Solutions to Exercise 4H Foundation questions

1a

$$I_n = \int \tan^n x \, dx$$

$$= \int \tan^{n-2} x \tan^2 x \, dx$$

$$= \int \tan^{n-2} x \, (\sec^2 x - 1) dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - I_{n-2}$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

Hence

$$I_n = \int u^{n-2} du - I_{n-2}$$

$$= \frac{u^{n-1}}{n-1} - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

1b

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_0 = \int \tan^0 x \, dx$$

$$= \int 1 \, dx$$

$$= x$$

$$I_2 = \frac{\tan^{2-1} x}{2-1} - I_0$$

$$= \tan x - x$$

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$$I_4 = \frac{\tan^{4-1} x}{4-1} - I_2$$

$$= \frac{1}{3}\tan^3 x - \tan x + x$$

$$I_6 = \frac{\tan^{6-1} x}{6-1} - I_4$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$$

2a

$$I_n = \int x^n e^x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^n, v' = e^x$

$u' = nx^{n-1}, v = e^x$

Hence

$$I_n = x^n e^x - n\int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

2b $I_n = x^n e^x - nI_{n-1}$

$$I_0 = \int x^0 e^x \, dx$$

$$= \int e^x \, dx$$

$$= e^x$$

$$I_1 = xe^x - I_0$$

$$= xe^x - e^x$$

$$I_2 = x^2 e^x - 2I_1$$

$$= x^2 e^x - 2xe^x + 2e^x$$

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$$I_3 = x^3e^x - 3I_2$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$
$$= (x^3 - 3x^2 + 6x - 6)e^x + C$$

3a

$$I_n = \int_1^e x(\ln x)^n \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = (\ln x)^n, v' = x$$

$$u' = \frac{n(\ln x)^{n-1}}{x}, v = \frac{1}{2}x^2$$

Hence

$$I_n = \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{1}{2}n\int_1^e x(\ln x)^{n-1} \, dx$$
$$= \left(\frac{1}{2}e^2 - 0\right) - \frac{1}{2}nI_{n-1}$$
$$= \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$$

3b

$$I_n = \int_1^e x(\ln x)^n \, dx$$

$$I_0 = \int_1^e x(\ln x)^0 \, dx$$
$$= \int_1^e x \, dx$$
$$= \left[\frac{1}{2}x^2\right]_1^e$$
$$= \frac{1}{2}e^2 - \frac{1}{2}$$

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$$I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$$

$$I_1 = \frac{1}{2}e^2 - \frac{1}{2}I_0$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$I_2 = \frac{1}{2}e^2 - I_1$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{1}{4}$$

$$= \frac{1}{4}e^2 - \frac{1}{4}$$

$$I_3 = \frac{1}{2}e^2 - \frac{3}{2}I_2$$

$$= \frac{1}{2}e^2 - \frac{3}{2}\left(\frac{1}{4}e^2 - \frac{1}{4}\right)$$

$$= \frac{4}{8}e^2 - \frac{3}{8}e^2 + \frac{3}{8}$$

$$= \frac{1}{8}e^2 + \frac{3}{8}$$

$$I_4 = \frac{1}{2}e^2 - 2I_3$$

$$= \frac{1}{2}e^2 - 2\left(\frac{1}{8}e^2 + \frac{3}{8}\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{3}{4}$$

$$= \frac{1}{4}e^2 - \frac{3}{4}$$

$$= \frac{1}{4}(e^2 - 3)$$

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4a

$$u_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \cos^{n-1} x \, , v' = \cos x$

$u' = -(n-1)\cos^{n-2} x \sin x \, , v = \sin x$

Hence

$$u_n = [\sin x \cos^{n-1} x]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (0-0) + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

4b

$$u_n = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, (1 - \cos^2 x) \, dx$$

$$= (n-1)\left(\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \cos^n x \, dx\right)$$

$$= (n-1)(u_{n-2} - u_n)$$

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4c

$$u_n = (n-1)(u_{n-2} - u_n)$$

$$= (n-1)u_{n-2} - (n-1)u_n$$

$$u_n + (n-1)u_n = (n-1)u_{n-2}$$

$$nu_n = (n-1)u_{n-2}$$

$$u_n = \frac{n-1}{n}u_{n-2}$$

$$u_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= 1$$

$$u_3 = \frac{3-1}{3}u_0$$

$$= \frac{2}{3}$$

$$u_5 = \frac{5-1}{5}u_3$$

$$= \frac{8}{15}$$

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Solutions to Exercise 4H Development questions

5a

$$T_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$$

$$T_n = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \times \sec^2 x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = \sec^{n-2} x \, , v' = \sec^2 x$$

$$u' = (n-2)\sec^{n-3} x \sec x \tan x \, , v = \tan x$$

Hence

$$T_n = [\sec^{n-2} x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)\sec^{n-3} x \sec x \tan x \times \tan x \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x \tan^2 x \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} (\sec^n x - \sec^{n-2} x) \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)(T_n - T_{n-2})$$

$$T_n = \sqrt{2}^{n-2} - \big((n-2)T_n - (n-2)T_{n-2}\big)$$

$$T_n = \sqrt{2}^{n-2} - (n-2)T_n + (n-2)T_{n-2}$$

$$T_n + (n-2)T_n = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n(1 + n - 2) = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n(n-1) = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n = \frac{\sqrt{2}^{n-2}}{n-1} + \frac{n-2}{n-1} T_{n-2}$$

Uncorrected proofs

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5b

$$T_n = \frac{\sqrt{2}^{n-2}}{n-1} + \frac{n-2}{n-1}T_{n-2}$$

$$T_0 = \int_0^{\frac{\pi}{4}} 1\,dx$$

$$T_0 = [x]_0^{\frac{\pi}{4}}$$

$$T_0 = \frac{\pi}{4}$$

$$T_2 = \frac{\sqrt{2}^{2-2}}{2-1} + \frac{2-2}{2-1}T_0$$

$$T_2 = 1 + 0 = 1$$

$$T_4 = \frac{\sqrt{2}^{4-2}}{4-1} + \frac{4-2}{4-1}T_2$$

$$T_4 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$T_6 = \frac{\sqrt{2}^{6-2}}{6-1} + \frac{6-2}{6-1}T_4$$

$$T_6 = \frac{4}{5} + \frac{4}{5} \times \frac{4}{3} = \frac{28}{15}$$

6a

$$C_n = \int_0^{\frac{\pi}{2}} x^n \cos x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = x^n, v' = \cos x$$

$$u' = nx^{n-1}, v = \sin x$$

Hence

$$C_n = [x^n \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x\,dx$$

Uncorrected proofs

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$$C_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$$

Consider $\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx$

$u = x^{n-1}, v' = \sin x$

$u' = (n-1)x^{n-2}, v = -\cos x$

Hence

$$\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx = [-x^{n-1}\cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)x^{n-2}\cos x\,dx$$

$$\int_0^{\frac{\pi}{2}} x^{n-1}\sin x\,dx = \int_0^{\frac{\pi}{2}} (n-1)x^{n-2}\cos x\,dx$$

Therefore

$$C_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} (n-1)x^{n-2}\cos x\,dx$$

$$C_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\cos x\,dx$$

$$C_n = \left(\frac{\pi}{2}\right)^n - n(n-1)C_{n-2}$$

6b

$$C_0 = \int_0^{\frac{\pi}{2}} \cos x\,dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= 1$$

$$C_2 = \left(\frac{\pi}{2}\right)^2 - 2(2-1)C_0$$

$$C_2 = \left(\frac{\pi}{2}\right)^2 - 2$$

$$C_4 = \left(\frac{\pi}{2}\right)^4 - 4(4-1)C_2$$

Uncorrected proofs

$$C_4 = \left(\frac{\pi}{2}\right)^4 - 12\left(\left(\frac{\pi}{2}\right)^2 - 2\right)$$

$$C_4 = \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$$

$$C_6 = \left(\frac{\pi}{2}\right)^6 - 6(6-1)C_4$$

$$C_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right)$$

$$C_6 = \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$$

7a

$$I_n = \int \frac{x^n}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}x^2}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}(1+x^2-1)}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}(1+x^2)}{1+x^2}\,dx - \int \frac{x^{n-2}}{1+x^2}\,dx$$

$$I_n = \int x^{n-2}\,dx - I_{n-2}$$

$$I_n = \frac{x^{n-1}}{n-1} - I_{n-2}$$

7b

$$I_1 = \int \frac{x}{1+x^2}\,dx$$

$$I_1 = \frac{1}{2}\ln(1+x^2)$$

$$I_3 = \frac{x^2}{3-1} - I_1$$

$$I_3 = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)$$

Uncorrected proofs

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$$I_5 = \frac{x^4}{5-1} - I_3$$

$$I_5 = \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(1+x^2) + C$$

8a

$$I_n = \int_0^1 (1-x^2)^n \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = (1-x^2)^n, v' = 1$$

$$u' = -2nx(1-x^2)^{n-1}, v = x$$

Hence

$$I_n = \int_0^1 (1-x^2)^n \, dx$$

$$I_n = [x(1-x^2)^n]_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1} \, dx$$

$$I_n = 2n\int_0^1 \big(1-(1-x^2)\big)(1-x^2)^{n-1} \, dx$$

$$I_n = 2n\int_0^1 \big(1-(1-x^2)\big)(1-x^2)^{n-1} \, dx$$

$$I_n = 2n\int_0^1 ((1-x^2)^{n-1} - (1-x^2)^n) \, dx$$

$$I_n = 2n(I_{n-1} - I_n)$$

$$I_n = 2nI_{n-1} - 2nI_n$$

$$I_n(1+2n) = 2nI_{n-1}$$

$$I_n = \frac{2nI_{n-1}}{1+2n}$$

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

Uncorrected proofs

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8b

$$I_0 = \int_0^1 (1-x^2)^0\,dx$$

$$= \int_0^1 1\,dx$$

$$= 1$$

$$I_1 = \frac{2(1)}{2(1)+1} I_0$$

$$I_1 = \frac{2}{3} \times 1 = \frac{2}{3}$$

$$I_2 = \frac{2(2)}{2(2)+1} I_1$$

$$I_2 = \frac{4}{5} \times \frac{2}{3} = \frac{8}{15}$$

$$I_3 = \frac{2(3)}{2(3)+1} I_2$$

$$I_3 = \frac{6}{7} \times \frac{8}{15} = \frac{16}{35}$$

$$I_4 = \frac{2(4)}{2(4)+1} I_3$$

$$I_4 = \frac{8}{9} \times \frac{16}{35} = \frac{128}{315}$$

9a

$$u_n = \int_0^1 x(1-x^3)^n\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = (1-x^3)^n, v' = x$$

$$u' = -3nx^2(1-x^3)^{n-1}, v = \frac{1}{2}x^2$$

Hence

$$u_n = \int_0^1 x(1-x^3)^n\,dx$$

Uncorrected proofs

$$u_n = \left[\frac{1}{2}x^2(1-x^3)^n,\right]_0^1 + \frac{3}{2}n\int_0^1 x^3(1-x^3)^{n-1}\,dx$$

$$u_n = \frac{3}{2}n\int_0^1 \big(1-(1-x^3)\big)(1-x^3)^{n-1}\,dx$$

$$u_n = \frac{3}{2}n\int_0^1 ((1-x^3)^{n-1} - (1-x^3)^n)\,dx$$

$$u_n = \frac{3}{2}n(u_{n-1} - u_n)$$

$$u_n = \frac{3}{2}nu_{n-1} - \frac{3}{2}nu_n$$

$$u_n\left(1+\frac{3}{2}n\right) = \frac{3}{2}nu_{n-1}$$

$$u_n = \frac{\frac{3}{2}n}{1+\frac{3}{2}n}u_{n-1} \times \frac{2}{2}$$

$$u_n = \frac{3n}{3n+2}u_{n-1}$$

9b

$$u_0 = \int_0^1 x(1-x^3)^0\,dx$$

$$= \int_0^1 x\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{2}$$

$$u_1 = \frac{3(1)}{3(1)+2}u_0$$

$$u_1 = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

$$u_2 = \frac{3(2)}{3(2)+2}u_1$$

Uncorrected proofs

$$u_2 = \frac{6}{8} \times \frac{3}{10} = \frac{9}{40}$$

$$u_3 = \frac{3(3)}{3(3)+2} u_2$$

$$u_3 = \frac{9}{11} \times \frac{9}{40} = \frac{81}{440}$$

$$u_4 = \frac{3(4)}{3(4)+2} u_3$$

$$u_4 = \frac{12}{14} \times \frac{81}{440} = \frac{243}{1540}$$

10a

$$J_n = \int \frac{x^n}{\sqrt{1-x^2}} dx$$

$$J_n = \int \frac{x^{n-1}x}{\sqrt{1-x^2}} dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = x^{n-1}, v' = \frac{x}{\sqrt{1-x^2}}$$

$$u' = (n-1)x^{n-2}, v = -\sqrt{1-x^2}$$

Hence

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}\sqrt{1-x^2}\, dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}\sqrt{1-x^2} \times \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int \frac{x^{n-2}(1-x^2)}{\sqrt{1-x^2}} dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int \left(\frac{x^{n-2}}{\sqrt{1-x^2}} - \frac{x^n}{\sqrt{1-x^2}}\right) dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)(J_{n-2} - J_n)$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2} - (n-1)J_n$$

$$J_n + (n-1)J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2}$$

Uncorrected proofs

$$nJ_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2}$$

$$J_n = \frac{1}{n}\left((n-1)J_{n-2} - x^{n-1}\sqrt{1-x^2}\right)$$

10b

$$J_2 = \int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$J_0 = \int \frac{x^0}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x$$

$$J_2 = \frac{1}{2}\left((2-1)J_0 - x^{2-1}\sqrt{1-x^2}\right)$$

$$J_2 = \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + C$$

11a

$$u_n = \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx$$

$$u_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \cos^2 x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = \sin^{n-1} x\,,\, v' = \sin x \cos^2 x$$

$$u' = (n-1)\sin^{n-2} x \cos x\,,\, v = -\frac{1}{3}\cos^3 x$$

Hence

$$u_n = \left[-\frac{1}{3}\sin^{n-1} x \cos^3 x\right]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x \cos^3 x\,dx$$

$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^4 x\,dx$$

Uncorrected proofs

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$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\cos^2 x\,dx$$

$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} (\sin^{n-2} x\cos^2 x - \sin^n x\cos^2 x)\,dx$$

$$u_n = \frac{n-1}{3}(u_{n-2} - u_n)$$

$$u_n = \frac{n-1}{3}u_{n-2} - \frac{n-1}{3}u_n$$

$$u_n + \frac{n-1}{3}u_n = \frac{n-1}{3}u_{n-2}$$

$$u_n\left(1 + \frac{n-1}{3}\right) = \frac{n-1}{3}u_{n-2}$$

$$u_n\left(\frac{n+2}{3}\right) = \frac{n-1}{3}u_{n-2}$$

$$u_n = \frac{n-1}{n+2}u_{n-2}$$

11b

$$u_0 = \int_0^{\frac{\pi}{2}} \sin^0 x\cos^2 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2x)\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$u_2 = \frac{2-1}{2+2}u_0$$

$$u_2 = \frac{1}{4}\times\frac{\pi}{4} = \frac{\pi}{16}$$

Uncorrected proofs

$$u_4 = \frac{4-1}{4+2}u_2$$

$$u_4 = \frac{3}{6} \times \frac{\pi}{16} = \frac{\pi}{32}$$

12a

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$$

$$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$

$$= \int_0^1 (1+x)^{-\frac{1}{2}}\,dx$$

$$= \left[2\sqrt{1+x}\right]_0^1$$

$$= 2\sqrt{2} - 2$$

12b

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$$

$$I_n = \int_0^1 \frac{x^{n-1}x}{\sqrt{1+x}}\,dx$$

$$I_n = \int_0^1 \frac{x^{n-1}(1+x-1)}{\sqrt{1+x}}\,dx$$

$$I_n = \int_0^1 \frac{x^{n-1}(1+x) - x^{n-1}}{\sqrt{1+x}}\,dx$$

$$I_n = \int_0^1 \left(x^{n-1}\sqrt{1+x} - \frac{x^{n-1}}{\sqrt{1+x}}\right)dx$$

$$I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx - I_{n-1}$$

$$I_{n-1} + I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

Uncorrected proofs

12c

$$I_{n-1} + I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

Consider $\int_0^1 x^{n-1}\sqrt{1+x}\,dx$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = \sqrt{1+x}, v' = x^{n-1}$$

$$u' = \frac{1}{2\sqrt{1+x}}, v = \frac{1}{n}x^n$$

Hence

$$\int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

$$= \left[\frac{1}{n}x^n\sqrt{1+x}\right]_0^1 - \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1+x}}dx$$

$$= \frac{1}{n}\sqrt{2} - \frac{1}{2n}I_n$$

Hence

$$I_{n-1} + I_n = \frac{1}{n}\sqrt{2} - \frac{1}{2n}I_n$$

$$I_n + \frac{1}{2n}I_n = \frac{1}{n}\sqrt{2} - I_{n-1}$$

$$I_n\left(1 + \frac{1}{2n}\right) = \frac{1}{n}\sqrt{2} - I_{n-1}$$

$$I_n = \frac{\frac{1}{n}\sqrt{2} - I_{n-1}}{1 + \frac{1}{2n}}$$

$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$$

Uncorrected proofs

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12d

$$I_1 = \frac{2\sqrt{2} - 2(1)I_0}{2(1) + 1}$$

$$= \frac{2\sqrt{2} - 2\left(2\sqrt{2} - 2\right)}{3}$$

$$= \frac{2\sqrt{2} - 4\sqrt{2} + 4}{3}$$

$$= \frac{4 - 2\sqrt{2}}{3}$$

$$I_2 = \frac{2\sqrt{2} - 2(2)I_1}{2(2) + 1}$$

$$= \frac{2\sqrt{2} - 4\left(\frac{4 - 2\sqrt{2}}{3}\right)}{5}$$

$$= \frac{2\sqrt{2} - \frac{16 - 8\sqrt{2}}{3}}{5}$$

$$= \frac{6\sqrt{2} - 16 + 8\sqrt{2}}{15}$$

$$= \frac{14\sqrt{2} - 16}{15}$$

13a

$$(1 + t^2)^{n-1} + t^2(1 + t^2)^{n-1}$$

$$= (1 + t^2)(1 + t^2)^{n-1}$$

$$= (1 + t^2)^n$$

13b

$$P_n = \int_0^x (1 + t^2)^n \, dt$$

$$P_n = \int_0^x ((1 + t^2)^{n-1} + t^2(1 + t^2)^{n-1}) \, dt$$

Uncorrected proofs

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$$P_n = \int_0^x (1+t^2)^{n-1}\,dt + \int_0^x t^2(1+t^2)^{n-1}\,dt$$

$$P_n = P_{n-1} + \int_0^x t^2(1+t^2)^{n-1}\,dt$$

Consider $\int_0^x t^2(1+t^2)^{n-1}\,dt$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = t, v' = t(1+t^2)^{n-1}$$

$$u' = 1, v = \frac{(1+t^2)^n}{2n}$$

Hence

$$\int_0^x t^2(1+t^2)^{n-1}\,dt$$

$$= \left[\frac{t(1+t^2)^n}{2n}\right]_0^x - \frac{1}{2n}\int_0^x (1+t^2)^n\,dt$$

Therefore

$$P_n = P_{n-1} + \left[\frac{t(1+t^2)^n}{2n}\right]_0^x - \frac{1}{2n}\int_0^x (1+t^2)^n\,dt$$

$$P_n = P_{n-1} + \frac{x(1+x^2)^n}{2n} - \frac{1}{2n}P_n$$

$$P_n + \frac{1}{2n}P_n = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n\left(1+\frac{1}{2n}\right) = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n\left(\frac{2n+1}{2n}\right) = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n = \frac{2n}{2n+1}\left(P_{n-1} + \frac{x(1+x^2)^n}{2n}\right)$$

$$P_n = \frac{1}{2n+1}((1+x^2)^n x + 2nP_{n-1})$$

Uncorrected proofs

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13c i

$$P_0 = \int_0^x (1+t^2)^0\,dt$$

$$= \int_0^x 1\,dt$$

$$= x$$

$$P_1 = \frac{1}{2(1)+1}((1+x^2)^1 x + 2(1)P_0)$$

$$= \frac{1}{3}\left((1+x^2)x + 2x\right)$$

$$P_2 = \frac{1}{2(2)+1}\left((1+x^2)^2 x + 2(2)\frac{1}{3}\left((1+x^2)x + 2x\right)\right)$$

$$= \frac{1}{5}\left((1+x^2)^2 x + \frac{4}{3}\left((1+x^2)x + 2x\right)\right)$$

$$= \frac{1}{5}\left((1+x^2)^2 x + \frac{4}{3}(1+x^2)x + \frac{8}{3}x\right)$$

$$= \frac{1}{5}\left((1+x^2)^2 x + \frac{4}{3}(1+x^2)x + \frac{8}{3}x\right)$$

$$P_3 = \frac{1}{2(3)+1}((1+x^2)^3 x + 2(3)P_2)$$

$$= \frac{1}{7}\left((1+x^2)^3 x + \frac{6}{5}\left((1+x^2)^2 x + \frac{4}{3}(1+x^2)x + \frac{8}{3}x\right)\right)$$

$$= \frac{1}{7}\left((1+x^2)^3 x + \frac{6}{5}(1+x^2)^2 x + \frac{24}{15}(1+x^2)x + \frac{16}{5}x\right)$$

$$= \frac{1}{7}(1+x^2)^3 x + \frac{6}{35}(1+x^2)^2 x + \frac{24}{105}(1+x^2)x + \frac{16}{35}x$$

$$P_4 = \frac{1}{2(4)+1}((1+x^2)^4 x + 2(4)P_3)$$

$$= \frac{1}{9}\left((1+x^2)^4 x + 8\left(\frac{1}{7}(1+x^2)^3 x + \frac{6}{35}(1+x^2)^2 x + \frac{24}{105}(1+x^2)x + \frac{16}{35}x\right)\right)$$

$$= \frac{1}{9}\left((1+x^2)^4 x + \frac{8}{7}(1+x^2)^3 x + \frac{48}{35}(1+x^2)^2 x + \frac{64}{35}(1+x^2)x + \frac{128}{35}x\right)$$

Uncorrected proofs

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13c ii

$$P_4 = \int_0^x (1+t^2)^4\,dt$$

$$= \int_0^x (1+4t^2+6t^4+4t^6+t^8)\,dt$$

$$= \left[t+\frac{4}{3}t^3+\frac{6}{5}t^5+\frac{4}{7}t^7+\frac{1}{9}t^9\right]_0^x$$

$$= x+\frac{4}{3}x^3+\frac{6}{5}x^5+\frac{4}{7}x^7+\frac{1}{9}x^9$$

13d

$$P_4 = \frac{1}{9}\left((1+x^2)^4x+\frac{8}{7}(1+x^2)^3x+\frac{48}{35}(1+x^2)^2x+\frac{64}{35}(1+x^2)x+\frac{128}{35}x\right)$$

$$= x+\frac{4}{3}x^3+\frac{6}{5}x^5+\frac{4}{7}x^7+\frac{1}{9}x^9$$

$$x+\frac{4}{3}x^3+\frac{6}{5}x^5+\frac{4}{7}x^7+\frac{1}{9}x^9$$

$$= \frac{1}{9}\left((1+x^2)^4x+\frac{8}{7}(1+x^2)^3x+\frac{48}{35}(1+x^2)^2x+\frac{64}{35}(1+x^2)x+\frac{128}{35}x\right)$$

$$x\left(1+\frac{4}{3}x^2+\frac{6}{5}x^4+\frac{4}{7}x^6+\frac{1}{9}x^8\right)$$

$$= \frac{1}{9}\left((1+x^2)^4x+\frac{8}{7}(1+x^2)^3x+\frac{48}{35}(1+x^2)^2x+\frac{64}{35}(1+x^2)x+\frac{128}{35}x\right)$$

Therefore

$$1+\frac{4}{3}x^2+\frac{6}{5}x^4+\frac{4}{7}x^6+\frac{1}{9}x^8$$

$$= \frac{1}{9}\left((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2+\frac{64}{35}(1+x^2)+\frac{128}{35}\right)$$

Uncorrected proofs

14a

$$T_n = \int_0^1 x^n \sqrt{1-x}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$$u = x^n, v' = \sqrt{1-x}$$

$$u' = nx^{n-1}, v = -\frac{2}{3}(1-x)\sqrt{1-x}$$

Hence

$$T_n = \left[-\frac{2}{3}x^n(1-x)\sqrt{1-x}\right]_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\, dx$$

$$T_n = \left[-\frac{2}{3}x^n(1-x)\sqrt{1-x}\right]_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}\sqrt{1-x} - x^n\sqrt{1-x}\, dx$$

$$T_n = \frac{2n}{3}(T_{n-1} - T_n)$$

$$T_n = \frac{2n}{3}T_{n-1} - \frac{2n}{3}T_n$$

$$T_n + \frac{2n}{3}T_n = \frac{2n}{3}T_{n-1}$$

$$T_n\left(1 + \frac{2n}{3}\right) = \frac{2n}{3}T_{n-1}$$

$$T_n\left(\frac{2n+3}{3}\right) = \frac{2n}{3}T_{n-1}$$

$$T_n = \frac{2n}{2n+3}T_{n-1}$$

14b

$$T_0 = \int_0^1 x^0\sqrt{1-x}\, dx$$

$$= \int_0^1 \sqrt{1-x}\, dx$$

$$= \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1$$

Uncorrected proofs

$$= \frac{2}{3}$$

$$T_1 = \frac{2(1)}{2(1)+3} T_0$$

$$T_1 = \frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$$

$$T_2 = \frac{2(2)}{2(2)+3} T_1$$

$$T_2 = \frac{4}{7} \times \frac{4}{15} = \frac{16}{105}$$

$$T_3 = \frac{2(3)}{2(3)+3} T_2$$

$$T_3 = \frac{6}{9} \times \frac{16}{105} = \frac{32}{315}$$

14c

$$T_n = \frac{n!\,(n+1)!}{(2n+3)!} 4^{n+1}$$

$$T_{n-1} = \frac{(n-1)!\,n!}{(2(n-1)+3)!} 4^n = \frac{(n-1)!\,n!}{(2n+1)!} 4^n$$

$$T_n = \frac{2n}{2n+3} T_{n-1}$$

$$\frac{n!\,(n+1)!}{(2n+3)!} 4^{n+1} = \frac{2n}{2n+3} T_{n-1}$$

$$\text{LHS} = \frac{n!\,(n+1)!}{(2n+3)!} 4^{n+1}$$

$$\text{RHS} = \frac{2n}{2n+3} T_{n-1}$$

$$= \frac{2n}{2n+3} \frac{(n-1)!\,n!}{(2n+1)!} 4^n$$

$$= \frac{2}{2n+3} \frac{n!\,n!}{(2n+1)!} 4^n \times \frac{2n+2}{2n+2}$$

$$= \frac{4(n+1)}{(2n+2)(2n+3)} \frac{n!\,n!}{(2n+1)!} 4^n$$

Uncorrected proofs

$$= \frac{(n+1)!\,n!}{(2n+3)!}4^{n+1}$$

$$= \text{LHS}$$

Thus, the formula has been proven by induction via the reduction formula.

Uncorrected proofs

Ex 2

Ex 4H

Q15 a) $I_n = \int_0^1 (1-x^2)^n \cdot 1 \, dx$

$= \left[(1-x^2)^n \cdot x\right]_0^1 - \int_0^1 (-2x)\, n (1-x^2)^{n-1} \cdot x \, dx$

(by parts with $u = (1-x^2)^n$, $v' = 1$)

$= 0 - 0 + 2n \int_0^1 x^2 (1-x^2)^{n-1} \, dx$

$= 2n \, J_{n-1}$

b) $J_{n-1} = \int_0^1 x^2 (1-x^2)^{n-1} \, dx$

$= -\int_0^1 (1-x^2)(1-x^2)^{n-1} \, dx + \int_0^1 (1-x^2)^{n-1} \, dx$

$= -I_n + I_{n-1}$

So $I_n = -2n \, I_n + 2n \, I_{n-1}$

$(2n+1) I_n = 2n \, I_{n-1}$

thus $I_n = \frac{2n}{2n+1} I_{n-1}$

c) From b) $J_{n-1} = I_{n-1} - I_n$

so $J_n = I_n - I_{n+1}$

$= I_n - \frac{2(n+1)}{2(n+1)+1} I_n$ by (b)

$= \frac{2n+3-2(n+1)}{2n+3} I_n$

$= \frac{1}{2n+3} I_n$

d) $J_n = \frac{1}{2n+3} I_n$ from (c)

$= \frac{1}{2n+3} \cdot 2n \, J_{n-1}$ from (a)

so $J_n = \frac{2n}{2n+3} J_{n-1}$

Ext 2

Ex 4H

Q16 a) $I_1 = \int_0^{\pi/4} \tan\theta \, d\theta$

$= \left[-\log(\cos\theta)\right]_0^{\pi/4}$ (see Box 9)

$= -\log\left(\frac{1}{\sqrt{2}}\right) + \cancel{\log 1}^{=0}$

$= \frac{1}{2}\log 2$

b) $I_n + I_{n-2} = \int_0^{\pi/4} (\tan^n\theta + \tan^{n-2}\theta)\, d\theta$

$= \int_0^{\pi/4} \tan^{n-2}\theta\,(\tan^2\theta + 1)\, d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \sec^2\theta \, d\theta$

$= \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\pi/4}$

$= \frac{1}{n-1} - 0 = \frac{1}{n-1}$.

c) $\tan^n\theta \leq \tan^{n-2}\theta$ for $0 \leq \theta \leq \frac{\pi}{4}$

with equality at the end points only,

hence $\int_0^{\pi/4} \tan^n\theta \, d\theta < \int_0^{\pi/4} \tan^{n-2}\theta \, d\theta$

i.e. $I_n < I_{n-2}$.

from b) $I_n + I_n < \frac{1}{n-1}$

so $I_n < \frac{1}{2(n-1)}$

and again from b)

$I_{n-2} + I_{n-2} > \frac{1}{n-1}$

so $I_{n-2} > \frac{1}{2(n-1)}$

or $I_n > \frac{1}{2(n+1)}$

combining these $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

Ex L 2

4H

Q16 d) from b) $I_5 = \frac{1}{4} - I_3$

$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$

$= I_1 - \frac{1}{4}$

$= \frac{1}{2}\log 2 - \frac{1}{4}$

and from c) $\frac{1}{12} < I_5 < \frac{1}{8}$

so $\frac{1}{12} < \frac{1}{2}\log 2 - \frac{1}{4} < \frac{1}{8}$

$\frac{1}{3} < \frac{1}{2}\log 2 < \frac{3}{8}$

so $\frac{2}{3} < \log 2 < \frac{3}{4}$ ∎

Ext 2
Ex 4H
Q17

$$I_n = \int_0^{\pi/3} \cos nx \sec x \, dx$$

a) $$I_n = \int_0^{\pi/3} \cos((n-2)x + 2x) \sec x \, dx$$

$$= \int_0^{\pi/3} \cos(n-2)x \cos 2x \sec x \, dx - \int_0^{\pi/3} \sin(n-2)x \sin 2x \sec x \, dx$$

but $\cos 2x = 2\cos^2 x - 1$ and $\sin 2x = 2\sin x \cos x$ so

$$I_n = 2\int_0^{\pi/3} \cos(n-2)x \cos x \, dx - \int_0^{\pi/3} \cos(n-2)x \sec x \, dx$$

$$- 2\int_0^{\pi/3} \sin(n-2)x \sin x \, dx$$

$$= 2\int_0^{\pi/3} \cos(n-1)x \, dx - I_{n-2}$$

$$= \frac{2}{n-1}\left[\sin(n-1)x\right]_0^{\pi/3} - I_{n-2}$$

$$= \frac{2}{n-1} \sin\frac{(n-1)\pi}{3} - I_{n-2}.$$

b) $$I_5 = \frac{2}{4}\sin\frac{4\pi}{3} - I_3$$

$$I_3 = \frac{2}{2}\sin\frac{2\pi}{3} - I_1$$

$$I_1 = \int_0^{\pi/3} \cos x \sec x \, dx = \frac{\pi}{3}.$$

so $$I_3 = +\frac{\sqrt{3}}{2} - \frac{\pi}{3}$$

& $$I_5 = \frac{1}{2}\cdot\left(\frac{-\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

$$= \frac{\pi}{3} - \frac{3\sqrt{3}}{4}$$