Solutions to Exercise 4G Foundation questions

1a

$$\int \cos x \, dx$$

$$= \sin x + C$$

1b

$$\int \sin x \, dx$$

$$= -\cos x + C$$

1c

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int \frac{\sin x}{\cos x} dx$$

$$= \int -\frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

1d

$$\int \cot x \, dx$$

$$= \int \frac{\cos x}{\sin x} dx$$

Let $u = \sin x$

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$du = \cos x \, dx$

Hence

$$\int \frac{\cos x}{\sin x} dx$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|\sin x| + C$$

2a

$$\int \cos x \sin^2 x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$$\int \cos x \sin^2 x \, dx$$

$$= \int u^2 \, du$$

$$= \frac{1}{3} u^3 + C$$

$$= \frac{1}{3} \sin^3 x + C$$

2b

$$\int \cos^2 x \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int \cos^2 x \sin x \, dx$$

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$$= -\int u^2 \, du$$

$$= -\frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}\cos^3 x + C$$

2c

$$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int (1 - \cos^2 x) \sin x \, dx$$

$$= \int (u^2 - 1) \, du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

2d

$$\int \cos^3 x \, dx$$

$$= \int \cos^2 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

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$du = \cos x \, dx$

Hence

$$\int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (1 - u^2) \, du$$

$$= u - \frac{1}{3}u^3 + C$$

$$= \sin x - \frac{1}{3}\sin^3 x + C$$

2e

$$\int \cos^5 x \, dx$$

$$= \int \cos^4 x \cos x \, dx$$

$$= \int (\cos^2 x)^2 \cos x \, dx$$

$$= \int (1 - \sin^2 x)^2 \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$$\int (1 - \sin^2 x)^2 \cos x \, dx$$

$$= \int (1 - u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$$

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2f

$$\int \sin^3 x \cos^3 x \, dx$$

$$= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$$\int \sin^3 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^3 (1 - u^2) \, du$$

$$= \int (u^3 - u^5) \, du$$

$$= \frac{1}{4} u^4 - \frac{1}{6} u^6 + C$$

$$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$$

3a

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right)$$

$$= \frac{\pi}{4}$$

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3b

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x \, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(1 + \cos 2x) \, dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + \cos 2x) \, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}$$

3c

$$\int_0^{\pi} \sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi} \sin^2 2x \, dx$$

$$= \frac{1}{4}\int_0^{\pi} \frac{1}{2}(1 - \cos 4x) \, dx$$

$$= \frac{1}{8}\int_0^{\pi} (1 - \cos 4x) \, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi}$$

$$= \frac{1}{8}\left((\pi - 0) - (0 - 0)\right)$$

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$$= \frac{\pi}{8}$$

4a

$$\int \sec^2 x \, dx$$

$$= \tan x + C$$

4b

$$\int \tan^2 x \, dx$$

$$= \int (\sec^2 x - 1) \, dx$$

$$= \tan x - x + C$$

4c

$$\int \sec^4 x \, dx$$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int (1 + \tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

Hence

$$\int (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int (1 + u^2) \, du$$

$$= u + \frac{1}{3}u^3 + C$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

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4d

$$\int \tan^4 x \, dx$$

$$= \int \tan^2 x \tan^2 x \, dx$$

$$= \int \tan^2 x (\sec^2 x - 1) \, dx$$

$$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx$$

$$= \int (\tan^2 x \sec^2 x - (\sec^2 x - 1)) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

Hence

$$\int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int u^2 \, du - \int (\sec^2 x - 1) \, dx$$

$$= \frac{1}{3} u^3 - (\tan x - x) + C$$

$$= \frac{1}{3} \tan^3 x - \tan x + x + C$$

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Solutions to Exercise 4G Development questions

5a

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \ dx$

$x = \frac{\pi}{2}, u = 0$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx$$

$$= -\int_1^0 u^3 \, du$$

$$= \int_0^1 u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_0^1$$

$$= \frac{1}{4}$$

5b

$$\int_0^{\frac{\pi}{6}} \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \cos x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \cos x \, (1 - \sin^2 x) \, dx$$

Let $u = \sin x$

$du = \cos x \ dx$

Uncorrected proofs

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$x = \frac{\pi}{6}, u = \frac{1}{2}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{6}} \cos x \,(1 - \sin^2 x)\, dx$$

$$= \int_0^{\frac{1}{2}} (1 - u^2)\, du$$

$$= \left[u - \frac{1}{3}u^3\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2} - \frac{1}{24}$$

$$= \frac{11}{24}$$

5c

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \; dx$

$x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{3}} \sin^3 x \cos x \, dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_0^{\frac{\sqrt{3}}{2}}$$

Uncorrected proofs

$$= \frac{1}{4}\left(\frac{9}{16}\right)$$

$$= \frac{9}{64}$$

5d

$$\int_0^{\frac{\pi}{3}} \sin^5 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \sin^4 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, (1 - \cos^2 x)^2 \, dx$$

Let $u = \cos x$

$du = -\sin x \ dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \sin x \, (1 - \cos^2 x)^2 \, dx$$

$$= -\int_1^{\frac{1}{2}} (1 - u^2)^2 \, du$$

$$= -\int_1^{\frac{1}{2}} (1 - 2u^2 + u^4) \, du$$

$$= \int_{\frac{1}{2}}^1 (1 - 2u^2 + u^4) \, du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{\frac{1}{2}}^1$$

$$= 1 - \frac{2}{3} + \frac{1}{5} - \frac{1}{2} + \frac{2}{3}\left(\frac{1}{8}\right) - \frac{1}{5}\left(\frac{1}{32}\right)$$

Uncorrected proofs

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$$= \frac{53}{480}$$

5e

$$\int_0^{\pi} \sin^3 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x \sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$$

Let $u = \cos x$

$du = -\sin x \; dx$

$x = \pi, u = -1$

$x = 0, u = 1$

Hence

$$\int_0^{\pi} \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$$

$$= -\int_1^{-1} (1 - u^2) u^2 \, du$$

$$= \int_{-1}^{1} (u^2 - u^4) \, du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_{-1}^{1}$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{4}{15}$$

Uncorrected proofs

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5f

$$\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x \sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x \sin^2 x (1 - \sin^2 x) \, dx$$

Let $u = \sin x$

$du = \cos x \ dx$

$x = \frac{\pi}{4}, u = \frac{1}{\sqrt{2}}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x (1 - \sin^2 x) \, dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} u^2(1 - u^2) \, du$$

$$= \int_0^{\frac{1}{\sqrt{2}}} (u^2 - u^4) \, du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5$$

$$= \frac{1}{6\sqrt{2}} - \frac{1}{20\sqrt{2}}$$

$$= \frac{10}{60\sqrt{2}} - \frac{3}{60\sqrt{2}}$$

$$= \frac{7}{60\sqrt{2}}$$

Uncorrected proofs

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6a

$$\int \cos^4 x \, dx$$

$$= \int \cos^2 x \cos^2 x \, dx$$

$$= \int \left(\frac{1}{2}(1 + \cos 2x)\right)\left(\frac{1}{2}(1 + \cos 2x)\right) dx$$

$$= \frac{1}{4}\int (1 + \cos 2x)(1 + \cos 2x) \, dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{32}(12x + 8\sin 2x + \sin 4x) + C$$

6b

$$\int \sin^4 x \, dx$$

$$= \int \sin^2 x \sin^2 x \, dx$$

$$= \int \left(\frac{1}{2}(1 - \cos 2x)\right)\left(\frac{1}{2}(1 - \cos 2x)\right) dx$$

$$= \frac{1}{4}\int (1 - \cos 2x)(1 - \cos 2x) \, dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

Uncorrected proofs

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$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C$$

6c

$$\int \sin^4 x \cos^4 x \, dx$$

$$= \int (\sin x \cos x)^4 \, dx$$

$$= \int \left(\frac{1}{2}\sin 2x\right)^4 dx$$

$$= \frac{1}{16}\int \sin^4 2x \, dx$$

$$= \frac{1}{16}\int \sin^2 2x \sin^2 2x \, dx$$

$$= \frac{1}{16}\int \left(\frac{1}{2}(1 - \cos 4x)\right)\left(\frac{1}{2}(1 - \cos 4x)\right) dx$$

$$= \frac{1}{64}\int (1 - \cos 4x)(1 - \cos 4x) \, dx$$

$$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x) \, dx$$

$$= \frac{1}{64}\int \left(1 - 2\cos 4x + \frac{1}{2}(1 + \cos 8x)\right) dx$$

$$= \frac{1}{64}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{64}\left(\frac{3}{2}x - \frac{1}{2}\sin 4x + \frac{1}{16}\sin 8x\right) + C$$

$$= \frac{1}{1024}(24x - 8\sin 4x + \sin 8x) + C$$

Uncorrected proofs

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7a

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \; dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x \, dx$$

$$= \int_0^{\sqrt{3}} u^2 \, du$$

$$= \left[\frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \sqrt{3}$$

7b

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \; dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = -\frac{\pi}{6}, u = -\frac{1}{\sqrt{3}}$

Hence

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x \, dx$$

Uncorrected proofs

$$= \int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{1}{4}\left(9 - \frac{1}{9}\right)$$

$$= \frac{1}{4}\left(\frac{80}{9}\right)$$

$$= \frac{20}{9}$$

$$= 2\frac{2}{9}$$

7c

$$\int_0^{\frac{\pi}{4}} \sec^4 x \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec^3 x \sec x \tan x \, dx$$

Let $u = \sec x$

$du = \sec x \tan x \ dx$

$x = \frac{\pi}{4}, u = \sqrt{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \sec^3 x \sec x \tan x \, dx$$

$$= \int_1^{\sqrt{2}} u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_1^{\sqrt{2}}$$

$$= \frac{1}{4}(4) - \frac{1}{4}$$

Uncorrected proofs

$= \frac{3}{4}$

7d

$$\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^3 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \tan^3 x - \tan^3 x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x \tan x \sec x \, (\sec^2 x - 1) - (\sec^2 x - 1) \tan x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \tan x \sec x \, (\sec^2 x - 1) \, dx - \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \tan x \sec x \, (\sec^2 x - 1) \, dx - \int_0^{\frac{\pi}{4}} \sec x \sec x \tan x \, dx + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

Let $u = \sec x$

$du = \sec x \tan x \ dx$

$x = \frac{\pi}{4}, u = \sqrt{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \sec x \tan x \sec x \, (\sec^2 x - 1) \, dx - \int_0^{\frac{\pi}{4}} \sec x \sec x \tan x \, dx + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_1^{\sqrt{2}} u(u^2 - 1) \, du - \int_1^{\sqrt{2}} u \, du + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_1^{\sqrt{2}} (u^3 - u) \, du - \int_1^{\sqrt{2}} u \, du + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_1^{\sqrt{2}} (u^3 - 2u) \, du + \int_0^{\frac{\pi}{4}} \tan x \, dx$$

Uncorrected proofs

$$= \left[\frac{1}{4}u^4 - u^2\right]_1^{\sqrt{2}} + [-\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}(4) - 2 - \frac{1}{4} + 1 - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{4} - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}$$

$$= \frac{1}{4}(2\ln 2 - 1)$$

8a

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

$$= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+t^2+2t}\,dt$$

$$= 2\int_0^1 \frac{1}{(t+1)^2}\,dt$$

Uncorrected proofs

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$$= 2\left[-\frac{1}{t+1}\right]_0^1$$

$$= 2\left(-\frac{1}{2}+1\right)$$

$$= 1$$

8b

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$$

$$= \int_0^1 \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{4(1+t^2)+5(1-t^2)}\,dt$$

$$= 2\int_0^1 \frac{1}{4+4t^2+5-5t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{9-t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{(3-t)(3+t)}\,dt$$

Uncorrected proofs

$$\frac{1}{(3-t)(3+t)} = \frac{\frac{1}{6}}{3-t} + \frac{\frac{1}{6}}{3+t} \quad \text{(using cover} - \text{up method)}$$

$$2\int_0^1 \frac{1}{(3-t)(3+t)}\,dt$$

$$= 2\int_0^1 \left(\frac{\frac{1}{6}}{3-t} + \frac{\frac{1}{6}}{3+t}\right)dt$$

$$= \frac{1}{3}\left[-\ln|3-t| + \ln|3+t|\right]_0^1$$

$$= \frac{1}{3}\left[\ln\left|\frac{3+t}{3-t}\right|\right]_0^1$$

$$= \frac{1}{3}\ln 2$$

8c

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = -\frac{\pi}{2}, t = -1$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$$

$$= \int_{-1}^{1} \frac{1}{5+3\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

Uncorrected proofs

$$= 2\int_{-1}^{1} \frac{1}{5(1+t^2)+6t}\,dt$$

$$= 2\int_{-1}^{1} \frac{1}{5+5t^2+6t}\,dt$$

$$= \frac{2}{5}\int_{-1}^{1} \frac{1}{t^2+\frac{6}{5}t+1}\,dt$$

$$= \frac{2}{5}\int_{-1}^{1} \frac{1}{\left(t+\frac{3}{5}\right)^2+\frac{16}{25}}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{\frac{4}{5}}{\left(t+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}\,dt$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{5\left(t+\frac{3}{5}\right)}{4}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{5t+3}{4}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\tan^{-1}(2)-\tan^{-1}\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\tan^{-1}(2)+\tan^{-1}\left(\frac{1}{2}\right)\right)$$

This can be further simplified by utilising the tangent addition formula to $\frac{\pi}{4}$, however the above expression is acceptable.

9a

$$\int_0^1 \sqrt{1-x^2}\,dx$$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$x = 1, \theta = \frac{\pi}{2}$

$x = 0, \theta = 0$

Hence

Uncorrected proofs

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

9b

$$\int_0^1 x^3\sqrt{1+x^2}\,dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$x = 1, \theta = \frac{\pi}{4}$

$x = 0, \theta = 0$

Hence

$$\int_0^1 x^3\sqrt{1+x^2}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^3\theta\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta$$

Uncorrected proofs

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$$= \int_0^{\frac{\pi}{4}} \tan^3\theta \sec^3\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec\theta \tan\theta (\sec^2\theta - 1)\sec^2\theta \, d\theta$$

Let $u = \sec\theta$

$du = \sec\theta \tan\theta \, d\theta$

$\theta = \frac{\pi}{4}, u = \sqrt{2}$

$\theta = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta (\sec^2\theta - 1)\sec^2\theta \, d\theta$$

$$= \int_1^{\sqrt{2}} (u^2 - 1)u^2 \, du$$

$$= \int_1^{\sqrt{2}} (u^4 - u^2) \, du$$

$$= \left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3}$$

$$= \frac{4\sqrt{2} - 1}{5} + \frac{1 - 2\sqrt{2}}{3}$$

$$= \frac{12\sqrt{2} - 3 + 5 - 10\sqrt{2}}{15}$$

$$= \frac{2\sqrt{2} + 2}{15}$$

$$= \frac{2}{15}(1 + \sqrt{2})$$

Uncorrected proofs

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9c

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$

Let $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$x = 1, \theta = \frac{\pi}{2}$

$x = 0, \theta = 0$

Hence

$$\int_0^1 x^2\sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\,d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{16}$$

10a

$$\int \sin x\cos x\,dx$$

Let $u = \sin x$

$du = \cos x\,dx$

Uncorrected proofs

Hence

$$\int \sin x \cos x \, dx$$

$$= \int u \, du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\sin^2 x + C$$

10b

$$\int \sin x \cos x \, dx$$

$$= \int \frac{1}{2}\sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + C$$

10c

$$\frac{1}{2}\sin^2 x + C$$

$$= \frac{1}{2}\left(\frac{1}{2}(1 - \cos 2x)\right) + C$$

$$= \frac{1}{4}(1 - \cos 2x) + C$$

$$= \frac{1}{4} - \frac{1}{4}\cos 2x + C$$

$\frac{1}{4} + C$ is still a constant, C

$$\frac{1}{4} - \frac{1}{4}\cos 2x + C$$

$$= -\frac{1}{4}\cos 2x + C$$

Uncorrected proofs

11a

$$\int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x\, (\tan^2 x + 1)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$$

Let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{4}, u = 1$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$$

$$= \int_0^1 u\, du$$

$$= \left[\frac{1}{2}u^2\right]_0^1$$

$$= \frac{1}{2}$$

11b

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos x - \cos^3 x)\, dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x\, (1 - \cos^2 x)\, dx$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x \sin^2 x\, dx$$

Let $u = \sin x$

Uncorrected proofs

$du = \cos x \, dx$

$x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2}$

$x = -\frac{\pi}{3}, u = -\frac{\sqrt{3}}{2}$

Hence

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x \sin^2 x \, dx$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} u^2 \, du$$

$$= \left[\frac{1}{3}u^3\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{3}\left(\frac{3\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right)$$

$$= \frac{2\sqrt{3}}{8}$$

$$= \frac{\sqrt{3}}{4}$$

12a

$$\int_0^{\frac{\pi}{3}} \sin^3 x \sec^2 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \sin^2 x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \, (1 - \cos^2 x)}{\cos^2 x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Uncorrected proofs

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{\sin x\,(1-\cos^2 x)}{\cos^2 x}\,dx$$

$$= -\int_1^{\frac{1}{2}} \frac{1-u^2}{u^2}\,du$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{u^2} - 1\right) du$$

$$= \left[-\frac{1}{u} - u\right]_{\frac{1}{2}}^{1}$$

$$= -2 - \left(-2 - \frac{1}{2}\right)$$

$$= -2 + \frac{5}{2}$$

$$= \frac{1}{2}$$

12b

$$\int_0^{\frac{\pi}{3}} \sin^3 x \sec^4 x\,dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^4 x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \sin^2 x}{\cos^4 x}\,dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x\,(1-\cos^2 x)}{\cos^4 x}\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

Uncorrected proofs

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{\sin x\,(1-\cos^2 x)}{\cos^4 x}\,dx$$

$$= -\int_1^{\frac{1}{2}} \frac{1-u^2}{u^4}\,du$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du$$

$$= \left[-\frac{1}{3u^3} + \frac{1}{u}\right]_{\frac{1}{2}}^{1}$$

$$= -\frac{1}{3} + 1 + \frac{1}{3}(8) - 2$$

$$= \frac{7}{3} - 1$$

$$= \frac{4}{3}$$

13a

$$\int \sin 3x \cos x\,dx$$

$$= \frac{1}{2}\int (\sin(3x - x) + \sin(3x + x))\,dx$$

$$= \frac{1}{2}\int (\sin 2x + \sin 4x)\,dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$$

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13b

$$\int \cos 3x \sin x \, dx$$

$$= \frac{1}{2}\int (\sin(x - 3x) + \sin(x + 3x)) \, dx$$

$$= \frac{1}{2}\int (-\sin 2x + \sin 4x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C$$

$$= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$$

13c

$$\int \cos 6x \cos 2x \, dx$$

$$= \frac{1}{2}\int (\cos(6x - 2x) + \cos(6x + 2x)) \, dx$$

$$= \frac{1}{2}\int (\cos 4x + \cos 8x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x\right) + C$$

$$= \frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$$

14a

$$\int_0^{\frac{\pi}{4}} \sin 3x \sin x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(3x - x) - \cos(3x + x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 2x - \cos 4x) \, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

Uncorrected proofs

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$$= \frac{1}{2}\left(\frac{1}{2} - 0 - 0 + 0\right)$$

$$= \frac{1}{4}$$

14b

$$\int_0^{\frac{\pi}{4}} \cos 4x \cos 2x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(4x - 2x) + \cos(4x + 2x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 2x + \cos 6x) \, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6} - 0 - 0\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)$$

$$= \frac{1}{6}$$

14c

$$\int_0^{\frac{\pi}{3}} \sin 4x \cos 2x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (\sin(4x - 2x) + \sin(4x + 2x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (\sin 2x + \sin 6x) \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x - \frac{1}{6}\cos 6x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)$$

Uncorrected proofs

$= \frac{1}{2}\left(\frac{3}{4}\right)$

$= \frac{3}{8}$

15a

$\int \frac{1}{1+\cos x}\,dx$

Let $t = \tan\left(\frac{x}{2}\right)$

$dx = \frac{2}{1+t^2}\,dt$

$\cos x = \frac{1-t^2}{1+t^2}$

Hence

$\int \frac{1}{1+\cos x}\,dx$

$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$

$= \int \frac{2}{1+t^2+1-t^2}\,dt$

$= \int \frac{2}{2}\,dt$

$= \int 1\,dt$

$= t + C$

$= \tan\frac{x}{2} + C$

15b

$\int \frac{1}{1+\sin x-\cos x}\,dx$

Let $t = \tan\left(\frac{x}{2}\right)$

Uncorrected proofs

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$$dx = \frac{2}{1+t^2}dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{1+\sin x - \cos x}dx$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}dt$$

$$= \int \frac{2}{1+t^2+2t-1+t^2}dt$$

$$= \int \frac{2}{2t^2+2t}dt$$

$$= \int \frac{1}{t(t+1)}dt$$

$$= \int \left(\frac{1}{t}-\frac{1}{t+1}\right)dt \quad \text{(using cover – up method)}$$

$$= \ln|t| - \ln|t+1| + C$$

$$= \ln\left|\frac{t}{t+1}\right| + C$$

$$= \ln\left|\frac{\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)+1}\right| + C$$

15c

$$\int \frac{1}{3\sin x + 4\cos x}dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}dt$$

$$\sin x = \frac{2t}{1+t^2}$$

Uncorrected proofs

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{3\sin x + 4\cos x}\,dx$$

$$= \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{6t + 4 - 4t^2}\,dt$$

$$= \int \frac{1}{3t + 2 - 2t^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\frac{3}{2}t + 1 - t^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{5}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}\,dt$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{5}{4} - t + \frac{3}{4}\right)\left(\frac{5}{4} + t - \frac{3}{4}\right)}\,dt$$

$$= \frac{1}{2}\int \frac{1}{(2-t)\left(\frac{1}{2}+t\right)}\,dt$$

$$= \frac{1}{2}\int \left(\frac{\frac{2}{5}}{\frac{1}{2}+t} + \frac{\frac{2}{5}}{2-t}\right)dt \quad \text{(using cover} - \text{up method)}$$

$$= \frac{1}{2}\int \left(\frac{\frac{4}{5}}{1+2t} + \frac{\frac{2}{5}}{2-t}\right)dt$$

$$= \frac{1}{5}\int \left(\frac{2}{1+2t} + \frac{1}{2-t}\right)dt$$

$$= \frac{1}{5}(\ln|1+2t| - \ln|2-t|) + C$$

Uncorrected proofs

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$$= \frac{1}{5}\ln\left|\frac{1+2t}{2-t}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{1+2\tan\left(\frac{x}{2}\right)}{2-\tan\left(\frac{x}{2}\right)}\right| + C$$

16a

$$\int \sec x \, dx$$

$$= \int \frac{1}{\cos x} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2} dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{\cos x} dx$$

$$= \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{1-t^2} dt$$

$$= \int \frac{2}{(1-t)(1+t)} dt$$

$$= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \quad \text{(using cover} - \text{up method)}$$

$$= -\ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C$$

Uncorrected proofs

16b

$$\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\times\frac{1+\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+2\tan\left(\frac{x}{2}\right)+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}+\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$\tan x=\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$

$$\cos x=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

Hence

$$\ln\left|\frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}+\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1}{\cos x}+\tan x\right|+C$$

$$=\ln|\sec x+\tan x|+C$$

Uncorrected proofs

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Ext 2

Ex 4G

Q17 $(\text{cis}\,\theta)^3 = \text{cis}\,3\theta$ by de Moivre

so $\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$

equate real parts to get

$$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$$

or $$\cos^3\theta = 3\cos\theta\sin^2\theta + \cos 3\theta$$

so $$\int \cos^2\theta\, d\theta = \int (3\cos\theta\sin^2\theta + \cos 3\theta)\, d\theta$$

$$= \sin^3\theta + \tfrac{1}{3}\sin 3\theta + C$$

Ext 2

Ex 4G

Q18 $I = \int \sec^3 x \, dx$

$= \int \sec^2 x . \sec x \, dx$

$= \tan x \sec x - \int \tan x . \sec x \tan x \, dx$ (by parts with $u = \sec x$, $v' = \sec^2 x$)

$= \tan x \sec x - \int \tan^2 x \sec x \, dx$

$= \tan x \sec x - \int \sec^3 x \, dx + \int \sec x \, dx$ (by pyth)

$= \tan x \sec x + \log|\sec x + \tan x| - I$

hence $2I = \tan x \sec x + \log|\sec x + \tan x| + 2C$

or $I = \frac{1}{2}\tan x \sec x + \frac{1}{2}\log|\sec x + \tan x| + C$

for some constant C.

Thus $\int_0^{\pi/4} \sec^3 x \, dx = \left[\frac{1}{2}\tan x \sec x + \frac{1}{2}\log(\sec x + \tan x)\right]_0^{\pi/4}$

$= \frac{1}{2} \cdot 1 \cdot \sqrt{2} + \frac{1}{2}\log(\sqrt{2}+1) - (0 + \frac{1}{2}\log(1+0))$

$= \frac{1}{\sqrt{2}} + \frac{1}{2}\log(\sqrt{2}+1)$ ≠