## Solutions to Exercise 4F Foundation questions

1a

$$\int xe^x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = e^x$

$u' = 1, v = e^x$

Hence

$$\int xe^x\,dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

1b

$$\int xe^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = e^{-x}$

$u' = 1, v = -e^{-x}$

Hence

$$\int xe^{-x}\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

$$= -e^{-x}(x+1) + C$$

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1c

$$\int (x+1)e^{3x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x+1, v' = e^{3x}$

$u' = 1, v = \frac{1}{3}e^{3x}$

Hence

$$\int (x+1)e^{3x}\,dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3}\int e^{3x}\,dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) + C$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{9}e^{3x} + C$$

$$= \frac{1}{9}e^{3x}(3x+3-1) + C$$

$$= \frac{1}{9}e^{3x}(3x+2) + C$$

1d

$$\int x\cos x\ dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \cos x$

$u' = 1, v = \sin x$

Hence

$$\int x\cos x\ dx$$

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$= x \sin x - \int \sin x \ dx$

$= x \sin x - (-\cos x) + C$

$= x \sin x + \cos x + C$

1e

$\int (x - 1) \sin 2x \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

Therefore

$u = x - 1, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$\int (x - 1) \sin 2x \, dx$

$= -\frac{1}{2}(x - 1) \cos 2x + \frac{1}{2}\int \cos 2x \, dx$

$= -\frac{1}{2}(x - 1) \cos 2x + \frac{1}{4}\sin 2x + C$

1f

$\int (2x - 3) \ \sec^2 x \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

Therefore

$u = 2x - 3, v' = \sec^2 x$

$u' = 2, v = \tan x$

Hence

$\int (2x - 3) \ \sec^2 x \, dx$

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$$= (2x - 3)\tan x - \int 2\tan x \, dx$$

$$= (2x - 3)\tan x + 2\int \frac{-\sin x}{\cos x} dx$$

$$= (2x - 3)\tan x + 2\ln|\cos x| + C$$

2a

$$\int_0^{\pi} x \sin x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sin x$

$u' = 1, v = -\cos x$

Hence

$$\int_0^{\pi} x \sin x \, dx$$

$$= [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx$$

$$= (-\pi\cos\pi - 0) + [\sin x]_0^{\pi}$$

$$= \pi + [\sin x]_0^{\pi}$$

$$= \pi + \sin\pi - \sin 0$$

$$= \pi$$

2b

$$\int_0^{\frac{\pi}{2}} x\cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \cos x$

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$u' = 1, v = \sin x$

Hence

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{\pi}{2} \sin \frac{\pi}{2} - 0 - [-\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \left(-\cos \frac{\pi}{2} + \cos 0\right)$$

$$= \frac{\pi}{2} - 1$$

2c

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sec^2 x$

$u' = 1, v = \tan x$

Hence

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \frac{\pi}{4} \tan \frac{\pi}{4} - 0 + \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} \, dx$$

$$= \frac{\pi}{4} + [\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \left(\ln\left(\cos \frac{\pi}{4}\right) - \ln(\cos 0)\right)$$

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$$= \frac{\pi}{4} + \left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right)$$

$$= \frac{\pi}{4} + \left(-\frac{1}{2}\ln 2 - 0\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2d

$$\int_0^1 x\, e^{2x}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = x, v' = e^{2x}$

$u' = 1, v = \frac{1}{2}e^{2x}$

Hence

$$\int_0^1 x\, e^{2x}\, dx$$

$$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\, dx$$

$$= \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^2 - \frac{1}{2}\left(\frac{1}{2}e^2 - \frac{1}{2}\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}(e^2 + 1)$$

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2e

$$\int_0^1 (1-x)e^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 1 - x, v' = e^{-x}$

$u' = -1, v = -e^{-x}$

Hence

$$\int_0^1 (1-x)e^{-x}\,dx$$

$$= [(1-x)(-e^{-x})]_0^1 - \int_0^1 e^{-x}\,dx$$

$$= [(x-1)e^{-x}]_0^1 - [-e^{-x}]_0^1$$

$$= 0 - (-1) + [e^{-x}]_0^1$$

$$= 1 + e^{-1} - 1$$

$$= \frac{1}{e}$$

2f

$$\int_{-2}^0 (x+2)e^{x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x + 2, v' = e^{x}$

$u' = 1, v = e^{x}$

Hence

$$\int_{-2}^0 (x+2)e^{x}\,dx$$

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$$= [(x+2)e^x]_{-2}^0 - \int_{-2}^0 e^x \, dx$$

$$= 2 - 0 - [e^x]_{-2}^0$$

$$= 2 - (1 - e^{-2})$$

$$= 1 + e^{-2}$$

3a

$$\int \ln x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln x, v' = 1$

$u' = \frac{1}{x}, v = x$

Hence

$$\int \ln x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

3b

$$\int \ln(x^2) \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln(x^2) = 2 \ln x, v' = 1$

$u' = \frac{2}{x}, v = x$

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Hence

$$\int \ln(x^2)\,dx$$

$$= x\ln(x^2) - \int 2\,dx$$

$$= x\ln(x^2) - 2x + C$$

$$= 2x\ln x - 2x + C$$

$$= 2x(\ln x - 1) + C$$

3c

$$\int \cos^{-1} x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = \cos^{-1} x\,, v' = 1$$

$$u' = \frac{-1}{\sqrt{1-x^2}}, v = x$$

Hence

$$\int \cos^{-1} x\,dx$$

$$= x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}dx$$

Let $u = 1 - x^2$

$$du = -2x\,dx$$

Therefore

$$\int \cos^{-1} x\,dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int \frac{1}{\sqrt{u}}du$$

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$$= x\cos^{-1}x - \frac{1}{2}\left(2\sqrt{u}\right) + C$$

$$= x\cos^{-1}x - \sqrt{u} + C$$

$$= x\cos^{-1}x - \sqrt{1-x^2} + C$$

4a

$$\int_0^1 \tan^{-1}x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = \tan^{-1}x\,, v' = 1$$

$$u' = \frac{1}{1+x^2}, v = x$$

Hence

$$\int_0^1 \tan^{-1}x\,dx$$

$$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1}(1) - 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}\left[\ln|1+x^2|\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

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4b

$$\int_1^e \ln x \, dx$$

$$= [x(\ln x - 1)]_1^e \qquad \text{(using the result from question 3a)}$$

$$= e(\ln e - 1) - (\ln 1 - 1)$$

$$= e(1 - 1) - (0 - 1)$$

$$= 1$$

4c

$$\int_1^e \ln \sqrt{x} \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = \ln \sqrt{x} = \frac{1}{2}\ln x \, , v' = 1$$

$$u' = \frac{1}{2x}, v = x$$

Hence

$$\int_1^e \ln \sqrt{x} \, dx$$

$$= \left[x \ln \sqrt{x}\right]_1^e - \int_1^e \frac{1}{2} dx$$

$$= \left[\frac{1}{2} x \ln x\right]_1^e - \left[\frac{1}{2} x\right]_1^e$$

$$= \left(\frac{1}{2} e - 0\right) - \left(\frac{1}{2} e - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Alternatively:

Since $\ln \sqrt{x} = \frac{1}{2}\ln x$

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$$\int_1^e \ln\sqrt{x}\,dx$$

$$= \frac{1}{2}\int_1^e \ln x\,dx$$

$$= \frac{1}{2} \times 1 \qquad \text{(using the result from question 4b)}$$

$$= \frac{1}{2}$$

5a

$$\int x \ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = \ln x\,, v' = x$$

$$u' = \frac{1}{x}, v = \frac{1}{2}x^2$$

Hence

$$\int x \ln x\,dx$$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

$$= \frac{1}{4}x^2(2\ln x - 1) + C$$

5b

$$\int x^2 \ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

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$u = \ln x\,, v' = x^2$

$u' = \frac{1}{x}, v = \frac{1}{3}x^3$

Hence

$$\int x^2 \ln x \, dx$$

$$= \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 \, dx$$

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C$$

$$= \frac{1}{9}x^3(3 \ln x - 1) + C$$

5c

$$\int \frac{\ln x}{x^2} dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln x\,, v' = \frac{1}{x^2}$

$u' = \frac{1}{x}, v = -\frac{1}{x}$

Hence

$$\int \frac{\ln x}{x^2} dx$$

$$= -\frac{1}{x}\ln x - \int \left(-\frac{1}{x^2}\right) dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C$$

$$= -\frac{1}{x}(\ln x + 1) + C$$

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## Solutions to Exercise 4F Development questions

6a

$$\int x^2 e^x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^2, v' = e^x$

$u' = 2x, v = e^x$

Hence

$$\int x^2 e^x \, dx$$

$$= x^2 e^x - \int 2xe^x \, dx$$

Consider $\int 2xe^x \, dx$,

$u = 2x, v' = e^x$

$u' = 2, v = e^x$

Hence

$$\int 2xe^x \, dx = 2xe^x - \int 2e^x \, dx$$

Therefore

$$\int x^2 e^x \, dx$$

$$= x^2 e^x - 2xe^x + \int 2e^x \, dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

6b

$$\int x^2 \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Uncorrected proofs

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Therefore

$u = x^2, v' = \cos x$

$u' = 2x, v = \sin x$

Hence

$$\int x^2 \cos x \, dx$$

$$= x^2 \sin x - \int 2x \sin x \, dx$$

Consider $\int 2x \sin x \, dx$,

$u = 2x, v' = \sin x$

$u' = 2, v = -\cos x$

Hence

$$\int 2x \sin x \, dx = -2x \cos x + \int 2 \cos x \, dx$$

Therefore

$$\int x^2 \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - \int 2 \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

6c

$$\int (\ln x)^2 \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = (\ln x)^2, v' = 1$

$u' = \dfrac{2 \ln x}{x}, v = x$

Hence

Uncorrected proofs

$$\int (\ln x)^2 \, dx$$

$$= x(\ln x)^2 - \int 2 \ln x \, dx$$

$$= x(\ln x)^2 - 2 \int \ln x \, dx$$

Consider $\int \ln x \, dx$,

$u = \ln x \, , v' = 1$

$u' = \frac{1}{x}, v = x$

Hence

$$\int \ln x \, dx = x \ln x - \int 1 \, dx$$

Therefore

$$\int (\ln x)^2 \, dx$$

$$= x(\ln x)^2 - 2x \ln x + 2 \int 1 \, dx$$

$$= x(\ln x)^2 - 2x \ln x + 2x + C$$

7a Using integration by parts:

$$\int_0^1 x(x-1)^5 \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = (x-1)^5$

$u' = 1, v = \frac{1}{6}(x-1)^6$

Hence

$$\int_0^1 x(x-1)^5 \, dx$$

Uncorrected proofs

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$$= \left[\frac{x}{6}(x-1)^6\right]_0^1 - \int_0^1 \frac{1}{6}(x-1)^6\,dx$$

$$= 0 - \left[\frac{1}{42}(x-1)^7\right]_0^1$$

$$= -\frac{1}{42}$$

By substitution:

$$\int_0^1 x(x-1)^5\,dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

$x = 1, u = 0$

$x = 0, u = -1$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_{-1}^0 (u+1)u^5\,du$$

$$= \int_{-1}^0 (u^6 + u^5)\,du$$

$$= \left[\frac{1}{7}u^7 + \frac{1}{6}u^6\right]_{-1}^0$$

$$= \left(\frac{1}{7}(0)^7 + \frac{1}{6}(0)^6\right) - \left(\frac{1}{7}(-1)^7 + \frac{1}{6}(-1)^6\right)$$

$$= \frac{1}{7} - \frac{1}{6}$$

$$= -\frac{1}{42}$$

Uncorrected proofs

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7b Using integration by parts:

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = x, v' = \sqrt{x+1}$$

$$u' = 1, v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

Hence

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$= \left[\frac{2}{3}x(x+1)^{\frac{3}{2}}\right]_0^1 - \int_0^1 \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} - \left[\frac{4}{15}(x+1)^{\frac{5}{2}}\right]_0^1$$

$$= \frac{4\sqrt{2}}{3} - \left(\frac{4}{15}(2)^{\frac{5}{2}} - \frac{4}{15}\right)$$

$$= \frac{4\sqrt{2}}{3} - \left(\frac{16\sqrt{2}}{15} - \frac{4}{15}\right)$$

$$= \frac{20\sqrt{2}}{15} - \frac{16\sqrt{2} - 4}{15}$$

$$= \frac{4\sqrt{2} + 4}{15}$$

$$= \frac{4}{15}(\sqrt{2} + 1)$$

By substitution:

$$\int_0^1 x\sqrt{x+1}\,dx$$

Let $u = \sqrt{x+1}$

$x = 1, u = \sqrt{2}$

$x = 0, u = 1$

Uncorrected proofs

$x = u^2 - 1$

$dx = 2u\,du$

Hence

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$= \int_1^{\sqrt{2}} (u^2 - 1)u \times 2u\,du$$

$$= 2\int_1^{\sqrt{2}} (u^4 - u^2)\,du$$

$$= 2\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= 2\left(\frac{4}{5}\sqrt{2} - \frac{2}{3}\sqrt{2} - \frac{1}{5} + \frac{1}{3}\right)$$

$$= \frac{8\sqrt{2} - 2}{5} - \frac{4\sqrt{2} - 2}{3}$$

$$= \frac{24\sqrt{2} - 6 - 20\sqrt{2} + 10}{15}$$

$$= \frac{4\sqrt{2} + 4}{15}$$

$$= \frac{4}{15}(\sqrt{2} + 1)$$

7c Using integration by parts:

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \sqrt{4-x}$

$u' = 1, v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$

Hence

Uncorrected proofs

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \left[-\frac{2}{3}x(4-x)^{\frac{3}{2}}\right]_0^4 + \int_0^4 \frac{2}{3}(4-x)^{\frac{3}{2}}\,dx$$

$$= 0 + \left[-\frac{4}{15}(4-x)^{\frac{5}{2}}\right]_0^4$$

$$= (0) - \left(-\frac{4}{15}(4-0)^{\frac{5}{2}}\right)$$

$$= \frac{4 \times 32}{15}$$

$$= \frac{128}{15}$$

By substitution:

$$\int_0^4 x\sqrt{4-x}\,dx$$

Let $u = \sqrt{4-x}$

$x = 4, u = 0$

$x = 0, u = 2$

$x = 4 - u^2$

$dx = -2u\,du$

Hence

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \int_2^0 (4-u^2)u \times -2u\,du$$

$$= 2\int_0^2 (4-u^2)u^2\,du$$

$$= 2\int_0^2 (4u^2 - u^4)\,du$$

$$= 2\left[\frac{4}{3}u^3 - \frac{1}{5}u^5\right]_0^2$$

Uncorrected proofs

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$$= 2\left[\frac{4}{3} \times 8 - \frac{1}{5} \times 32\right]$$

$$= \frac{128}{15}$$

8a

$$\int e^x \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = e^x, v' = \cos x$

$u' = e^x, v = \sin x$

Hence

$$\int e^x \cos x \, dx$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

Consider $\int e^x \sin x \, dx$,

$u = e^x, v' = \sin x$

$u' = e^x, v = -\cos x$

Hence

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Therefore

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

Therefore

$$\int e^x \cos x \, dx$$

Uncorrected proofs

$$= \frac{e^x \sin x + e^x \cos x}{2} + C$$

8b

$$\int e^{-x} \sin x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = e^{-x}, v' = \sin x$

$u' = -e^{-x}, v = -\cos x$

Hence

$$\int e^{-x} \sin x \, dx$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

Consider $\int e^{-x} \cos x \, dx$,

$u = e^{-x}, v' = \cos x$

$u' = -e^{-x}, v = \sin x$

Hence

$$\int e^{-x} \cos x \, dx = e^{-x} \sin x + \int e^{-x} \sin x \, dx$$

Therefore

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$2 \int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x$$

Therefore

$$\int e^{-x} \sin x \, dx$$

$$= -\frac{e^{-x} \cos x + e^{-x} \sin x}{2} + C$$

Uncorrected proofs

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9a

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = e^{2x}, v' = \cos x$

$u' = 2e^{2x}, v = \sin x$

Hence

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$$

$$= [e^{2x}\sin x]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx$$

Consider $\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx$,

$u = e^{2x}, v' = \sin x$

$u' = 2e^{2x}, v = -\cos x$

Hence

$$\int_0^{\frac{\pi}{2}} e^{2x}\sin x\,dx = [-e^{2x}\cos x]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$$

Therefore

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = [e^{2x}\sin x]_0^{\frac{\pi}{2}} - 2[-e^{2x}\cos x]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx$$

$$5\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = [e^{2x}\sin x]_0^{\frac{\pi}{2}} - 2[-e^{2x}\cos x]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x\,dx = \frac{1}{5}\left(e^{\pi} - 2(0-1)\right)$$

$$= \frac{1}{5}(e^{\pi} + 2)$$

Uncorrected proofs

9b

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = e^x, v' = \sin 2x$$

$$u' = e^x, v = -\frac{1}{2}\cos 2x$$

Hence

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx$$

$$= \left[-e^x \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx$$

Consider $\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx$,

$$u = e^x, v' = \cos 2x$$

$$u' = e^x, v = \frac{1}{2}\sin 2x$$

Hence

$$\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx = \left[e^x \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx$$

Therefore

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx = \left[-e^x \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\left[e^x \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}} - \frac{1}{4}\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx$$

$$\frac{5}{4}\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx = \left[-e^x \frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}} + \frac{1}{2}\left[e^x \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x \, dx = \frac{4}{5}\left(\left((0) - \left(-\frac{1}{2}\right)\right) + \frac{1}{2}\left(\left(\frac{1}{2}e^{\frac{\pi}{4}}\right) - (0)\right)\right)$$

$$= \frac{4}{5}\left(\frac{1}{2} + \frac{1}{4}e^{\frac{\pi}{4}}\right)$$

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$$= \frac{2}{5} + \frac{1}{5}e^{\frac{\pi}{4}}$$

$$= \frac{1}{5}\left(e^{\frac{\pi}{4}} + 2\right)$$

10a

$$\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \sin^{-1} x \, , v' = 1$

$u' = \dfrac{1}{\sqrt{1-x^2}}, v = x$

Hence

$$\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \, dx$$

$$= [x \sin^{-1} x]_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{\sqrt{3}}{2} \times \frac{\pi}{3} - \left[-\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\sqrt{3}}{2} \times \frac{\pi}{3} + \left[\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi\sqrt{3}}{6} + \frac{1}{2} - 1$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}$$

$$= \frac{3\pi}{6\sqrt{3}} - \frac{3\sqrt{3}}{6\sqrt{3}}$$

$$= \frac{3\pi - 3\sqrt{3}}{6\sqrt{3}}$$

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$$= \frac{1}{2\sqrt{3}}(\pi - \sqrt{3})$$

10b

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = \cos^{-1} x \, , v' = 1$$

$$u' = \frac{-1}{\sqrt{1-x^2}}, v = x$$

Hence

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$$

$$= [x \cos^{-1} x]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} - \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{-x}{\sqrt{1-x^2}} dx$$

$$= \left(\frac{\sqrt{3}}{2} \times \frac{\pi}{6}\right) - \left(-\frac{\sqrt{3}}{2} \times \frac{5\pi}{6}\right) - \left[\sqrt{1-x^2}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi\sqrt{3}}{12} + \frac{5\pi\sqrt{3}}{12} - \frac{1}{2} + \frac{1}{2}$$

$$= \frac{6\pi\sqrt{3}}{12}$$

$$= \frac{\pi\sqrt{3}}{2}$$

Uncorrected proofs

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10c

$$\int_0^1 4x\tan^{-1}x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \tan^{-1}x\,, v' = 4x$

$u' = \dfrac{1}{x^2+1}, v = 2x^2$

Hence

$$\int_0^1 4x\tan^{-1}x\,dx$$

$$= [2x^2\tan^{-1}x]_0^1 - 2\int_0^1 \frac{x^2}{x^2+1}dx$$

$$= \frac{\pi}{2} - 2\int_0^1 \frac{x^2+1-1}{x^2+1}dx$$

$$= \frac{\pi}{2} - 2\int_0^1 1 - \frac{1}{x^2+1}dx$$

$$= \frac{\pi}{2} - 2[x - \tan^{-1}x]_0^1$$

$$= \frac{\pi}{2} - 2\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - 2 + \frac{\pi}{2}$$

$$= \pi - 2$$

11a

$$\int_0^{\pi} x^2\cos 2x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x^2, v' = \cos 2x$

Uncorrected proofs

$u' = 2x, v = \frac{1}{2}\sin 2x$

Hence

$\int_0^{\pi} x^2 \cos 2x \, dx$

$= \left[x^2 \frac{1}{2}\sin 2x\right]_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx$

Consider $\int_0^{\pi} x \sin 2x \, dx$,

$u = x, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$\int_0^{\pi} x \sin 2x \, dx = \left[-x \frac{1}{2}\cos 2x\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$

Therefore

$\int_0^{\pi} x^2 \cos 2x \, dx$

$= \left[x^2 \frac{1}{2}\sin 2x\right]_0^{\pi} - \left[-x \frac{1}{2}\cos 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$

$= \left[x^2 \frac{1}{2}\sin 2x\right]_0^{\pi} + \left[x \frac{1}{2}\cos 2x\right]_0^{\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi}$

$= 0 + \frac{\pi}{2} - \frac{1}{2}(0)$

$= \frac{\pi}{2}$

11b

$\int_0^{\pi} x^2 \sin \frac{1}{2}x \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

Therefore

Uncorrected proofs

$u = x^2, v' = \sin\frac{1}{2}x$

$u' = 2x, v = -2\cos\frac{1}{2}x$

Hence

$\int_0^{\pi} x^2 \sin\frac{1}{2}x\,dx$

$= \left[-2x^2\cos\frac{1}{2}x\right]_0^{\pi} + 4\int_0^{\pi} x\cos\frac{1}{2}x\,dx$

Consider $\int_0^{\pi} x\cos\frac{1}{2}x\,dx$,

$u = x, v' = \cos\frac{1}{2}x$

$u' = 1, v = 2\sin\frac{1}{2}x$

Hence

$\int_0^{\pi} x\cos\frac{1}{2}x\,dx = \left[2x\sin\frac{1}{2}x\right]_0^{\pi} - 2\int_0^{\pi} \sin\frac{1}{2}x\,dx$

Therefore

$\int_0^{\pi} x^2 \sin\frac{1}{2}x\,dx$

$= \left[-2x^2\cos\frac{1}{2}x\right]_0^{\pi} + 4\left(\left[2x\sin\frac{1}{2}x\right]_0^{\pi} - 2\int_0^{\pi} \sin\frac{1}{2}x\,dx\right)$

$= 0 + 8\pi - 8\left[-2\cos\frac{1}{2}x\right]_0^{\pi}$

$= 8\pi + 16\left[\cos\frac{1}{2}x\right]_0^{\pi}$

$= 8\pi + 16(-1)$

$= 8\pi - 16$

Uncorrected proofs

11c

$$\int_1^e \sin(\ln x)\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \sin(\ln x)\,, v' = 1$

$u' = \dfrac{\cos(\ln x)}{x}, v = x$

Hence

$$\int_1^e \sin(\ln x)\,dx$$

$$= [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\,dx$$

Consider $\int_1^e \cos(\ln x)\,dx$,

$u = \cos(\ln x)\,, v' = 1$

$u' = \dfrac{-\sin(\ln x)}{x}, v = x$

Hence

$$\int_1^e \cos(\ln x)\,dx = [x\cos(\ln x)]_1^e + \int_1^e \sin(\ln x)\,dx$$

Therefore

$$\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e - \int_1^e \sin(\ln x)\,dx$$

$$2\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e$$

$$\int_1^e \sin(\ln x)\,dx$$

$$= \frac{1}{2}([x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e)$$

$$= \frac{1}{2}(e\sin 1 - 0 - e\cos 1 + 1)$$

Uncorrected proofs

$$= \frac{1}{2}e(\sin 1 - \cos 1) + \frac{1}{2}$$

11d

$$\int_1^e \cos(\ln x)\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$$u = \cos(\ln x)\,, v' = 1$$

$$u' = \frac{-\sin(\ln x)}{x}, v = x$$

Hence

$$\int_1^e \cos(\ln x)\, dx$$

$$= [x\cos(\ln x)]_1^e + \int_1^e \sin(\ln x)\, dx$$

Consider $\int_1^e \sin(\ln x)\, dx$,

$$u = \sin(\ln x)\,, v' = 1$$

$$u' = \frac{\cos(\ln x)}{x}, v = x$$

Hence

$$\int_1^e \sin(\ln x)\, dx = [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\, dx$$

Therefore

$$\int_1^e \cos(\ln x)\, dx = [x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\, dx$$

$$2\int_1^e \cos(\ln x)\, dx = [x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e$$

$$\int_1^e \cos(\ln x)\, dx$$

Uncorrected proofs

$$= \frac{1}{2}([x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e)$$

$$= \frac{1}{2}(e\cos 1 - 1 + e\sin 1)$$

$$= \frac{1}{2}e(\cos 1 + \sin 1) - \frac{1}{2}$$

12a

$$\int x\ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \ln x, v' = x$

$u' = \frac{1}{x}, v = \frac{1}{2}x^2$

Hence

$$\int x\ln x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

12b

$$\int x(\ln x)^2\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = (\ln x)^2, v' = x$

Uncorrected proofs

$u' = \dfrac{2\ln x}{x}, v = \dfrac{1}{2}x^2$

Hence

$\int x(\ln x)^2\,dx$

$= \dfrac{1}{2}x^2(\ln x)^2 - \int x\ln x\,dx$

$= \dfrac{1}{2}x^2(\ln x)^2 - \dfrac{1}{2}x^2\ln x + \dfrac{1}{4}x^2 + C$

13a

$\int_0^{\frac{\pi}{2}} x\sin x\cos x\,dx$

$= \dfrac{1}{2}\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

$\int uv'\,dx = uv - \int u'v\,dx$

Therefore

$u = x, v' = \sin 2x$

$u' = 1, v = -\dfrac{1}{2}\cos 2x$

Hence

$\dfrac{1}{2}\int_0^{\frac{\pi}{2}} x\sin 2x\,dx$

$= \dfrac{1}{2}\left(\left[-\dfrac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}} + \dfrac{1}{2}\int_0^{\frac{\pi}{2}}\cos 2x\,dx\right)$

$= \dfrac{1}{2}\left(\dfrac{\pi}{4} + \dfrac{1}{2}\left[\dfrac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}\right)$

$= \dfrac{\pi}{8} - 0$

$= \dfrac{\pi}{8}$

Uncorrected proofs

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13b

$$\int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} x\left(\frac{1}{2}(1 - \cos 2x)\right) dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (x - x\cos 2x) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x \, dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x \, dx$$

$$= \frac{1}{4}[x^2]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x \, dx$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \cos 2x$

$u' = 1, v = \frac{1}{2}\sin 2x$

Hence

$$\int_0^{\frac{\pi}{2}} x\cos 2x \, dx$$

$$= \left[\frac{1}{2}x\sin 2x\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x \, dx$$

$$= 0 - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}[\cos 2x]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}(-1 - 1)$$

Uncorrected proofs

$$= -\frac{1}{2}$$

Therefore

$$\int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x \cos 2x \, dx$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi^2}{16} + \frac{1}{4}$$

$$= \frac{1}{16}(\pi^2 + 4)$$

13c

$$\int_0^{\frac{\pi}{4}} x \tan^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} x(\sec^2 x - 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} (x \sec^2 x - x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx - \left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$$

$$= \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx - \frac{\pi^2}{32}$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sec^2 x$

$u' = 1, v = \tan x$

Hence

Uncorrected proofs

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} + [\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - 0 + \ln\left(\frac{1}{\sqrt{2}}\right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Therefore

$$\int_0^{\frac{\pi}{4}} x \tan^2 x \, dx$$

$$= \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$

13d

$$\int_0^{\pi} x^2(\cos^2 x - \sin^2 x) \, dx$$

$$= \int_0^{\pi} x^2 \cos 2x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = x^2, v' = \cos 2x$$

$$u' = 2x, v = \frac{1}{2}\sin 2x$$

Hence

$$\int_0^{\pi} x^2 \cos 2x \, dx$$

$$= \left[\frac{1}{2}x^2 \sin 2x\right]_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx$$

Uncorrected proofs

Consider $\int_0^{\pi} x \sin 2x \, dx$,

$u = x, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$$\int_0^{\pi} x \sin 2x \, dx = \left[-\frac{1}{2}x \cos 2x\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

Therefore

$$\int_0^{\pi} x^2 \cos 2x \, dx$$

$$= \left[\frac{1}{2}x^2 \sin 2x\right]_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx$$

$$= 0 - \left[-\frac{1}{2}x \cos 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

$$= \frac{1}{2}[x \cos 2x]_0^{\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{1}{2}(\pi - 0) - 0$$

$$= \frac{\pi}{2}$$

Uncorrected proofs

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Ex 2

Ex 4f

Q14 a) $I = \int \sqrt{a^2 - x^2}\, dx$

$= x\sqrt{a^2-x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}\, dx$ (by parts with $u = \sqrt{a^2-x^2}$, $v' = 1$)

$= x\sqrt{a^2-x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2-x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2-x^2}}\, dx$

$= x\sqrt{a^2-x^2} - \underbrace{\int \sqrt{a^2-x^2}\, dx}_{= I} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$

so $2I = x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + 2c$. (for some constant $c$)

thus $I = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$.

b) $I = \int \log\left(x + \sqrt{x^2+a^2}\right) dx$

$= x \log\left(x + \sqrt{x^2+a^2}\right) - \int \frac{x}{\sqrt{x^2+a^2}}\, dx$ (by parts with $u = \log(\text{-//-})$, $v' = 1$)

$= x \log\left(x + \sqrt{x^2+a^2}\right) - \sqrt{x^2+a^2} + C$

c) $I = \int \log\left(x + \sqrt{x^2-a^2}\right) dx$

$= x \log\left(x + \sqrt{x^2-a^2}\right) - \int \frac{x}{\sqrt{x^2-a^2}}\, dx$ (by parts with $u = \log(\text{-//-})$, $v' = 1$)

$= x \log\left(x + \sqrt{x^2-a^2}\right) - \sqrt{x^2-a^2} + C$.

Ext 2

Ex 4F

Q15 a) First note that $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$.

$$\text{so } \int x\sin x\cos 3x\,dx = \frac{1}{2}\int x(\sin 4x + \sin(-2x))\,dx$$

$$= \frac{1}{2}\int (x\sin 4x - x\sin 2x)\,dx$$

$$\text{now } \int x\sin ax\,dx = -\frac{x}{a}\cos ax + \frac{1}{a}\int \cos ax\,dx \quad \text{(by parts)}$$

$$= -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax \ (+C)$$

$$\text{so } \int x\sin x\cos 3x\,dx = -\frac{x}{8}\cos 4x + \frac{1}{32}\sin 4x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + C$$

$$= \frac{1}{32}\left(\sin 4x - 4\sin 2x + 8x\cos 2x - 4x\cos 4x\right) + C$$

b) First note $\cos(A+B) - \cos(A-B) = 2\cos A\cos B$.

$$\text{so } \int x\cos 2x\cos x\,dx = \frac{1}{2}\int x(\cos 3x + \cos x)\,dx$$

$$\text{now } \int x\cos ax\,dx = \frac{x}{a}\sin ax - \frac{1}{a}\int \sin ax\,dx \quad \text{(by parts)}$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \ (+C)$$

$$\text{thus } \int x\cos 2x\cos x\,dx = \frac{x}{6}\sin 3x + \frac{1}{18}\cos 3x + \frac{x}{2}\sin x + \frac{1}{2}\cos x + C$$

$$= \frac{1}{18}(3x\sin 3x + \cos 3x + 9x\sin x + 9\cos x) + C$$

c) using the identity in a) above

$$\int e^x\sin 2x\cos x\,dx = \frac{1}{2}\int e^x(\sin 3x + \sin x)\,dx$$

$$\text{Now } I = \int e^x\sin ax\,dx = e^x\sin ax - a\int e^x\cos ax\,dx \quad \text{(by parts)}$$

$$= e^x\sin ax - a\left(e^x\cos ax + a\int e^x\sin ax\,dx\right)$$

(by parts again)

$$\text{so } I = e^x(\sin ax - a\cos ax) - a^2 I$$

Ext 2

Ex 4F

Q15 c)

(cta)

$$(1+a^2)I = e^x(\sin ax - a\cos ax)$$

$$I = \frac{1}{1+a^2} e^x(\sin ax - a\cos ax).$$

so $\int e^x \sin 2x \cos x \, dx = \frac{1}{2}\int e^x(\sin 3x + \sin x)\, dx$

$$= \frac{1}{2}\left[\frac{1}{10} e^x(\sin 3x - 3\cos 3x) + \frac{1}{2} e^x(\sin x - \cos x)\right] + C$$

$$= \frac{1}{20}\left[e^x(\sin 3x - 3\cos 3x) + 5e^x(\sin x - \cos x)\right] + C.$$

Ext 2

Ex 4F

Q16 a) $I = \int_0^{\frac{1}{2}} x \sin^{-1} x \, dx$

$= \left[\frac{x^2}{2} \sin^{-1} x\right]_0^{1/2} - \frac{1}{2} \int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} \, dx$ (by parts)

in the integral put $x = \sin\theta$ with $0 \le \theta \le \frac{\pi}{6}$

so $dx = \cos\theta \, d\theta$

and $\sqrt{1-x^2} = |\cos\theta|$

$= \cos\theta$ (for this domain)

so $I = \left(\frac{1}{8} \cdot \frac{\pi}{6} - 0\right) - \frac{1}{2} \int_0^{\pi/6} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \, d\theta$

$= \frac{\pi}{48} - \frac{1}{2} \int_0^{\pi/6} \sin^2\theta \, d\theta$

$= \frac{\pi}{48} - \frac{1}{4} \int_0^{\pi/6} (1 - \cos 2\theta) \, d\theta$

$= \frac{\pi}{48} - \frac{1}{4} \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$

$= \frac{\pi}{48} - \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + 0$

$= \frac{\sqrt{3}}{16} - \frac{\pi}{48}$

b) $I = \int_0^1 x^2 \tan^{-1} x \, dx$

$= \left[\frac{x^3}{3} \tan^{-1} x\right]_0^1 - \frac{1}{3} \int_0^1 \frac{x^3}{1+x^2} \, dx$ (by parts)

$= \left(\frac{\pi}{12} - 0\right) - \frac{1}{3} \int_0^1 \frac{x^3 + x}{x^2+1} \, dx + \frac{1}{3} \int_0^1 \frac{x}{x^2+1} \, dx$

$= \frac{\pi}{12} - \frac{1}{3} \int_0^1 x \, dx + \frac{1}{6} \int_0^1 \frac{2x}{x^2+1} \, dx$

$= \frac{\pi}{12} - \frac{1}{6} \left[x^2\right]_0^1 + \frac{1}{6} \left[\log(x^2+1)\right]_0^1$

$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6} \log 2$

Ext 2

Ex 4F

Q17 $\lim_{N\to\infty} \int_0^N te^{-st}\,dt$

$= \lim_{N\to\infty} \left[\frac{te^{-st}}{-s}\right]_0^N + \frac{1}{s}\int_0^N e^{-st}\,dt$ (by parts)

$= \lim_{N\to\infty} \frac{Ne^{-sN}}{-s} - \frac{1}{s^2}\left[e^{-st}\right]_0^N$

$= \lim_{N\to\infty} \frac{Ne^{-sN}}{-s} - \frac{e^{-sN}}{s^2} + \frac{1}{s^2}$

$= 0 - 0 + \frac{1}{s^2}$ (since $s>0$)

$= \frac{1}{s^2}$