sqrt{a^2-x^2}, v' = 1$$

$$u' = \frac{-x}{\sqrt{a^2-x^2}}, v = x$$

Hence

$$\int \sqrt{a^2-x^2}\, dx$$

Uncorrected proofs

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\,dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\,dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\,dx + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$2\int \sqrt{a^2 - x^2}\,dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

11a

$$\int_0^1 \frac{5 - 5x^2}{(1 + 2x)(1 + x^2)}\,dx$$

$$\frac{5 - 5x^2}{(1 + 2x)(1 + x^2)} = \frac{A}{1 + 2x} + \frac{Bx + C}{1 + x^2}$$

$$5 - 5x^2 = A(1 + x^2) + (Bx + C)(1 + 2x)$$

$$5 - 5x^2 = A + Ax^2 + Bx + C + 2Bx^2 + 2Cx$$

$$5 - 5x^2 = (A + 2B)x^2 + (B + 2C)x + A + C$$

Equating coefficients gives:

$$A + 2B = -5$$

$$B + 2C = 0$$

$$A + C = 5$$

$$B = -2C$$

$$A = 5 - C$$

$$\therefore 5 - C + 2(-2C) = -5$$

$$5 - C - 4C = -5$$

$$-5C = -10$$

Uncorrected proofs

$\therefore C = 2$

$\therefore B = -4$

$\therefore A = 3$

Hence

$$\int_0^1 \frac{5 - 5x^2}{(1 + 2x)(1 + x^2)} dx$$

$$= \int_0^1 \left(\frac{3}{1 + 2x} + \frac{-4x + 2}{1 + x^2}\right) dx$$

$$= \int_0^1 \left(\frac{3}{2} \times \frac{2}{1 + 2x} - 2 \times \frac{2x}{1 + x^2} + \frac{2}{1 + x^2}\right) dx$$

$$= \left[\frac{3}{2} \ln|1 + 2x| - 2 \ln|1 + x^2| + 2 \tan^{-1} x\right]_0^1$$

$$= \frac{3}{2} \ln 3 - 2 \ln 2 + \frac{\pi}{2}$$

$$= \ln \sqrt{27} - \ln 4 + \frac{\pi}{2}$$

$$= \ln \sqrt{27} - \ln \sqrt{16} + \frac{\pi}{2}$$

$$= \frac{1}{2} \ln \frac{27}{16} + \frac{\pi}{2}$$

$$= \frac{1}{2}\left(\pi + \ln \frac{27}{16}\right)$$

11b

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \cos x + 2 \sin x} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1 + t^2} dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

Uncorrected proofs

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x+2\sin x}\,dx$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+2\times\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2\times\frac{1-t^2}{1+t^2}}{1+t^2+1-t^2+4t}\,dt$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+2t}\,dt$$

$$= \int_0^1 \frac{1-t^2}{(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{5}\int_0^1 \frac{5-5t^2}{(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{5}\left(\frac{1}{2}\left(\pi+\ln\frac{27}{16}\right)\right)$$

$$= \frac{1}{10}\left(\pi+\ln\frac{27}{16}\right)$$

12a

$$8\sin x+\cos x-2 = P(3\sin x+2\cos x-1)+Q(3\cos x-2\sin x)$$

$$8\sin x+\cos x-2 = 3P\sin x+2P\cos x-P+3Q\cos x-2Q\sin x$$

Equating coefficients gives:

$$P = 2$$

$$3P-2Q = 8$$

$$2P+3Q = 1$$

$$-2Q = 2$$

Uncorrected proofs

$\therefore Q = -1$

12b

$$\int \frac{8\sin x + \cos x - 2}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int \frac{2(3\sin x + 2\cos x - 1) - (3\cos x - 2\sin x)}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int \left(2 - \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\right)dx$$

Let $u = 3\sin x + 2\cos x - 1$

$du = 3\cos x - 2\sin x\,dx$

Hence

$$\int \left(2 - \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\right)dx$$

$$= \int 2\,dx - \int \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int 2\,dx - \int \frac{1}{u}\,du$$

$$= 2x - \ln|u| + C$$

$$= 2x - \ln|3\sin x + 2\cos x - 1| + C$$

13a

$$T_n = \int_0^{\pi} \sin^n x\,dx$$

$$T_n = \int_0^{\pi} \sin^{n-1} x \sin x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$u = \sin^{n-1} x\,, v' = \sin x$

$u' = (n-1)\sin^{n-2} x\cos x\,, v = -\cos x$

Uncorrected proofs

Hence

$$T_n = [-\sin^{n-1} x \cos x]_0^{\pi} + \int_0^{\pi} (n-1)\sin^{n-2} x \cos x \cos x \, dx$$

$$T_n = (n-1)\int_0^{\pi} \sin^{n-2} x \cos^2 x \, dx$$

$$T_n = (n-1)\int_0^{\pi} \sin^{n-2} x \, (1 - \sin^2 x) \, dx$$

$$T_n = (n-1)\int_0^{\pi} (\sin^{n-2} x - \sin^n x) \, dx$$

$$T_n = (n-1)(T_{n-2} - T_n)$$

$$T_n = (n-1)T_{n-2} - (n-1)T_n$$

$$T_n + (n-1)T_n = (n-1)T_{n-2}$$

$$nT_n = (n-1)T_{n-2}$$

$$T_n = \frac{n-1}{n} T_{n-2}$$

13b

$$T_0 = \int_0^{\pi} \sin^0 x \, dx$$

$$= \int_0^{\pi} 1 \, dx$$

$$= \pi$$

$$T_1 = \int_0^{\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi}$$

$$= 2$$

$$T_2 = \frac{2-1}{2} T_0$$

$$T_2 = \frac{1}{2} \times \pi = \frac{\pi}{2}$$

$$T_4 = \frac{4-1}{4} T_2$$

Uncorrected proofs

$$T_4 = \frac{3}{4} \times \frac{\pi}{2} = \frac{3\pi}{8}$$

$$T_6 = \frac{6-1}{6} T_4$$

$$T_6 = \frac{5}{6} \times \frac{3\pi}{8} = \frac{5\pi}{16}$$

$$T_3 = \frac{3-1}{3} T_1$$

$$T_3 = \frac{2}{3} \times 2 = \frac{4}{3}$$

$$T_5 = \frac{5-1}{5} T_3$$

$$T_5 = \frac{4}{5} \times \frac{4}{3} = \frac{16}{15}$$

Therefore

$$T_5 T_6 = \frac{5\pi}{16} \times \frac{16}{15}$$

$$= \frac{\pi}{3}$$

14a

$$I_n = \int_1^e (\ln x)^n \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = (\ln x)^n, v' = 1$$

$$u' = \frac{n(\ln x)^{n-1}}{x}, v = x$$

Hence

$$I_n = [x(\ln x)^n]_1^e - \int_1^e n(\ln x)^{n-1} \, dx$$

$$I_n = e - n \int_1^e (\ln x)^{n-1} \, dx$$

$$I_n = e - nI_{n-1}$$

Uncorrected proofs

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14b

$$I_0 = \int_1^e (\ln x)^0 \, dx$$

$$= \int_1^e 1 \, dx$$

$$= e - 1$$

$$I_1 = e - (1)I_0$$

$$= 1$$

$$I_2 = e - (2)I_1$$

$$= e - 2$$

$$I_3 = e - (3)I_2$$

$$= e - 3e + 6$$

$$= 6 - 2e$$

Uncorrected proofs

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Ex 6 2

Ex 4I

Q 15

$$I_n = \int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx \quad ; \quad n = 1, 2, 3, \ldots$$

a) $$I_1 = \int_0^1 \frac{1}{x+1}\,dx$$

$$= \left[\log(x+1)\right]_0^1$$

$$= \log 2 - \cancel{\log 1} = 0$$

$$= \log 2.$$

b) $$I_{n+1} = \int_0^1 \frac{x^n}{(x+1)^{n+1}}\,dx$$

$$= \left[\frac{x^n (x+1)^{-n}}{-n}\right]_0^1 + \int_0^1 \frac{n x^{n-1} (x+1)^{-n}}{n}\,dx$$ [by parts with $u = x^n$, $v' = (x+1)^{-(n+1)}$]

$$= -\frac{1}{n 2^n} - 0 + \int_0^1 \frac{x^{n-1}}{(x+1)^n}\,dx$$

$$= I_n - \frac{1}{n 2^n}$$

c) $$I_{n+1} = \int_0^1 \frac{x^n}{(x+1)^{n+1}}\,dx$$

$$= \int_0^1 \frac{x}{x+1} \cdot \frac{x^{n-1}}{(x+1)^n}\,dx$$

$$< \int_0^1 \frac{1}{2} \cdot \frac{x^{n-1}}{(x+1)^n}\,dx$$

$$< \frac{1}{2} I_n$$

d) From b) & c) $$I_n - \frac{1}{n 2^n} < \frac{1}{2} I_n$$

$$\frac{1}{2} I_n < \frac{1}{n 2^n}$$

$$\text{so } I_n < \frac{1}{n 2^{n-1}}$$

Ext 2

Ex 4I

Q15 (cont'd)

e) $I_3 = I_2 - \frac{1}{8}$ by (b)

$= I_1 - \frac{1}{2} - \frac{1}{8}$

$= I_1 - \frac{5}{8} < \frac{1}{12}$ by (d)

so $\log 2 < \frac{5}{8} + \frac{1}{12}$

ie $\log 2 < \frac{17}{24}$

also $I_4 = I_3 - \frac{1}{24}$ by (b)

$= \log 2 - \frac{5}{8} - \frac{1}{24}$ from above

$= \log 2 - \frac{2}{3}$

but $I_n > 0$ for all n so

$\log 2 > \frac{2}{3}$.

hence $\frac{2}{3} < \log 2 < \frac{17}{24}$.

Ext 2

Ex 4 I

Q16 $\displaystyle\int_0^{\pi} \frac{\cos x + 2\sin x}{5 + 3\cos x}\, dx$

$$= \frac{1}{3}\int_0^{\pi} \frac{3\cos x}{5 + 3\cos x}\, dx + \frac{2}{3}\int_0^{\pi} \frac{3\sin x}{5 + 3\cos x}\, dx$$

$$= \frac{1}{3}\int_0^{\pi} \frac{5 + 3\cos x}{5 + 3\cos x}\, dx + \frac{5}{3}\int_0^{\pi} \frac{1}{5 + 3\cos x}\, dx - \frac{2}{3}\int_0^{\pi} \frac{-3\sin x}{5 + 3\cos x}\, dx$$

$$= \frac{1}{3}\left[x\right]_0^{\pi} - \frac{5}{3}\cdot\frac{\pi}{4} - \frac{2}{3}\left[\log(5 + 3\cos x)\right]_0^{\pi}$$

$$= \frac{\pi}{3} - \frac{5\pi}{12} - \frac{2}{3}(\log 2 - \log 8)$$

$$= -\frac{\pi}{12} + \frac{2}{3}(3\log 2 - \log 2)$$

$$= -\frac{\pi}{12} + \frac{4}{3}\log 2$$

$$= \frac{1}{12}(16\log 2 - \pi)$$

Ext 2

Ex 4I

Q17 a) $u = t - t^{-1}$

$u^2 = t^2 + t^{-2} - 2$

$du = (1 + t^{-2})\,dt$

so $\int \frac{1+t^2}{1+t^4}\,dt = \int \frac{1+t^{-2}}{t^2+t^{-2}}\,dt$

$= \int \frac{du}{u^2+2}$

$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t-t^{-1}}{\sqrt{2}}\right) + C$

$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}(t^2-1)}{2t}\right) + C$

b) Let $\frac{1+t^2}{1+t^4} = \frac{At+B}{1+\sqrt{2}t+t^2} + \frac{Ct+D}{1-\sqrt{2}t+t^2}$

But this function is even so

$\frac{At+B}{1+\sqrt{2}t+t^2} + \frac{Ct+D}{1-\sqrt{2}t+t^2} = \frac{-At+B}{1-\sqrt{2}t+t^2} + \frac{-Ct+D}{1+\sqrt{2}t+t^2}$

thus $A = -C$ and $B = D$.

Also

$1+t^2 = (At+B)(1-\sqrt{2}t+t^2) + (Ct+D)(1+\sqrt{2}t+t^2)$

<u>at t=0</u> $1 = B + D$

$= 2B$

so $B = D = \frac{1}{2}$

<u>at t=1</u> $2 = (A+B)(2-\sqrt{2}) + (C+D)(2+\sqrt{2})$

$= A(2-\sqrt{2}) + \frac{1}{2}(2-\sqrt{2}) - A(2+\sqrt{2}) + \frac{1}{2}(2+\sqrt{2})$ (since $A=-C$)

$= -2A\sqrt{2} + 2$

Hence $A = -C = 0$.

Ex 12
Ex 4I
Q17
(ctd)

Thus $$\int \frac{1+t^2}{1+t^4}\,dt = \frac{1}{2}\int\left(\frac{1}{1+\sqrt{2}t+t^2} + \frac{1}{1-\sqrt{2}t+t^2}\right)dt$$

$$= \frac{1}{2}\int\left(\frac{1}{\frac{1}{2}+\left(t+\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{\frac{1}{2}+\left(t-\frac{1}{\sqrt{2}}\right)^2}\right)dt$$

$$= \frac{1}{2}\left[\sqrt{2}\tan^{-1}\left(\sqrt{2}\left(t+\frac{1}{\sqrt{2}}\right)\right) + \sqrt{2}\tan^{-1}\left(\sqrt{2}\left(t-\frac{1}{\sqrt{2}}\right)\right)\right] + C$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}(t\sqrt{2}+1) + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t-1) + C$$

c) If $pq \leq 0$ then $0 \in [p, q]$

and the substitution $u = t - t^{-1}$

is undefined when $t = 0$.

Ext 2

Ex 4 I

Q 18

$$I = \int_0^{\log 2} \frac{1}{5\cosh x - 3\sinh x}\, dx$$

$$= \int_0^{\log 2} \frac{2e^x}{5(e^{2x}+1) - 3(e^{2x}-1)}\, dx$$

$$= \int_0^{\log 2} \frac{2e^x}{2e^{2x} + 8}\, dx$$

$$= \int_0^{\log 2} \frac{e^x}{(e^x)^2 + 2^2}\, dx$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{e^x}{2}\right)\right]_0^{\log 2}$$

$$= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1}\frac{1}{2}\right)$$

$$= \frac{1}{2}(\alpha - \beta)$$

now $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$$= \frac{1 - \frac{1}{2}}{1 + 1 \times \frac{1}{2}}$$

$$= \frac{2-1}{2+1}$$

$$= \frac{1}{3}$$

so $\alpha - \beta = \tan^{-1}\frac{1}{3}$

hence $I = \frac{1}{2}\tan^{-1}\frac{1}{3}$.

Ext 2

Ex 4I

Q19 a) $I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

put $x = \tan\theta$ with $0 \le \theta \le \frac{\pi}{4}$

then $dx = \sec^2\theta \cdot d\theta = (1+\tan^2\theta)\,d\theta$

or $\frac{dx}{1+x^2} = d\theta$

so $I = \int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$

b) Put $u = \frac{\pi}{4} - \theta$

$du = -d\theta$

when $\theta = 0$, $u = \frac{\pi}{4}$

$\theta = \frac{\pi}{4}$, $u = 0$.

so $I = \int_{\pi/4}^{0} \log\left(1+\tan\left(\frac{\pi}{4}-u\right)\right)(-du)$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan u}{1+\tan u}\right)du$

$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan u+1-\tan u}{1+\tan u}\right)du$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan u}\right)du$

$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$ (since θ is a dummy variable)

c) $I = \int_0^{\pi/4} \left(\log 2 - \log(1+\tan\theta)\right)d\theta$

$= \int_0^{\pi/4} \log 2\,d\theta - I$

$2I = \frac{\pi}{4}\log 2$

so $I = \frac{\pi}{8}\log 2$.

Ext 2
Ex 4I
Q19 (ctd) further $I = \int_0^1 \log(1+x) \cdot \frac{1}{1+x^2}\,dx$

$$= \left[\log(1+x)\tan^{-1}x\right]_0^1 - \int_0^1 \frac{\tan^{-1}x}{1+x}\,dx$$

thus $\frac{\pi}{8}\log 2 = \frac{\pi}{4}\log 2 - \int_0^1 \frac{\tan^{-1}x}{1+x}\,dx$

or $\int_0^1 \frac{\tan^{-1}x}{1+x}\,dx = \frac{\pi}{8}\log 2$

so $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx = \int_0^1 \frac{\tan^{-1}x}{1+x}\,dx \qquad (!)$

Here is a more obscure, but elegant & equivalent solution.

$$I = \int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$$

put $x = \frac{1-u}{1+u}$ [essentially the $\tan(\frac{\pi}{4}-\theta)$ step above]

at $u=0 \quad x=1$
$u=1 \quad x=0$

and $\frac{dx}{du} = \frac{(1+u)(-1)-(1-u)}{(1+u)^2}$

so $dx = \frac{-2\,du}{(1+u)^2}$

thus $I = \int_1^0 \frac{\log\left(1+\frac{1-u}{1+u}\right)}{1+\left(\frac{1-u}{1+u}\right)^2} \cdot \frac{(-2)\,du}{(1+u)^2}$

$$= 2\int_0^1 \frac{\log\left(\frac{2}{1+u}\right)\,du}{(1+u)^2+(1-u)^2}$$

$$= 2\int_0^1 \frac{\log 2 - \log(1+u)}{2(1+u^2)}\,du$$

$$= \int_0^1 \frac{\log 2}{1+u^2}\,du - \int_0^1 \frac{\log(1+u)}{1+u^2}\,du$$

$$= \left[\log 2 \cdot \tan^{-1}u\right]_0^1 - I$$

$2I = \frac{\pi}{4}\log 2$ so $I = \frac{\pi}{8}\log 2$.