Solutions to Exercise 4B Foundation questions

1a

$$\int \frac{x}{x-1}\,dx$$
$$= \int \frac{x-1+1}{x-1}\,dx$$
$$= \int \left(\frac{x-1}{x-1}+\frac{1}{x-1}\right)dx$$
$$= \int \left(1+\frac{1}{x-1}\right)dx$$
$$= x+\ln|x-1|+C$$

1b

$$\int \frac{x-1}{x+1}\,dx$$
$$= \int \frac{x+1-2}{x+1}\,dx$$
$$= \int \left(\frac{x+1}{x+1}-\frac{2}{x+1}\right)dx$$
$$= \int \left(1-\frac{2}{x+1}\right)dx$$
$$= x-2\ln|x+1|+C$$

1c

$$\int \frac{x+1}{x-1}\,dx$$
$$= \int \frac{x-1+2}{x-1}\,dx$$
$$= \int \left(\frac{x-1}{x-1}+\frac{2}{x-1}\right)dx$$
$$= \int \left(1+\frac{2}{x-1}\right)dx$$
$$= x+2\ln|x-1|+C$$

© Cambridge University Press 2019

2a

$$\int_0^1 \frac{x-1}{x+1}\,dx$$

$$= \int_0^1 \frac{x+1-2}{x+1}\,dx$$

$$= \int_0^1 \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right)dx$$

$$= \int_0^1 \left(1 - \frac{2}{x+1}\right)dx$$

$$= [x - 2\ln|x+1|]_0^1$$

$$= (1 - 2\ln 2) - (0 - 2\ln 1)$$

$$= 1 - \ln 2^2 - 0 + 0$$

$$= 1 - \ln 4$$

2b

$$\int_0^2 \frac{x}{2x+1}\,dx$$

$$= \int_0^2 \left(\frac{1}{2} + \frac{-\frac{1}{2}}{(2x+1)}\right)dx \qquad \text{(by long division)}$$

$$= \int_0^2 \left(\frac{1}{2} - \frac{1}{2(2x+1)}\right)dx$$

Let $f(x) = 2x+1, f'(x) = 2$

Hence,

$$\int_0^2 \left(\frac{1}{2} - \frac{1}{2(2x+1)}\right)dx$$

$$= \int_0^2 \left(\frac{1}{2} - \frac{2}{4(2x+1)}\right)dx$$

$$= \int_0^2 \frac{1}{2}\,dx - \frac{1}{4}\int_0^2 \frac{2}{2x+1}\,dx$$

$$= \left[\frac{x}{2} - \frac{1}{4}\ln|2x+1|\right]_0^2$$

© Cambridge University Press 2019

$$= \left(1 - \frac{1}{4}\ln 5\right) - \left(0 - \frac{1}{4}\ln 1\right)$$

$$= 1 - \frac{1}{4}\ln 5$$

2c

$$\int_0^1 \frac{3 - x^2}{1 + x^2}\,dx$$

$$= \int_0^1 \left(-1 + \frac{4}{1 + x^2}\right) dx \qquad \text{(by long division)}$$

$$= \int_0^1 -1\,dx + 4\int_0^1 \frac{1}{1 + x^2}\,dx$$

$$= [-x]_0^1 + 4[\tan^{-1} x]_0^1$$

$$= -1 + 0 + 4\tan^{-1} 1 - 0$$

$$= -1 + 4 \times \frac{\pi}{4}$$

$$= -1 + \pi$$

$$= \pi - 1$$

3a

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1 - x}{\sqrt{1 - x^2}}\,dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{1 - x^2}} - \frac{x}{\sqrt{1 - x^2}}\right) dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1 - x^2}}\,dx - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}}\,dx$$

If $y = (1 - x^2)^{\frac{1}{2}}$, then

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

© Cambridge University Press 2019

Hence,

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^{\frac{\sqrt{3}}{2}} \frac{-x}{\sqrt{1-x^2}}\,dx$$

$$= \left[\sin^{-1} x\right]_0^{\frac{\sqrt{3}}{2}} + \left[\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \left(\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0\right) + \left(\frac{1}{2} - 1\right)$$

$$= \frac{\pi}{3} - \frac{1}{2}$$

3b

$$\int_0^1 \frac{2x+1}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{2x}{1+x^2} + \frac{1}{1+x^2}\right)dx$$

$$= \int_0^1 \frac{2x}{1+x^2}\,dx + \int_0^1 \frac{1}{1+x^2}\,dx$$

If $y = 1 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^1 \frac{2x}{1+x^2}\,dx + \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= [\ln(1+x^2)]_0^1 + [\tan^{-1}(x)]_0^1$$

$$= (\ln 2 - \ln 1) + (\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \ln 2 - 0 + \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} + \ln 2$$

© Cambridge University Press 2019

3c

$$\int_0^1 \frac{1-x}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

If $y = 1 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= [\tan^{-1} x]_0^1 - \frac{1}{2}[\ln(1+x^2)]_0^1 \qquad (\text{since } 1 + x^2 > 0)$$

$$= (\tan^{-1} 1 - \tan^{-1} 0) - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - 0 - \frac{\ln 2}{2} + 0$$

$$= \frac{\pi}{4} - \frac{2\ln 2}{4}$$

$$= \frac{1}{4}(\pi - \ln 2^2)$$

$$= \frac{1}{4}(\pi - \ln 4)$$

3d

$$\int_0^2 \frac{1+x}{4+x^2}\,dx$$

$$= \int_0^2 \left(\frac{1}{4+x^2} + \frac{x}{4+x^2}\right) dx$$

© Cambridge University Press 2019

$$= \int_0^2 \frac{1}{4+x^2}\,dx + \int_0^2 \frac{x}{4+x^2}\,dx$$

If $y = 4 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^2 \frac{1}{4+x^2}\,dx + \int_0^2 \frac{x}{4+x^2}\,dx$$

$$= \int_0^2 \frac{1}{4+x^2}\,dx + \frac{1}{2}\int_0^2 \frac{2x}{4+x^2}\,dx$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 + \frac{1}{2}[\ln(4+x^2)]_0^2 \qquad (\text{since } 4 + x^2 > 0)$$

$$= \frac{1}{2}[(\tan^{-1} 1 - \tan^{-1} 0) + (\ln 8 - \ln 4)]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0 + \ln 8 - \ln 4\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \ln 2^3 - \ln 2^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + 3\ln 2 - 2\ln 2\right]$$

$$= \frac{\pi}{8} + \frac{1}{2}\ln 2$$

4a

$$y = \log\left(x + \sqrt{x^2 + a^2}\right)$$

Let $u = x + (x^2 + a^2)^{\frac{1}{2}}$

$$\frac{du}{dx} = 1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \times 2x$$

$$= 1 + \frac{x}{(x^2 + a^2)^{\frac{1}{2}}}$$

Hence,

$$y = \log u$$

© Cambridge University Press 2019

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times \left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$\frac{dy}{dx} = \frac{\left(\frac{x + (x^2 + a^2)^{\frac{1}{2}}}{(x^2 + a^2)^{\frac{1}{2}}}\right)}{x + (x^2 + a^2)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{(x^2 + a^2)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}}$$

4b

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx$$

$$= \log\left(x + \sqrt{x^2 + a^2}\right) + C$$

(since $\sqrt{x^2 + a^2} \geq |x|$ and so $x + \sqrt{x^2 + a^2}$ is not negative)

4c i

$$\int \frac{1}{\sqrt{x^2 + 3}} dx$$

$$= \int \frac{1}{\sqrt{x^2 + \left(\sqrt{3}\right)^2}} dx$$

$$= \log\left(x + \sqrt{x^2 + 3}\right) + C$$

(since $\sqrt{x^2 + 3} \geq |x|$ and so $x + \sqrt{x^2 + 3}$ is not negative)

© Cambridge University Press 2019

4c ii

$$\int_{-4}^{4} \frac{1}{\sqrt{x^2+9}}$$

$$= \int_{-4}^{4} \frac{1}{\sqrt{x^2+3^2}}$$

$$= \left[\log\left(x+\sqrt{x^2+9}\right)\right]_{-4}^{4}$$

$$= \log 9 - \log 1$$

$$= \log 9 - 0$$

$$= \log 9$$

$$= \log 3^2$$

$$= 2\log 3$$

© Cambridge University Press 2019

Solutions to Exercise 4B Development questions

5a

$$\int \frac{x^3}{x^2+1}\,dx$$

$x^3 = x(x^2+1) - x$

Hence

$$\int \frac{x^3}{x^2+1}\,dx$$

$$= \int \frac{x(x^2+1)-x}{x^2+1}\,dx$$

$$= \int \left(x - \frac{x}{x^2+1}\right)dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$$

Note: Modulus function not needed since $x^2 + 1$ is always greater than 0.

5b

$$\int \frac{x^3}{x+1}\,dx$$

$x^3 = (x^3+1) - 1$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

Hence

$$\int \frac{x^3}{x+1}\,dx$$

$$= \int \frac{(x^3+1)-1}{x+1}\,dx$$

$$= \int \frac{(x+1)(x^2-x+1)-1}{x+1}\,dx$$

$$= \int \left(x^2 - x + 1 - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1| + C$$

Uncorrected proofs

© Cambridge University Press 2019

Chapter 4 worked solutions – Integration

5c i

$$\int \frac{x^3}{x-1}dx$$

$$x^3 = x^3 - 1 + 1$$

$$x^3 - 1 = (x-1)(x^2+x+1)$$

Hence

$$\int \frac{x^3}{x-1}dx$$

$$= \int \frac{(x-1)(x^2+x+1)+1}{x-1}dx$$

$$= \int \left(x^2+x+1+\frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + C$$

5c ii

$$\int \frac{x^4}{x^2+1}dx$$

$$x^4 = x^4 - 1 + 1$$

$$x^4 - 1 = (x^2-1)(x^2+1)$$

Hence

$$\int \frac{x^4}{x^2+1}dx$$

$$= \int \frac{(x^2-1)(x^2+1)+1}{x^2+1}dx$$

$$= \int \left(x^2-1+\frac{1}{x^2+1}\right)dx$$

$$= \frac{1}{3}x^3 - x + \tan^{-1}x + C$$

Uncorrected proofs

© Cambridge University Press 2019

5c iii

$$\int \frac{1}{1+e^x}\,dx$$

$$= \int \frac{1}{1+e^x} \times \frac{e^{-x}}{e^{-x}}\,dx$$

$$= \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

Let $u = e^{-x} + 1$

$du = -e^{-x}dx$

Hence

$$= \int \frac{e^{-x}}{e^{-x}+1}\,dx$$

$$= -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln(e^{-x}+1) + C$$

5c iv

$$\int \frac{x}{\sqrt{2+x}}\,dx$$

Let $u = 2 + x$

$du = dx$

Hence

$$\int \frac{x}{\sqrt{2+x}}\,dx$$

$$= \int \frac{u-2}{\sqrt{u}}\,du$$

$$= \int \left(\frac{u}{u^{\frac{1}{2}}} - \frac{2}{u^{\frac{1}{2}}}\right)du$$

$$= \int \left(u^{\frac{1}{2}} - 2u^{-\frac{1}{2}}\right)du$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{2}{3}u^{\frac{3}{2}} - 4u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(2+x)^{\frac{3}{2}} - 4(2+x)^{\frac{1}{2}} + C$$

Note: Square root notation is also fine.

5c v

$$\int \frac{x}{\sqrt{1-x}}\,dx$$

Let $u = 1 - x$

$du = -dx$

Hence

$$\int \frac{x}{\sqrt{1-x}}\,dx$$

$$= -\int \frac{1-u}{\sqrt{u}}\,du$$

$$= -\int \left(\frac{1}{u^{\frac{1}{2}}} - \frac{u}{u^{\frac{1}{2}}}\right)du$$

$$= -\int \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right)\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(1-x)^{\frac{3}{2}} - 2(1-x)^{\frac{1}{2}} + C$$

5c vi

$$\int \frac{x^3}{x^2+4}\,dx$$

$$= \int \frac{x^3 + 4x - 4x}{x^2+4}\,dx$$

$$= \int \frac{x(x^2+4) - 4x}{x^2+4}\,dx$$

Uncorrected proofs

$$= \int \left(x - \frac{4x}{x^2+4}\right) dx$$

$$= \frac{1}{2}x^2 - 2\ln|x^2+4| + C$$

6a

$$\int_1^2 \frac{e^{2x}+1}{e^{2x}-1} dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} \times \frac{e^{-x}}{e^{-x}} dx$$

$$= \int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

Let $u = e^x - e^{-x}$

$du = e^x + e^{-x} dx$

$x = 2, u = e^2 - e^{-2}$

$x = 1, u = e - e^{-1}$

Hence

$$\int_1^2 \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$$

$$= \int_{e-e^{-1}}^{e^2-e^{-2}} \frac{1}{u} du$$

$$= [\ln|u|]_{e-e^{-1}}^{e^2-e^{-2}}$$

$$= \ln(e^2 - e^{-2}) - \ln(e - e^{-1})$$

$$= \ln\left(\frac{e^2 - e^{-2}}{e - e^{-1}}\right)$$

6b

$$\int_0^1 \frac{e^x}{e^x + e^{-x}} dx$$

$$= \int_0^1 \frac{e^x}{e^x + e^{-x}} \times \frac{e^x}{e^x} dx$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}\,dx$$

$$= \frac{1}{2}[\ln|e^{2x}+1|]_0^1$$

$$= \frac{1}{2}(\ln(e^2+1) - \ln(2))$$

$$= \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$$

6c

$$\int_1^{\sqrt{3}} \frac{2+\frac{1}{x}}{x+\frac{1}{x}}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{2x+1}{x^2+1}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{2x}{x^2+1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx$$

$$= [\ln(x^2+1)]_1^{\sqrt{3}} + [\tan^{-1}x]_1^{\sqrt{3}}$$

$$= \ln 4 - \ln 2 + \tan^{-1}\sqrt{3} - \tan^{-1}1$$

$$= \ln 2 + \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \ln 2 + \frac{\pi}{12}$$

7a

$$\int \frac{x^2+x+1}{x+1}\,dx$$

$$= \int \left(\frac{x^2}{x+1}+1\right)dx$$

$$= \int \left(\frac{x^2-1+1}{x+1}+1\right)dx$$

$$= \int \left(\frac{(x-1)(x+1)+1}{x+1}+1\right)dx$$

Uncorrected proofs

$$= \int \left(x - 1 + \frac{1}{x+1} + 1\right) dx$$

$$= \int \left(x + \frac{1}{x+1}\right) dx$$

$$= \frac{1}{2}x^2 + \ln|x+1| + C$$

7b

$$\int \frac{x^3 - 2x^2 + 3}{x-2} dx$$

$$= \int \frac{x^2(x-2) + 3}{x-2} dx$$

$$= \int \left(x^2 + \frac{3}{x-2}\right) dx$$

$$= \frac{1}{3}x^3 + 3\ln|x-2| + C$$

7c

$$\int \frac{(x+1)^2}{1+x^2} dx$$

$$= \int \frac{x^2 + 2x + 1}{1+x^2} dx$$

$$= \int \left(1 + \frac{2x}{1+x^2}\right) dx$$

$$= x + \ln(1+x^2) + C$$

8a $y = \ln\left(x + \sqrt{x^2 - a^2}\right), x > |a|$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - a^2}} \times \left(1 + \frac{x}{\sqrt{x^2 - a^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2 - a^2}} \times \left(\frac{x + \sqrt{x^2 - a^2}}{\sqrt{x^2 - a^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - a^2}}$$

Uncorrected proofs

8b

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$$

8c i

$$\int \frac{1}{\sqrt{x^2 - 5}} dx$$

$$= \int \frac{1}{\sqrt{x^2 - \sqrt{5}^2}} dx$$

$$= \ln\left(x + \sqrt{x^2 - 5}\right) + C$$

8c ii

$$\int_{\sqrt{5}}^{3} \frac{1}{\sqrt{x^2 - 4}} dx$$

$$= \left[\ln\left(x + \sqrt{x^2 - 4}\right)\right]_{\sqrt{5}}^{3}$$

$$= \ln(3 + \sqrt{5}) - \ln(\sqrt{5} + 1)$$

$$= \ln\left(\frac{\sqrt{5} + 3}{\sqrt{5} + 1}\right)$$

$$= \ln\left(\frac{\sqrt{5} + 1 + 2}{\sqrt{5} + 1}\right)$$

$$= \ln\left(1 + \frac{2}{\sqrt{5} + 1} \times \frac{\sqrt{5} - 1}{\sqrt{5} - 1}\right)$$

$$= \ln\left(1 + \frac{\sqrt{5} - 1}{2}\right)$$

$$= \ln\left(\frac{\sqrt{5} + 1}{2}\right)$$

Uncorrected proofs

© Cambridge University Press 2019

Ext 2

Ex 4B

Q 9

$$\int \frac{dx}{x+\sqrt{x}} = 2\int \frac{\frac{1}{2}\,1/\sqrt{x}}{\sqrt{x}+1}\,dx$$

$$= 2\log(\sqrt{x}+1) + C \qquad (\text{note } x>0)$$

Note: there is no need for any absolute value here since $\sqrt{x}+1 \geq 0$ for all $x \geq 0$.

In these solutions, absolute values will only be included when there is need.

Note: This can be done by substitution (as in 4C)

Put $u = \sqrt{x}$ then $u^2 = x$

so $2u\,du = dx$

hence $$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\,du}{u^2+u}$$

$$= 2\int \frac{du}{u+1} \qquad (\text{for } u>0)$$

$$= 2\log(u+1) + C$$

$$= 2\log(\sqrt{x}+1) + C \quad \text{as before.}$$

Ex 2

Ex 4B

Q10 $$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}\left(x+\sqrt{x^2-a^2}\right)}\,dx$$

$$= \int \frac{1+\frac{x}{\sqrt{x^2-a^2}}}{\left(x+\sqrt{x^2-a^2}\right)}\,dx$$

$$= \log\left|x+\sqrt{x^2-a^2}\right| + c$$

for $|x| > |a|$