Solutions to Exercise 4C Foundation questions

1a Using reverse chain rule:

$\int 2x(x^2+1)^4\,dx$

Let $f(x) = x^2 + 1$

$f'(x) = 2x$

$\int 2x(x^2+1)^4\,dx$

$= \frac{1}{5}(x^2+1)^5 + C$

Using substitution:

Let $u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Thus

$\int 2x(x^2+1)^4\,dx$

$= \int u^4\,du$

$= \frac{1}{5}u^5 + C$

$= \frac{1}{5}(x^2+1)^5 + C$

1b Using reverse chain rule:

$\int 3x^2(1+x^3)^6\,dx$

Let $f(x) = 1 + x^3$

$f'(x) = 3x^2$

$\int 3x^2(1+x^3)^6\,dx$

$= \frac{1}{7}(1+x^3)^7 + C$

Using substitution:

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Let $u = 1+ x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\, dx$

Thus

$\int 3x^2(1 + x^3)^6\, dx$

$= \int u^6\, du$

$= \frac{1}{7}u^7 + C$

$= \frac{1}{7}(1+x^3)^7 + C$

1c Using reverse chain rule:

$\int \frac{6x^2}{(1 + x^3)^2}\, dx$

Let $f(x) = 1 + x^3$

$f'(x) = 3x^2$

$\int \frac{6x^2}{(1 + x^3)^2}\, dx$

$= 2\int \frac{3x^2}{(1 + x^3)^2}\, dx$

$= \frac{2}{-1}(1 + x^3)^{-1} + C$

$= -\frac{2}{1 + x^3} + C$

Using substitution:

Let $u = 1 + x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

Thus

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$$\int \frac{6x^2}{(1+x^3)^2}\,dx$$

$$= 2\int \frac{3x^2}{(1+x^3)^2}\,dx$$

$$= 2\int u^{-2}\,du$$

$$= -2u^{-1} + C$$

$$= -\frac{2}{1+x^3} + C$$

1d Using reverse chain rule:

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

Let $f(x) = 3 - x^2$

$f'(x) = -2x$

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

$$= -2\int (-2x(3-x^2)^{-5})\,dx$$

$$= \frac{-2}{-4}(3-x^2)^{-4} + C$$

$$= \frac{1}{2(3-x^2)^4} + C$$

Using substitution:

Let $u = 3 - x^2$

$$\frac{du}{dx} = -2x$$

$$du = -2x\,dx$$

Thus

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

$$= -2\int \frac{-2x}{(3-x^2)^5}\,dx$$

$$= -2\int u^{-5}\,du$$

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$$= \frac{-2}{-4}u^{-4} + C$$

$$= \frac{1}{2(3-x^2)^4} + C$$

1e Using reverse chain rule:

$$\int \frac{x}{\sqrt{x^2-2}}dx$$

Let $f(x) = x^2 - 2$

$f'(x) = 2x$

$$\int \frac{x}{\sqrt{x^2-2}}dx$$

$$= \frac{1}{2}\int 2x(x^2-2)^{-\frac{1}{2}}dx$$

$$= \frac{1}{2} \times 2(x^2-2)^{\frac{1}{2}} + C$$

$$= \sqrt{x^2-2} + C$$

Using substitution:

Let $u = x^2 - 2$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

Thus,

$$\int \frac{x}{\sqrt{x^2-2}}dx$$

$$= \frac{1}{2}\int \frac{2x}{\sqrt{x^2-2}}dx$$

$$= \frac{1}{2}\int u^{-\frac{1}{2}}du$$

$$= \frac{1}{2} \times 2u^{\frac{1}{2}} + C$$

$$= \sqrt{x^2-2} + C$$

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1f Using reverse chain rule:

$$\int \frac{x^3}{\sqrt{1+x^4}}dx$$

Let $f(x) = 1 + x^4$

$f'(x) = 4x^3$

$$\int \frac{x^3}{\sqrt{1+x^4}}dx$$

$$= \frac{1}{4}\int 4x^3(1+x^4)^{-\frac{1}{2}}dx$$

$$= \frac{1}{4} \times 2(1+x^4)^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{1+x^4} + C$$

Using substitution:

Let $u = 1 + x^4$

$$\frac{du}{dx} = 4x^3$$

$$du = 4x^3 dx$$

Thus,

$$\int \frac{x^3}{\sqrt{1+x^4}}dx$$

$$= \frac{1}{4}\int \frac{4x^3}{\sqrt{1+x^4}}dx$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}du$$

$$= \frac{1}{4} \times 2u^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{1+x^4} + C$$

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2a

$$\int \frac{\cos x}{\sin^3 x}\, dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x\, dx$$

Hence,

$$\int \frac{\cos x}{\sin^3 x}\, dx$$

$$= \int \frac{1}{u^3}\, du$$

$$= \int u^{-3}\, du$$

$$= \frac{u^{-2}}{-2} + C$$

$$= \frac{-1}{2\sin^2 x} + C$$

2b

$$\int \frac{\sec^2 x}{(1+\tan x)^2}\, dx$$

Let $u = 1 + \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x\, dx$$

Hence,

$$\int \frac{\sec^2 x}{(1+\tan x)^2}\, dx$$

$$= \int \frac{1}{u^2}\, du$$

$$= \int u^{-2}\, du$$

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$= -u^{-1} + C$

$= \dfrac{-1}{1 + \tan x} + C$

2c

$\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

Let $u = \ln x$

$\dfrac{du}{dx} = \dfrac{1}{x}$

$du = \dfrac{dx}{x}$

Hence,

$\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

$= \displaystyle\int u^2\,du$

$= \dfrac{1}{3}u^3 + C$

$= \dfrac{1}{3}(\ln x)^3 + C$

2d

$\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Let $u = \sqrt{x}$

$\dfrac{du}{dx} = \dfrac{1}{2\sqrt{x}}$

$du = \dfrac{dx}{2\sqrt{x}}$

Hence,

$\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

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$$= 2\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$$

$$= 2\int \cos u\;\,du$$

$$= 2\sin u + C$$

$$= 2\sin\sqrt{x} + C$$

2e

$$\int \frac{x}{1+x^4}dx$$

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$du = 2x\,dx$$

$$\int \frac{x}{1+x^4}dx$$

$$= \frac{1}{2}\int \frac{2x}{1^2+(x^2)^2}dx$$

$$= \frac{1}{2}\int \frac{1}{1^2+u^2}du$$

$$= \frac{1}{2}\times\frac{1}{1}\tan^{-1}u + C$$

$$= \frac{1}{2}\tan^{-1}x^2 + C$$

2f

$$\int \frac{x^2}{\sqrt{1-x^6}}dx$$

Let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\,dx$$

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$$\int \frac{x^2}{\sqrt{1-x^6}}dx$$

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{1-(x^3)^2}}dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1^2-u^2}}du$$

$$= \frac{1}{3} \times \frac{1}{1}\sin^{-1} u + C$$

$$= \frac{1}{3}\sin^{-1} x^3 + C$$

3a

$$\int_0^1 x^3(1+3x^4)^2\,dx$$

Let $u = 1 + 3x^4$

$$\frac{du}{dx} = 12x^3$$

$$du = 12x^3 dx$$

When $x = 1, u = 4.$

When $x = 0, u = 1.$

Hence

$$\int_0^1 x^3(1+3x^4)^2\,dx$$

$$= \frac{1}{12}\int_0^1 12x^3(1+3x^4)^2\,dx$$

$$= \frac{1}{12}\int_1^4 u^2\,du$$

$$= \left[\frac{1}{36}u^3\right]_1^4$$

$$= \frac{64}{36} - \frac{1}{36}$$

$$= \frac{63}{36}$$

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$= \frac{7}{4}$

3b

$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

When $x = 1, u = 3.$

When $x = 0, u = 4.$

Hence

$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int_4^3 \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int_3^4 \frac{du}{\sqrt{u}}$

$= \left[\frac{1}{2} \times 2u^{\frac{1}{2}}\right]_3^4$

$= \left[\sqrt{u}\right]_3^4$

$= 2 - \sqrt{3}$

3c

$\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$

Let $u = x^2 + 2x + 3$

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$$\frac{du}{dx} = 2x + 2 = 2(x + 1) du = 2(x + 1)\, dx$$

When $x = 4, u = 27.$

When $x = 3, u = 18.$

Hence

$$\int_3^4 \frac{x + 1}{\sqrt{x^2 + 2x + 3}} dx$$

$$= \frac{1}{2} \int_3^4 \frac{2(x + 1)}{\sqrt{x^2 + 2x + 3}} dx$$

$$= \frac{1}{2} \int_{18}^{27} \frac{du}{\sqrt{u}}$$

$$= \left[\frac{1}{2} \times 2u^{\frac{1}{2}}\right]_{18}^{27}$$

$$= \left[\sqrt{u}\right]_{18}^{27}$$

$$= \sqrt{27} - \sqrt{18}$$

$$= 3\sqrt{3} - 3\sqrt{2}$$

$$= 3\left(\sqrt{3} - \sqrt{2}\right)$$

3d

$$\int_0^{\frac{\pi}{2}} \sin^4 x \ \cos x dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x \, dx$$

When $x = \frac{\pi}{2}, u = 1.$

When $x = 0, u = 0.$

Hence,

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$$\int_0^{\frac{\pi}{2}} \sin^4 x \ \cos x dx$$

$$= \int_0^1 u^4 \, du$$

$$= \left[\frac{1}{5}u^5\right]_0^1$$

$$= \frac{1}{5} - 0$$

$$= \frac{1}{5}$$

3e

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \ dx$$

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x \, dx$$

When $x = \frac{\pi}{4}, u = 1.$

When $x = 0, u = 0.$

Hence,

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \ dx$$

$$= \int_0^1 u^2 \, du$$

$$= \left[\frac{1}{3}u^3\right]_0^1$$

$$= \frac{1}{3} - 0$$

$$= \frac{1}{3}$$

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3f

$$\int_1^{e^2} \frac{\ln x}{x}\, dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

When $x = e^2, u = 2$.

When $x = 1, u = 0$.

Hence,

$$\int_1^{e^2} \frac{\ln x}{x}\, dx$$

$$= \int_0^2 u\ du$$

$$= \left[\frac{1}{2}u^2\right]_0^2$$

$$= \frac{1}{2} \times 4 - 0$$

$$= 2$$

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Solutions to Exercise 4C Development questions

4a

$$\int_0^1 x(x-1)^5\,dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

$x = 1, u = 0$

$x = 0, u = -1$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_{-1}^0 (u+1)u^5\,du$$

$$= \int_{-1}^0 (u^6 + u^5)\,du$$

$$= \left[\frac{1}{7}u^7 + \frac{1}{6}u^6\right]_{-1}^0$$

$$= \left(\frac{1}{7}(0)^7 + \frac{1}{6}(0)^6\right) - \left(\frac{1}{7}(-1)^7 + \frac{1}{6}(-1)^6\right)$$

$$= \frac{1}{7} - \frac{1}{6}$$

$$= -\frac{1}{42}$$

4b

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_0^1 x(x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1)\,dx$$

Uncorrected proofs

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$$= \int_0^1 (x^6 - 5x^5 + 10x^4 - 10x^3 + 5x^2 - x)\, dx$$

$$= \left[\frac{1}{7}x^7 - \frac{5}{6}x^6 + 2x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{7} - \frac{5}{6} + 2 - \frac{5}{2} + \frac{5}{3} - \frac{1}{2}$$

$$= -\frac{1}{42}$$

5a

$$\int x\sqrt{x+1}\, dx$$

Let $u = \sqrt{x+1}$

$x = u^2 - 1$

$dx = 2u\, du$

Hence

$$\int x\sqrt{x+1}\, dx$$

$$= \int (u^2 - 1)u \times 2u\, du$$

$$= 2\int (u^4 - u^2)\, du$$

$$= 2\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C$$

$$= \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + C$$

$$= \frac{2}{15}\left(3(x+1)^2\sqrt{x+1} - 5(x+1)\sqrt{x+1}\right) + C$$

$$= \frac{2}{15}\left((x+1)\sqrt{x+1}\left(3(x+1)\sqrt{x+1} - 5\right)\right) + C$$

$$= \frac{2}{15}(x+1)\sqrt{x+1}(3x-2) + C$$

$$= \frac{2}{15}(3x-2)(x+1)\sqrt{x+1} + C$$

Uncorrected proofs

5b

$$\int \frac{1}{1+\sqrt{x}}\,dx$$

Let $u = 1 + \sqrt{x}$

$x = (u-1)^2$

$dx = 2(u-1)\,du$

Hence

$$\int \frac{1}{1+\sqrt{x}}\,dx$$

$$= 2\int \frac{1}{u}(u-1)\,du$$

$$= 2\int \left(1 - \frac{1}{u}\right)\,du$$

$$= 2(u - \ln|u|) + C$$

$$= 2 + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$$

$$= 2\left(1 + \sqrt{x} - \ln\left(1+\sqrt{x}\right)\right) + C$$

5c

$$\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$$

Let $u = x^{\frac{1}{4}}$

$x = u^4$

$dx = 4u^3\,du$

Hence

$$\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$$

$$= 4\int \frac{u^3}{1+u}\,du$$

$u^3 = u^3 + 1 - 1$

$u^3 + 1 = (u+1)(u^2 - u + 1)$

Uncorrected proofs

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$u^3 = (u+1)(u^2-u+1) - 1$

Hence

$$4\int \frac{u^3}{1+u}\,du$$

$$= 4\int \frac{(u+1)(u^2-u+1)-1}{1+u}\,du$$

$$= 4\int \left(u^2-u+1-\frac{1}{1+u}\right)du$$

$$= 4\left(\frac{1}{3}u^3-\frac{1}{2}u^2+u-\ln|1+u|\right)+C$$

$$= \frac{4}{3}x^{\frac{3}{4}}-2x^{\frac{1}{2}}+4x^{\frac{1}{4}}-4\ln\left(1+x^{\frac{1}{4}}\right)+C$$

$$= 4\left(\frac{1}{3}x^{\frac{3}{4}}-\frac{1}{2}x^{\frac{1}{2}}+x^{\frac{1}{4}}-\ln\left(1+x^{\frac{1}{4}}\right)\right)+C$$

$$= 4\left(\frac{1}{3}x^{\frac{3}{4}}-\frac{1}{2}\sqrt{x}+x^{\frac{1}{4}}-\ln\left(1+x^{\frac{1}{4}}\right)\right)+C$$

5d

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$$

Let $u = \sqrt{e^{2x}-1}$

$e^{2x} = u^2+1$

$$du = \frac{e^{2x}}{\sqrt{e^{2x}-1}}\,dx$$

$$du = \frac{u^2+1}{u}\,dx$$

$$dx = \frac{u}{u^2+1}\,du$$

Hence,

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$$

$$= \int \frac{1}{u}\times\frac{u}{u^2+1}\,du$$

Uncorrected proofs

$$= \int \frac{1}{u^2 + 1} du$$

$$= \tan^{-1}(u) + C$$

$$= \tan^{-1}\left(\sqrt{e^{2x} - 1}\right) + C$$

6a

$$\int_0^1 \frac{2 - x}{(2 + x)^3} dx$$

Let $u = 2 + x$

$x = u - 2$

$x = 1, u = 3$

$x = 0, u = 2$

$dx = du$

Hence

$$\int_0^1 \frac{2 - x}{(2 + x)^3} dx$$

$$= \int_2^3 \frac{2 - u + 2}{u^3} du$$

$$= \int_2^3 (4u^{-3} - u^{-2})\, du$$

$$= [-2u^{-2} + u^{-1}]_2^3$$

$$= -\frac{2}{9} + \frac{1}{3} + \frac{1}{2} - \frac{1}{2}$$

$$= \frac{1}{9}$$

Uncorrected proofs

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6b

$$\int_0^4 x\sqrt{4-x}\,dx$$

Let $u = \sqrt{4-x}$

$x = 4, u = 0$

$x = 0, u = 2$

$x = 4 - u^2$

$dx = -2u\,du$

Hence

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \int_2^0 (4-u^2)u \times -2u\,du$$

$$= 2\int_0^2 (4-u^2)u^2\,du$$

$$= 2\int_0^2 (4u^2 - u^4)\,du$$

$$= 2\left[\frac{4}{3}u^3 - \frac{1}{5}u^5\right]_0^2$$

$$= 2\left[\frac{4}{3} \times 8 - \frac{1}{5} \times 32\right]$$

$$= \frac{128}{15}$$

6c

$$\int_0^4 \frac{1}{5+\sqrt{x}}dx$$

Let $u = \sqrt{x}$

$x = 4, u = 2$

$x = 0, u = 0$

$x = u^2$

Uncorrected proofs

$dx = 2u\,du$

Hence

$$\int_0^4 \frac{1}{5+\sqrt{x}}dx$$

$$= \int_0^2 \frac{1}{5+u} \times 2u\,du$$

$$= 2\int_0^2 \frac{u}{5+u}du$$

$$= 2\int_0^2 \frac{u+5-5}{5+u}du$$

$$= 2\int_0^2 \left(1-\frac{5}{5+u}\right)du$$

$$= 2[u - 5\ln|5+u|]_0^2$$

$$= 2(2 - 5\ln 7 - 0 + 5\ln 5)$$

$$= 4 + 10\ln\left(\frac{5}{7}\right)$$

6d

$$\int_4^{12} \frac{1}{(4+x)\sqrt{x}}dx$$

Let $u = \sqrt{x}$

$x = 12, u = 2\sqrt{3}$

$x = 4, u = 2$

$x = u^2$

$dx = 2u\,du$

Hence

$$\int_4^{12} \frac{1}{(4+x)\sqrt{x}}dx$$

$$= \int_2^{2\sqrt{3}} \frac{1}{(4+u^2)u} \times 2u\,du$$

Uncorrected proofs

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$$= \int_2^{2\sqrt{3}} \frac{2}{4+u^2}\,du$$

$$= \left[\tan^{-1}\left(\frac{u}{2}\right)\right]_2^{2\sqrt{3}}$$

$$= \tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

7a

$$\int \frac{1}{(1+x)\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

Hence

$$\int \frac{1}{(1+x)\sqrt{x}}\,dx$$

$$= \int \frac{1}{(1+u^2)u}\,2u\,du$$

$$= \int \frac{2}{(1+u^2)}\,du$$

$$= 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}\left(\sqrt{x}\right) + C$$

7b

$$\int \frac{x}{\sqrt{x+1}}\,dx$$

Let $u = \sqrt{x+1}$

$x = u^2 - 1$

Uncorrected proofs

$dx = 2u\ du$

Hence

$$\int \frac{x}{\sqrt{x+1}}dx$$

$$= \int \frac{u^2-1}{u}2u\ du$$

$$= 2\int (u^2-1)\ du$$

$$= 2\left(\frac{1}{3}u^3 - u\right) + C$$

$$= \frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1} + C$$

$$= \left(\frac{2}{3}(x+1) - 2\right)\sqrt{x+1} + C$$

$$= \left(\frac{2}{3}x - \frac{4}{3}\right)\sqrt{x+1} + C$$

$$= \frac{2}{3}(x-2)\sqrt{x+1} + C$$

8a

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta\ d\theta$

Hence

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$

$$= \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}}\sec^2\theta\ d\theta$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}}d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta}d\theta$$

Uncorrected proofs

$$= \int \cos\theta \ d\theta$$

$$= \sin\theta + C$$

$$\theta = \tan^{-1} x$$

$$\sin\theta + C$$

$$= \sin(\tan^{-1} x) + C$$

$$= \frac{x}{\sqrt{x^2 + 1}} + C$$

8b

$$\int \frac{x^2}{\sqrt{4 - x^2}} dx$$

Let $x = 2\sin\theta$

$$dx = 2\cos\theta \ d\theta$$

Hence

$$\int \frac{x^2}{\sqrt{4 - x^2}} dx$$

$$= \int \frac{4\sin^2\theta}{\sqrt{4 - 4\sin^2\theta}} 2\cos\theta \ d\theta$$

$$= \int \frac{8\sin^2\theta\cos\theta}{\sqrt{4(1 - \sin^2\theta)}} d\theta$$

$$= \int \frac{8\sin^2\theta\cos\theta}{2\cos\theta} d\theta$$

$$= \int 4\sin^2\theta \ d\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

Hence

$$\int 4\sin^2\theta \ d\theta$$

$$= \int (2 - 2\cos 2\theta) \ d\theta$$

Uncorrected proofs

$= 2\theta - \sin 2\theta + C$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$

Hence

$2\theta - \sin 2\theta + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{1-\frac{x^2}{4}} + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{\frac{4-x^2}{4}} + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + C$

8c

$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

Let $x = 5\cos\theta$

$dx = -5\sin\theta\ d\theta$

Hence

$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

$= \int \frac{1}{25\cos^2\theta\sqrt{25-25\cos^2\theta}}(-5\sin\theta)\,d\theta$

$= \int \frac{-5\sin\theta}{25\cos^2\theta\sqrt{25\sin^2\theta}}\,d\theta$

$= \int \frac{-5\sin\theta}{25\cos^2\theta\,5\sin\theta}\,d\theta$

$= \int \frac{-1}{25\cos^2\theta}\,d\theta$

$= -\frac{1}{25}\int \sec^2\theta\ d\theta$

Uncorrected proofs

$$= -\frac{1}{25}\tan\theta + C$$

$$\theta = \cos^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

Hence

$$-\frac{1}{25}\tan\theta + C$$

$$= -\frac{1}{25}\frac{\sqrt{1-\cos^2\theta}}{\cos\theta} + C$$

$$= -\frac{1}{25}\frac{\sqrt{1-\frac{x^2}{25}}}{\frac{x}{5}} + C$$

$$= \frac{-\sqrt{\frac{25-x^2}{25}}}{5x} + C$$

$$= \frac{-\frac{1}{5}\sqrt{25-x^2}}{5x} + C$$

$$= \frac{-\sqrt{25-x^2}}{25x} + C$$

8d

$$\int \frac{1}{x^2\sqrt{1+x^2}}dx$$

Let $x = \tan\theta$

$$dx = \sec^2\theta\, d\theta$$

Hence

$$\int \frac{1}{x^2\sqrt{1+x^2}}dx$$

$$= \int \frac{1}{\tan^2\theta\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta$$

Uncorrected proofs

$$= \int \frac{\sec^2\theta}{\tan^2\theta \sec\theta} d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$= \int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta} d\theta$$

Let $u = \sin\theta$

$du = \cos\theta \, d\theta$

Hence

$$\int \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{\sin(\tan^{-1} x)} + C$$

$$= -\frac{1}{\sin(\tan^{-1} x)} + C$$

$$= \frac{-\sqrt{x^2+1}}{x} + C$$

9a

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}} dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$x = \sqrt{2}, \theta = \tan^{-1}\sqrt{2}$

$x = 0, \theta = 0$

Uncorrected proofs

Hence

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \frac{\tan^3\theta}{\sqrt{\tan^2\theta+1}}\sec^2\theta\,d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \frac{\tan^3\theta}{\sec\theta}\sec^2\theta\,d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \tan^3\theta\sec\theta\,d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \sec\theta\tan\theta\tan^2\theta\,d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \sec\theta\tan\theta(\sec^2\theta-1)\,d\theta$$

Let $u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$\theta = \tan^{-1}\sqrt{2}, u = \sqrt{3}$

$\theta = 0, u = 1$

Hence

$$\int_0^{\tan^{-1}\sqrt{2}} \sec\theta\tan\theta(\sec^2\theta-1)\,d\theta$$

$$= \int_1^{\sqrt{3}} u^2 - 1\,du$$

$$= \left[\frac{1}{3}u^3 - u\right]_1^{\sqrt{3}}$$

$$= \sqrt{3} - \sqrt{3} - \left(\frac{1}{3} - 1\right)$$

$$= \frac{2}{3}$$

Uncorrected proofs

9b

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$$x^3 = x(x^2+1) - x$$

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$$= \int_0^{\sqrt{2}} \frac{x(x^2+1) - x}{\sqrt{x^2+1}}\,dx$$

$$= \int_0^{\sqrt{2}} \left(x\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}\right)dx$$

From here on, the integral can be evaluated by a substitution $u = x^2$.

10a

$$\int_1^2 \sqrt{4-x^2}\,dx$$

Let $x = 2\sin\theta$

$$dx = 2\cos\theta\,d\theta$$

$$x = 2, \theta = \frac{\pi}{2}$$

$$x = 1, \theta = \frac{\pi}{6}$$

Hence

$$\int_1^2 \sqrt{4-x^2}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2}\,2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2\theta)}\,2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos\theta\,2\cos\theta\,d\theta$$

Uncorrected proofs

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Hence

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)$$

$$= \pi + 0 - \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

10b

$$\int_1^2 \sqrt{4 - x^2} \, dx$$

The function represents a semi-circle of radius 2 centred at the origin.

The area between $x = 1$ and $x = 2$ represents half of the minor segment created of a circle of radius 2.

The formula for the area of a segment:

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

$r = 2, \cos\left(\frac{\theta}{2}\right) = \frac{1}{2}$ (draw out the diagram to see the triangle)

Uncorrected proofs

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$\theta = \frac{2\pi}{3}$

Hence

$\int_1^2 \sqrt{4 - x^2}\, dx$

$= \frac{1}{2} \times$ Area of segment

$= \frac{1}{2} \times \frac{1}{2} 2^2 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

11a

$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$

Let $u = \frac{\pi}{2} - x$

$x = \frac{\pi}{2} - u$

$du = -dx$

$x = \frac{\pi}{2}, u = 0$

$x = 0, u = \frac{\pi}{2}$

Hence

$I = -\int_{\frac{\pi}{2}}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\sin\left(\frac{\pi}{2} - u\right) + \cos\left(\frac{\pi}{2} - u\right)} du$

$I = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\cos u + \sin u} du$

u is a dummy variable and can be replaced by x.

$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$

Uncorrected proofs

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11b

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$$

Hence

$$I + I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\, dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x}\, dx$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$2I = \int_0^{\frac{\pi}{2}} 1\, dx$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

12a

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

Let $u = \cos x$

$du = -\sin x\, dx$

$x = \pi, u = -1$

$x = 0, u = 1$

Hence

$$\int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

$$= -\int_1^{-1} \frac{1}{1 + u^2}\, du$$

$$= \int_{-1}^{1} \frac{1}{1 + u^2}\, du$$

Uncorrected proofs

$= [\tan^{-1} u]_{-1}^{1}$

$= \tan^{-1} 1 - \tan^{-1}(-1)$

$= \frac{\pi}{4} + \frac{\pi}{4}$

$= \frac{\pi}{2}$

12b i

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

Let $u = \pi - x$

$x = \pi - u$

$du = -dx$

$x = \pi, u = 0$

$x = 0, u = \pi$

Hence

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = -\int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1 + \cos^2(\pi - u)} du$$

$$I = \int_0^{\pi} \frac{(\pi - u)\sin(u)}{1 + \cos^2(u)} du$$

$$I = \int_0^{\pi} \frac{\pi \sin(u)}{1 + \cos^2(u)} du - \int_0^{\pi} \frac{u \sin(u)}{1 + \cos^2(u)} du$$

$$I = \pi \int_0^{\pi} \frac{\sin(u)}{1 + \cos^2(u)} du - I$$

$$I = \pi \times \frac{\pi}{2} - I$$

$$I = \frac{\pi^2}{2} - I$$

Uncorrected proofs

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12b ii

$$I = \frac{\pi^2}{2} - I$$

$$2I = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4}$$

Uncorrected proofs

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Ex 12

Ex 4C

Q 13 a) put $x = \sin\theta$, $0 \le \theta \le \pi/2$ so that $\cos\theta \ge 0$.

$dx = \cos\theta\, d\theta$

at $x = 0$ $\theta = 0$

at $x = \frac{1}{2}$ $\theta = \frac{\pi}{6}$

so $I = \int_0^{\pi/6} \frac{x^2}{\sqrt{1-x^2}}\, dx$

$= \int_0^{\pi/6} \frac{\sin^2\theta \cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$

$= \int_0^{\pi/6} \frac{\sin^2\theta \cos\theta\, d\theta}{\cos^2\theta}$ (by Pythagoras)

$= \int_0^{\pi/6} \frac{\sin^2\theta \cos\theta\, d\theta}{\cos\theta}$ (since $\cos\theta \ge 0$)

$= \int_0^{\pi/6} \sin^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\pi/6} 1 - \cos 2\theta\, d\theta$ (double angle)

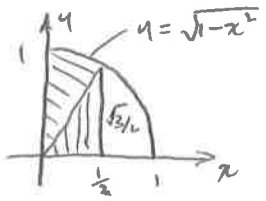
$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$

$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right) - 0$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

b) $I = \int_0^{1/2} \frac{x^2-1}{\sqrt{1-x^2}}\, dx + \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$

$= -\int_0^{1/2} \sqrt{1-x^2}\, dx + \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$



$= -$ sector $- \Delta + \left[\sin^{-1} x\right]_0^{1/2}$

$= -\frac{1}{2}\cdot 1^2 \cdot \frac{\pi}{6} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$.

Ext 2

Ex 4C

Q 14 $\quad I = \int \frac{dx}{x\sqrt{x^2-1}}$

a) put $u = \sqrt{x^2-1}$ so that $x^2 = u^2+1$

$$du = \frac{x}{\sqrt{x^2-1}}\,dx$$

so $$I = \int \frac{x\,dx}{x^2\sqrt{x^2-1}}$$

$$= \int \frac{du}{u^2+1}$$

$$= \tan^{-1} u + C_1$$

$$= \tan^{-1}\sqrt{x^2-1} + C_1$$

b) put $u = -\sqrt{x^2-1}$ so that $x^2 = u^2+1$ again

$$du = \frac{-x\,dx}{\sqrt{x^2-1}}$$

so $$I = -\int \frac{-x\,dx}{x^2\sqrt{x^2-1}}$$

$$= -\int \frac{du}{u^2+1}$$

$$= -\tan^{-1} u + C_2$$

$$= -\tan^{-1}\left(-\sqrt{x^2-1}\right) + C_2$$

$$= \tan^{-1}\sqrt{x^2-1} + C_2 \quad \text{(since } \tan^{-1} \text{ is odd)}$$

Ext 2
Ex 4C
Q15 a) $I = \int_{2+\varepsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$

put $x = 2\sec\theta$, $0 \le \theta < \pi/2$, so that $x > 0$.

so that $\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4}$

$= \sqrt{4\tan^2\theta}$

$= 2\tan\theta$ for this domain

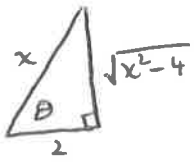
$dx = 2\sec\theta\tan\theta\, d\theta$

so $\int \frac{dx}{x^2\sqrt{x^2-4}} = \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta\, 2\tan\theta}$

$= \frac{1}{4}\int \cos\theta\, d\theta$

$= \frac{1}{4}\sin\theta + C$

$= \frac{1}{4}\sin\frac{\sqrt{x^2-4}}{x} + C$



Thus $I = \left[\frac{\sqrt{x^2-4}}{4x}\right]_{2+\varepsilon}^{4}$

$= \frac{\sqrt{12}}{16} - \frac{\sqrt{4\varepsilon+\varepsilon^2}}{4(2+\varepsilon)} = \frac{\sqrt{3}}{8} - \frac{\sqrt{4\varepsilon+\varepsilon^2}}{4(2+\varepsilon)}$

b) $\lim_{\varepsilon\to 0^+} I = \lim_{\varepsilon\to 0^+} \frac{\sqrt{3}}{8} - \frac{\sqrt{4\varepsilon+\varepsilon^2}}{4(2+\varepsilon)}$

$= \frac{\sqrt{3}}{8} - 0$

$= \frac{\sqrt{3}}{8}$.

Ext 2

Ex 4C

Q15 Here is another way to do part a)

$$I = \int \frac{dx}{x^2\sqrt{x^2-4}} \qquad \text{with } x > 2$$

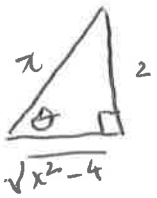
$$= \int \frac{dx}{x^3\sqrt{1-\frac{4}{x^2}}} \qquad (\text{since } x>0,\ \sqrt{x^2} = x)$$

now put $\theta = \sin^{-1}\frac{2}{x}$ so that $\cos\theta > 0$

then $\sin\theta = \frac{2}{x}$

$$\cos\theta\, d\theta = -\frac{2}{x^2}\, dx$$

$$-\frac{1}{2}\cos\theta\, d\theta = \frac{1}{x^2}\, dx$$



so $I = \frac{1}{4}\int \frac{2}{x}\cdot\frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}\cdot\frac{dx}{x^2}$

$$= \frac{1}{2}\int \sin\theta\,\frac{1}{\cos\theta}\cdot\left(-\frac{1}{2}\right)\cos\theta\, d\theta$$

$$= -\frac{1}{4}\int \sin\theta\, d\theta$$

$$= \frac{1}{4}\cos\theta + c$$

$$= \frac{1}{4}\,\frac{\sqrt{x^2-4}}{x}$$

so $\displaystyle\int_{2+\varepsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}} = \left[\frac{\sqrt{x^2-4}}{4x}\right]_{2+\varepsilon}^{4}$

$$= \frac{\sqrt{12}}{16} - \frac{\sqrt{\varepsilon(4+\varepsilon)}}{4(2+\varepsilon)} \qquad \left(\text{diff of 2 sq. in numerator}\right)$$

$$= \frac{\sqrt{3}}{8} - \frac{\sqrt{\varepsilon(4+\varepsilon)}}{4(2+\varepsilon)} \qquad (\text{as before})$$