Solutions to Exercise 4E Foundation questions

1a

$$\int \frac{1}{9+x^2}\,dx$$

$$= \int \frac{1}{3^2+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{3}{3^2+x^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

1b

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2-x^2}}\,dx$$

$$= \sin^{-1}\frac{x}{3} + C$$

1c

$$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$$

Let $\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$$\frac{1}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$$1 = A(x+3) + B(x-3)$$

When $x = 3$,

$$1 = 6A$$

$$A = \frac{1}{6}$$

When $x = -3$,

$$1 = -6B$$

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$$B = -\frac{1}{6}$$

Hence,

$$\int \frac{1}{x^2 - 9}\,dx$$

$$= \int \left(\frac{1}{6(x-3)} - \frac{1}{6(x+3)}\right) dx$$

$$= \frac{1}{6}\int \left(\frac{1}{x-3} - \frac{1}{x+3}\right) dx$$

$$= \frac{1}{6}(\ln|x-3| - \ln|x+3|) + C$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

1d

$$\frac{1}{9-x^2} = \frac{1}{(3-x)(3+x)}$$

Let $\frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$

$$\frac{1}{(3-x)(3+x)} = \frac{A(3+x) + B(3-x)}{(3-x)(3+x)}$$

$$1 = A(3+x) + B(3-x)$$

When $x = 3$,

$$1 = 6A$$

$$A = \frac{1}{6}$$

When $x = -3$,

$$1 = 6B$$

$$B = \frac{1}{6}$$

Hence,

$$\int \frac{1}{9-x^2}\,dx$$

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$$= \int \left(\frac{1}{6(3-x)} + \frac{1}{6(3+x)}\right) dx$$

$$= \frac{1}{6}\int \left(-\frac{1}{x-3} + \frac{1}{x+3}\right) dx$$

$$= \frac{1}{6}(-\ln|3-x| + \ln|3+x|) + C$$

$$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$$

Alternatively:

$$\int \frac{1}{9-x^2} dx$$

$$= -\int \frac{1}{x^2-9} dx$$

$$= -\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C \qquad \text{(using answer from part c)}$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|^{-1} + C$$

$$= \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$$

$$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$$

1e

$$\int \frac{1}{\sqrt{9+x^2}} dx$$

$$= \int \frac{1}{\sqrt{3^2+x^2}} dx$$

$$= \ln\left(x + \sqrt{9+x^2}\right) + C \qquad \text{(using standard integral)}$$

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Chapter 4 worked solutions – Integration

1f

$$\int \frac{1}{\sqrt{x^2-9}}\,dx$$
$$= \int \frac{1}{\sqrt{x^2-3^2}}\,dx$$
$$= \ln\left|x+\sqrt{x^2-9}\right| + C \qquad \text{(using standard integral)}$$

2a

$$\int \frac{1}{x^2+4x+5}\,dx$$
$$= \int \frac{1}{(x^2+4x+4)+1}\,dx$$
$$= \int \frac{1}{(x+2)^2+1^2}\,dx$$
$$= \tan^{-1}(x+2) + C$$

2b

$$\int \frac{1}{x^2-4x+20}\,dx$$
$$= \int \frac{1}{(x^2-4x+4)+16}\,dx$$
$$= \int \frac{1}{(x-2)^2+4^2}\,dx$$
$$= \frac{1}{4}\int \frac{4}{(x-2)^2+4^2}\,dx$$
$$= \frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right) + C$$

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2c

$$\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-8x-9)}}\,dx$$

$$= \int \frac{1}{\sqrt{-\left((x^2-8x+16)-25\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{-((x-4)^2-25)}}\,dx$$

$$= \int \frac{1}{\sqrt{25-(x-4)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5^2-(x-4)^2}}\,dx$$

$$= \sin^{-1}\frac{x-4}{5} + C$$

2d

$$\int \frac{1}{\sqrt{20-8x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2+8x-20)}}\,dx$$

$$= \int \frac{1}{\sqrt{-\left((x^2+8x+16)-36\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{-((x+4)^2-36)}}\,dx$$

$$= \int \frac{1}{\sqrt{36-(x+4)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{6^2-(x+4)^2}}\,dx$$

$$= \sin^{-1}\frac{x+4}{6} + C$$

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2e

$$\int \frac{1}{\sqrt{x^2-6x+13}}\,dx$$

$$= \int \frac{1}{\sqrt{(x^2-6x+9)+4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2+4}}\,dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2+2^2}}\,dx$$

$$= \ln\left((x-3)+\sqrt{(x-3)^2+2^2}\right)+C \qquad \text{(using standard integral)}$$

$$= \ln\left(x-3+\sqrt{x^2-6x+13}\right)+C$$

Note: we don't need to take absolute values here, because the log expression is always positive:

$$(x-3)^2+2^2 > (x-3)^2 \geq 0$$

$$\sqrt{(x-3)^2+2^2} > \sqrt{(x-3)^2}$$

$$\sqrt{(x-3)^2+2^2} > |(x-3)|$$

$$\sqrt{(x-3)^2+2^2}+(x-3) > 0$$

2f

$$\int \frac{1}{\sqrt{4x^2+8x+6}}\,dx$$

$$= \int \frac{1}{\sqrt{4\left(x^2+2x+\frac{3}{2}\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{4\left((x^2+2x+1)+\frac{1}{2}\right)}}\,dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(x+1)^2+\frac{1}{2}}}\,dx$$

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$$= \frac{1}{2}\ln\left|(x+1) + \sqrt{(x+1)^2 + \frac{1}{2}}\right| + C \qquad \text{(using standard integral)}$$

$$= \frac{1}{2}\ln\left|x + 1 + \sqrt{x^2 + 2x + \frac{3}{2}}\right| + C$$

3a

$$\int_1^3 \frac{1}{x^2 - 2x + 5}\,dx$$

$$= \int_1^3 \frac{1}{(x^2 - 2x + 1) + 4}\,dx$$

$$= \int_1^3 \frac{1}{(x-1)^2 + 4}\,dx$$

$$= \int_1^3 \frac{1}{(x-1)^2 + 2^2}\,dx$$

$$= \frac{1}{2}\int_1^3 \frac{2}{(x-1)^2 + 2^2}\,dx$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x-1}{2}\right]_1^3$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{8}$$

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3b

$$\int_1^5 \frac{4}{x^2-6x+13}\,dx$$

$$= \int_1^5 \frac{4}{(x^2-6x+9)+4}\,dx$$

$$= \int_1^5 \frac{4}{(x-3)^2+4}\,dx$$

$$= 2\int_1^5 \frac{2}{(x-3)^2+2^2}\,dx$$

$$= 2\left[\tan^{-1}\frac{x-3}{2}\right]_1^5$$

$$= 2(\tan^{-1}1 - \tan^{-1}(-1))$$

$$= 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= 2 \times \frac{\pi}{2}$$

$$= \pi$$

3c

$$\int_{-1}^0 \frac{1}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-(x^2+2x-3)}}\,dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-\left((x^2+2x+1)-4\right)}}\,dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-((x-1)^2-4)}}\,dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{4-(x-1)^2}}\,dx$$

$$= \frac{1}{2}\int_{-1}^0 \frac{2}{\sqrt{2^2-(x-1)^2}}\,dx$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{x-1}{2}\right]_{-1}^0$$

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$$= \frac{1}{2}\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right)$$
$$= \frac{1}{2}\left(-\frac{\pi}{6} - \left(-\frac{\pi}{2}\right)\right)$$
$$= \frac{1}{2}\left(-\frac{\pi}{6} + \frac{\pi}{2}\right)$$
$$= \frac{\pi}{6}$$

3d

$$\int_0^1 \frac{3}{\sqrt{3 + 4x - 4x^2}}\,dx$$
$$= \int_0^1 \frac{3}{\sqrt{-(4x^2 - 4x - 3)}}\,dx$$
$$= \int_0^1 \frac{3}{\sqrt{-\left((4x^2 - 4x + 1) - 4\right)}}\,dx$$
$$= \int_0^1 \frac{3}{\sqrt{-((2x-1)^2 - 4)}}\,dx$$
$$= \int_0^1 \frac{3}{\sqrt{4 - (2x-1)^2}}\,dx$$
$$= \frac{3}{2}\int_0^1 \frac{2}{\sqrt{2^2 - (2x-1)^2}}\,dx$$
$$= \frac{3}{2}\left[\sin^{-1}\frac{2x-1}{2}\right]_0^1$$
$$= \frac{3}{2}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$$
$$= \frac{3}{2}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)$$
$$= \frac{3}{2}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$
$$= \frac{3}{2} \times \frac{\pi}{3}$$
$$= \frac{\pi}{2}$$

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3e

$$\int_{-1}^{3} \frac{1}{\sqrt{x^2+2x+10}}\,dx$$
$$= \int_{-1}^{3} \frac{1}{\sqrt{(x^2+2x+1)+9}}\,dx$$
$$= \int_{-1}^{3} \frac{1}{\sqrt{(x+1)^2+3^2}}\,dx$$
$$= \left[\ln\left((x+1)+\sqrt{(x+1)^2+9}\right)\right]_{-1}^{3} \qquad \text{(using standard integral)}$$
$$= \ln 9 - \ln 3$$
$$= \ln\left(\frac{9}{3}\right)$$
$$= \ln 3$$

3f

$$\int_{\frac{1}{2}}^{1} \frac{2}{\sqrt{x^2-x+1}}\,dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{2}{\sqrt{\left(x^2-x+\frac{1}{4}\right)+\frac{3}{4}}}\,dx$$
$$= 2\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}\,dx$$
$$= 2\left[\ln\left(\left(x-\frac{1}{2}\right)+\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\right)\right]_{\frac{1}{2}}^{1}$$
$$= 2\left(\ln\left(\frac{3}{2}\right)-\ln\left(\frac{\sqrt{3}}{2}\right)\right)$$
$$= 2\ln\left(\frac{3}{2}\right)-2\ln\left(\frac{\sqrt{3}}{2}\right)$$
$$= \ln\left(\frac{3}{2}\right)^2-\ln\left(\frac{\sqrt{3}}{2}\right)^2$$

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$$= \ln\left(\frac{9}{4}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{9}{4} \div \frac{3}{4}\right)$$

$$= \ln 3$$

Solutions to Exercise 4E Development questions

4a

$$\int \frac{2x+1}{x^2+2x+2}\,dx$$

$$= \int \frac{2x+1}{x^2+2x+1+1}\,dx$$

$$= \int \frac{2x+1}{(x+1)^2+1}\,dx$$

Let $u = x+1$

$du = dx$

$x = u-1$

Hence

$$\int \frac{2x+1}{(x+1)^2+1}\,dx$$

$$= \int \frac{2(u-1)+1}{u^2+1}\,du$$

$$= \int \frac{2u-1}{u^2+1}\,du$$

$$= \int \frac{2u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$$= \ln|u^2+1| - \tan^{-1}u + C$$

$$= \ln((x+1)^2+1) - \tan^{-1}(x+1) + C$$

$$= \ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

4b

$$\int \frac{x}{x^2+2x+10}\,dx$$

$$= \int \frac{x}{x^2+2x+1+9}\,dx$$

$$= \int \frac{x}{(x+1)^2+9}\,dx$$

Let $u = x+1$

Uncorrected proofs

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$du = dx$

$x = u - 1$

Hence

$$\int \frac{x}{(x+1)^2+9}\,dx$$

$$= \int \frac{u-1}{u^2+9}\,du$$

$$= \int \frac{u}{u^2+9}\,du - \int \frac{1}{u^2+9}\,du$$

$$= \frac{1}{2}\int \frac{2u}{u^2+9}\,du - \frac{1}{3}\int \frac{3}{u^2+9}\,du$$

$$= \frac{1}{2}\ln(u^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{2}\ln((x+1)^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$$

$$= \frac{1}{2}\ln(x^2+2x+10) - \frac{1}{3}\tan^{-1}\left(\frac{x+1}{3}\right) + C$$

4c

$$\int \frac{x}{\sqrt{6x-x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{-(x^2-6x)}}\,dx$$

$$= \int \frac{x}{\sqrt{-((x-3)^2-9)}}\,dx$$

$$= \int \frac{x}{\sqrt{9-(x-3)^2}}\,dx$$

Let $u = x - 3$

$du = dx$

$x = u + 3$

Hence

$$\int \frac{x}{\sqrt{9-(x-3)^2}}\,dx$$

Uncorrected proofs

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$$= \int \frac{u+3}{\sqrt{9-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{9-u^2}}\,du + \int \frac{3}{\sqrt{9-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{9-u^2}}\,du + 3\sin^{-1}\left(\frac{u}{3}\right)$$

Let $w = 9 - u^2$

$dw = -2u\,du$

Hence

$$\int \frac{u}{\sqrt{9-u^2}}\,du$$

$$= -\frac{1}{2}\int \frac{1}{\sqrt{w}}\,dw$$

$$= -\frac{1}{2} \times 2\sqrt{w}$$

$$= -\sqrt{w}$$

Hence

$$\int \frac{u}{\sqrt{9-u^2}}\,du + 3\sin^{-1}\left(\frac{u}{3}\right)$$

$$= -\sqrt{w} + 3\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= -\sqrt{9-u^2} + 3\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 3\sin^{-1}\left(\frac{x-3}{3}\right) + C$$

4d

$$\int \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$$

$$= \int \frac{x+3}{\sqrt{-(x^2+2x-4)}}\,dx$$

$$= \int \frac{x+3}{\sqrt{-((x+1)^2-5)}}\,dx$$

Uncorrected proofs

$$= \int \frac{x+3}{\sqrt{5-(x+1)^2}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = u - 1$

Hence

$$\int \frac{x+3}{\sqrt{5-(x+1)^2}}\,dx$$

$$= \int \frac{u-1+3}{\sqrt{5-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{5-u^2}}\,du + \int \frac{2}{\sqrt{5-u^2}}\,du$$

$$= -\sqrt{5-u^2} - 2\sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= -\sqrt{4-2x-x^2} - 2\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$$

4e

$$\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$$

$$= \int \frac{x}{\sqrt{(x+1)^2+9}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = u - 1$

Hence

$$\int \frac{x}{\sqrt{(x+1)^2+9}}\,dx$$

$$= \int \frac{u-1}{\sqrt{u^2+9}}\,du$$

$$= \int \frac{u}{\sqrt{u^2+9}}\,du - \int \frac{1}{\sqrt{u^2+9}}\,du$$

Uncorrected proofs

$$= \sqrt{u^2 + 9} - \ln\left(u + \sqrt{u^2 + 9}\right) + C$$

$$= \sqrt{x^2 + 2x + 10} - \ln\left(x + 1 + \sqrt{x^2 + 2x + 10}\right) + C$$

4f

$$\int \frac{x + 3}{\sqrt{x^2 - 2x - 4}} dx$$

$$= \int \frac{x + 3}{\sqrt{(x - 1)^2 - 5}} dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

Hence

$$\int \frac{x + 3}{\sqrt{(x - 1)^2 - 5}} dx$$

$$= \int \frac{u + 4}{\sqrt{u^2 - 5}} du$$

$$= \int \frac{u}{\sqrt{u^2 - 5}} du + \int \frac{4}{\sqrt{u^2 - 5}} du$$

$$= \sqrt{u^2 - 5} + 4\ln\left|u + \sqrt{u^2 - 5}\right| + C$$

$$= \sqrt{x^2 - 2x - 4} + 4\ln\left|x - 1 + \sqrt{x^2 - 2x - 4}\right| + C$$

5a

$$\int_0^2 \frac{x + 1}{x^2 + 4} dx$$

$$= \int_0^2 \frac{x}{x^2 + 4} + \frac{1}{x^2 + 4} dx$$

$$= \int_0^2 \frac{x}{x^2 + 4} + \frac{1}{2}\frac{2}{x^2 + 4} dx$$

$$= \left[\frac{1}{2}\ln|x^2 + 4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

Uncorrected proofs

$$= \frac{1}{2}\ln 8 + \frac{\pi}{8} - \frac{1}{2}\ln 4$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{8}$$

5b

$$\int_1^2 \frac{x+1}{x^2-4x+5}\,dx$$

$$= \int_1^2 \frac{x+1}{(x-2)^2+1}\,dx$$

Let $u = x - 2$

$du = dx$

$x = 2, u = 0$

$x = 1, u = -1$

$x = u + 2$

Hence

$$\int_1^2 \frac{x+1}{(x-2)^2+1}\,dx$$

$$= \int_{-1}^0 \frac{u+3}{u^2+1}\,du$$

$$= \int_{-1}^0 \left(\frac{u}{u^2+1} + \frac{3}{u^2+1}\right)du$$

$$= \left[\frac{1}{2}\ln(u^2+1) + 3\tan^{-1}u\right]_{-1}^0$$

$$= -\frac{1}{2}\ln 2 - 3\tan^{-1}(-1)$$

$$= -\frac{1}{2}\ln 2 + \frac{3\pi}{4}$$

$$= -\frac{1}{4}\times 2\ln 2 + \frac{3\pi}{4}$$

$$= -\frac{1}{4}\ln 4 + \frac{3\pi}{4}$$

Uncorrected proofs

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$= \frac{1}{4}(3\pi - \ln 4)$

5c

$$\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$$

$$= \int_1^2 \frac{2x-3}{(x-1)^2+1}\,dx$$

Let $u = x - 1$

$du = dx$

$x = 2, u = 1$

$x = 1, u = 0$

$x = u + 1$

Hence

$$\int_1^2 \frac{2x-3}{(x-1)^2+1}\,dx$$

$$= \int_0^1 \frac{2(u+1)-3}{u^2+1}\,du$$

$$= \int_0^1 \frac{2u-1}{u^2+1}\,du$$

$$= \int_0^1 \left(\frac{2u}{u^2+1} - \frac{1}{u^2+1}\right)du$$

$$= [\ln(u^2+1) - \tan^{-1}u]_0^1$$

$$= \ln 2 - \frac{\pi}{4}$$

5d

$$\int_{-1}^0 \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int_{-1}^0 \frac{x}{\sqrt{-(x^2+2x-3)}}\,dx$$

Uncorrected proofs

$$= \int_{-1}^{0} \frac{x}{\sqrt{-((x+1)^2-4)}}\,dx$$

$$= \int_{-1}^{0} \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = 0, u = 1$

$x = -1, u = 0$

$x = u - 1$

Hence

$$\int_{-1}^{0} \frac{x}{\sqrt{4-(x+1)^2}}\,dx$$

$$= \int_{0}^{1} \frac{u-1}{\sqrt{4-u^2}}\,du$$

$$= \int_{0}^{1} \left(\frac{u}{\sqrt{4-u^2}} - \frac{1}{\sqrt{4-u^2}}\right) du$$

$$= \left[-\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right)\right]_0^1$$

$$= -\sqrt{3} - \frac{\pi}{6} + 2$$

5e

$$\int_{-1}^{3} \frac{1-2x}{\sqrt{x^2+2x+3}}\,dx$$

$$= \int_{-1}^{3} \frac{1-2x}{\sqrt{(x+1)^2+2}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = 3, u = 4$

$x = -1, u = 0$

$x = u - 1$

Uncorrected proofs

Hence

$$\int_{-1}^{3} \frac{1-2x}{\sqrt{(x+1)^2+2}}\,dx$$

$$= \int_{0}^{4} \frac{1-2(u-1)}{\sqrt{u^2+2}}\,du$$

$$= \int_{0}^{4} \frac{3-2u}{\sqrt{u^2+2}}\,du$$

$$= \int_{0}^{4} \left(\frac{3}{\sqrt{u^2+2}} - \frac{2u}{\sqrt{u^2+2}}\right) du$$

$$= \left[3\ln\left|u+\sqrt{u^2+2}\right| - 2\sqrt{u^2+2}\right]_0^4$$

$$= 3\ln\left(4+3\sqrt{2}\right) - 6\sqrt{2} - 3\ln(\sqrt{2}) + 2\sqrt{2}$$

$$= 3\ln\left(\frac{4+3\sqrt{2}}{\sqrt{2}}\right) - 4\sqrt{2}$$

$$= 3\ln\left(\frac{4}{\sqrt{2}}+3\right) - 4\sqrt{2}$$

$$= 3\ln\left(2\sqrt{2}+3\right) - 4\sqrt{2}$$

5f

$$\int_{0}^{1} \frac{x+3}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int_{0}^{1} \frac{x+3}{\sqrt{(x+2)^2-3}}\,dx$$

Let $u = x+2$

$du = dx$

$x = 1, u = 3$

$x = 0, u = 2$

$x = u-2$

Hence

$$\int_{0}^{1} \frac{x+3}{\sqrt{(x+2)^2-3}}\,dx$$

Uncorrected proofs

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$$= \int_2^3 \frac{u+1}{\sqrt{u^2-3}}\,du$$

$$= \int_2^3 \left(\frac{u}{\sqrt{u^2-3}} + \frac{1}{\sqrt{u^2-3}}\right) du$$

$$= \left[\sqrt{u^2-3} + \ln\left|u+\sqrt{u^2-3}\right|\right]_2^3$$

$$= \sqrt{6} + \ln\left(3+\sqrt{6}\right) - 1 - \ln 3$$

$$= \ln\left(\frac{3+\sqrt{6}}{3}\right) + \sqrt{6} - 1$$

$$= \ln\left(1+\sqrt{\frac{2}{3}}\right) + \sqrt{6} - 1$$

6a

$$\int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x - \sqrt{1-x^2} + C$$

6b

$$\int \sqrt{\frac{3-x}{2+x}}\,dx$$

$$= \int \sqrt{\frac{3-x}{2+x}} \times \sqrt{\frac{3-x}{3-x}}\,dx$$

Uncorrected proofs

$$= \int \frac{3-x}{\sqrt{6+x-x^2}}\,dx$$

$$= \int \frac{3-x}{\sqrt{-(x^2-x-6)}}\,dx$$

$$= \int \frac{3-x}{\sqrt{-(x^2-x-6)}}\,dx$$

$$= \int \frac{3-x}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx$$

Let $u = x - \frac{1}{2}$

$du = dx$

$x = u + \frac{1}{2}$

Hence

$$\int \frac{3-x}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$= \int \frac{3-u-\frac{1}{2}}{\sqrt{\frac{25}{4}-u^2}}\,du$$

$$= \int \frac{\frac{5}{2}-u}{\sqrt{\frac{25}{4}-u^2}}\,du$$

$$= \int \left(\frac{\frac{5}{2}}{\sqrt{\frac{25}{4}-u^2}} - \frac{u}{\sqrt{\frac{25}{4}-u^2}} \right) du$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2u}{5}\right) + \sqrt{\frac{25}{4}-u^2} + C$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x-1}{5}\right) + \sqrt{6+x-x^2} + C$$

Uncorrected proofs

6c

$$\int \sqrt{\frac{x-1}{x+1}}\,dx$$

$$= \int \sqrt{\frac{x-1}{x+1}} \times \sqrt{\frac{x-1}{x-1}}\,dx$$

$$= \int \frac{x-1}{\sqrt{x^2-1}}\,dx$$

$$= \int \left(\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right)dx$$

$$= \sqrt{x^2-1} - \ln\left|x+\sqrt{x^2-1}\right| + C$$

7a

$$\int_{-1}^{0} \sqrt{\frac{1-x}{x+3}}\,dx$$

$$= \int_{-1}^{0} \sqrt{\frac{1-x}{x+3}} \times \sqrt{\frac{1-x}{1-x}}\,dx$$

$$= \int_{-1}^{0} \frac{1-x}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int_{-1}^{0} \frac{1-x}{\sqrt{4-(x+1)^2}}\,dx$$

Let $u = x+1$

$du = dx$

$x = 0, u = 1$

$x = -1, u = 0$

$x = u - 1$

$$= \int_{0}^{1} \frac{1-u+1}{\sqrt{4-u^2}}\,du$$

$$= \int_{0}^{1} \frac{2-u}{\sqrt{4-u^2}}\,du$$

Uncorrected proofs

$$= \int_0^1 \left(\frac{2}{\sqrt{4-u^2}} - \frac{u}{\sqrt{4-u^2}}\right) du$$

$$= \left[2\sin^{-1}\left(\frac{u}{2}\right) + \sqrt{4-u^2}\right]_0^1$$

$$= \frac{\pi}{3} + \sqrt{3} - 2$$

7b

$$\int_{-1}^0 \sqrt{\frac{x+2}{1-x}}\, dx$$

$$= \int_{-1}^0 \sqrt{\frac{x+2}{1-x}} \times \sqrt{\frac{x+2}{x+2}}\, dx$$

$$= \int_{-1}^0 \frac{x+2}{\sqrt{2-x-x^2}}\, dx$$

$$= \int_{-1}^0 \frac{x+2}{\sqrt{2-x-x^2}}\, dx$$

$$= \int_{-1}^0 \frac{x+2}{\sqrt{\frac{9}{4} - \left(x+\frac{1}{2}\right)^2}}\, dx$$

Let $u = x + \frac{1}{2}$

$du = dx$

$x = 0, u = \frac{1}{2}$

$x = -1, u = -\frac{1}{2}$

$x = u - \frac{1}{2}$

Hence

$$\int_{-1}^0 \frac{x+2}{\sqrt{\frac{9}{4} - \left(x+\frac{1}{2}\right)^2}}\, dx$$

Uncorrected proofs

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u - \frac{1}{2} + 2}{\sqrt{\frac{9}{4} - u^2}}\, du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u + \frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}}\, du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{u}{\sqrt{\frac{9}{4} - u^2}} + \frac{\frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}} \right) du$$

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u}{\sqrt{\frac{9}{4} - u^2}}\, du$ is an odd function. Therefore it equals 0.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{u}{\sqrt{\frac{9}{4} - u^2}} + \frac{\frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}} \right) du$$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}}\, du$ This is an even function.

$$= \int_{0}^{\frac{1}{2}} \frac{3}{\sqrt{\frac{9}{4} - u^2}}\, du$$

$$= \left[3 \sin^{-1}\left(\frac{2u}{3}\right) \right]_0^{\frac{1}{2}}$$

$$= 3 \sin^{-1}\left(\frac{1}{3}\right)$$

Uncorrected proofs

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7c

$$\int_0^1 \sqrt{\frac{x+1}{x+3}}\,dx$$

$$= \int_0^1 \sqrt{\frac{x+1}{x+3}} \times \sqrt{\frac{x+1}{x+1}}\,dx$$

$$= \int_0^1 \frac{x+1}{\sqrt{x^2+4x+3}}\,dx$$

$$= \int_0^1 \frac{x+1}{\sqrt{(x+2)^2-1}}\,dx$$

Let $u = x+2$

$du = dx$

$x = 1, u = 3$

$x = 0, u = 2$

$x = u-2$

Hence

$$\int_0^1 \frac{x+1}{\sqrt{(x+2)^2-1}}\,dx$$

$$= \int_2^3 \frac{u-2+1}{\sqrt{u^2-1}}\,du$$

$$= \int_2^3 \frac{u-1}{\sqrt{u^2-1}}\,du$$

$$= \int_2^3 \left(\frac{u}{\sqrt{u^2-1}} - \frac{1}{\sqrt{u^2-1}}\right)du$$

$$= \left[\sqrt{u^2-1} - \ln\left|u+\sqrt{u^2-1}\right|\right]_2^3$$

$$= 2\sqrt{2} - \ln\left(3+2\sqrt{2}\right) - \sqrt{3} + \ln\left(2+\sqrt{3}\right)$$

$$= 2\sqrt{2} - \sqrt{3} + \ln\left(\frac{2+\sqrt{3}}{3+2\sqrt{2}}\right)$$

Uncorrected proofs

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Ex 2
Ex 4f2

Q8 a) rationalising the numerator

$$\sqrt{\frac{x}{4-x}} = \frac{x}{\sqrt{4x-x^2}} \quad \text{for } 0 < x < 4$$

and is undefined at $x=0$, the lower limit of the integral.

b) $$\int_\varepsilon^2 \sqrt{\frac{x}{4-x}}\,dx = \int_\varepsilon^2 \frac{x}{\sqrt{4x-x^2}}\,dx$$

$$= \int_\varepsilon^2 \frac{2}{\sqrt{4-(x-2)^2}}\,dx - \int_\varepsilon^2 \frac{2-x}{\sqrt{4x-x^2}}\,dx$$

$$= \left[2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2}\right]_\varepsilon^2$$

$$= \left(2\sin^{-1} 0 - \sqrt{4}\right) - \left(2\sin^{-1}\frac{\varepsilon-2}{2} - \sqrt{4\varepsilon-\varepsilon^2}\right)$$

so taking the limit as $\varepsilon \to 0^+$

$$\int_0^2 \sqrt{\frac{x}{4-x}}\,dx = \lim_{\varepsilon\to 0^+} \int_\varepsilon^2 \sqrt{\frac{x}{4-x}}\,dx$$

$$= \lim_{\varepsilon\to 0^+} -2 - 2\sin^{-1}\left(\frac{\varepsilon-2}{2}\right) + \sqrt{4\varepsilon-\varepsilon^2}$$

$$= -2 - 2\sin^{-1}(-1) + 0$$

$$= \pi - 2.$$

Ext 2

Ex 4F2

Q9 a) $RHS = (x+1)(x^2+2x+2) + x - 1$

$= x^3 + 2x^2 + 2x$
$\quad + x^2 + 2x + 2$
$\quad + x - 1$

$= x^3 + 3x^2 + 5x + 1$

$= LHS.$

b) $\int_{-1}^{0} \frac{x^3+3x^2+5x+1}{\sqrt{x^2+2x+2}}\,dx = \int_{-1}^{0} \frac{(x+1)(x^2+2x+2)}{\sqrt{x^2+2x+2}} + \frac{x-1}{\sqrt{x^2+2x+2}}\,dx$

$= \int_{-1}^{0} (x+1)\sqrt{x^2+2x+2}\,dx$
$\quad + \int_{-1}^{0} \frac{x+1}{\sqrt{x^2+2x+2}}\,dx - 2\int_{-1}^{0} \frac{dx}{\sqrt{(x+1)^2+1}}$

$= \frac{1}{3}\left[(x^2+2x+2)^{3/2}\right]_{-1}^{0} + \left[(x^2+2x+2)^{1/2}\right]_{-1}^{0}$
$\quad - 2\left[\ln\left(x+1+\sqrt{x^2+2x+2}\right)\right]_{-1}^{0}$

$= \frac{1}{3}\left(2^{3/2} - 1\right) + \left(2^{1/2} - 1\right)$
$\quad - \left(2\ln(1+\sqrt{2}) - 2\ln 1\right)$

$= \frac{1}{3}(5\sqrt{2} - 4) - 2\ln(1+\sqrt{2})$