Solutions to Exercise 4A Foundation questions

Let C be a constant.

1a

$$\int e^{4x}\,dx$$

$$= \frac{1}{4}e^{4x} + C$$

1b

$$\int \sin 5x\,dx$$

$$= -\frac{1}{5}\cos 5x + C$$

1c

$$\int \sec^2 \frac{1}{2}x\,dx$$

$$= 2\tan\frac{1}{2}x + C$$

1d

$$\int \frac{1}{3x-4}dx$$

Let $u = 3x - 4$

$$\frac{du}{dx} = 3$$

Hence,

$$\int \frac{1}{3x-4}dx$$

$$= \frac{1}{3}\int \frac{3}{u}dx$$

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$$= \frac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|3x - 4| + C$$

1e

$$\int \frac{2}{\sqrt{x}}\,dx$$

$$= 2\int x^{-\frac{1}{2}}\,dx$$

$$= 2 \times \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= 4\sqrt{x} + C$$

1f

$$\int 3^x\,dx$$

$$= \frac{3^x}{\ln 3} + C$$

2a

$$\int \frac{1}{(2x-1)^2}\,dx$$

$$= \int (2x-1)^{-2}\,dx$$

$$= \frac{(2x-1)^{-1}}{2 \times -1} + C$$

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$$= -\frac{1}{2(2x-1)} + C$$

2b

$$\int \frac{1}{\sqrt{25-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5^2-x^2}}\,dx$$

Let $f(x) = x, f'(x) = 1$

Hence,

$$\int \frac{1}{\sqrt{5^2-x^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{5}\right) + C$$

2c

$$\int x^2 e^{x^3}\,dx$$

$f(x) = x^3, f'(x) = 3x^2$

Hence,

$$\int x^2 e^{x^3}\,dx$$

$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$

$$= \frac{1}{3}\int f'(x) e^{f(x)}\,dx$$

$$= \frac{1}{3}\frac{e^{f(x)}}{\ln e} + C$$

$$= \frac{1}{3}e^{x^3} + C$$

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2d

$$\int \frac{1}{9+x^2}dx$$

$$= \int \frac{1}{3^2+x^2}dx$$

Let $f(x) = x, f'(x) = 1$

Hence,

$$\int \frac{1}{3^2+x^2}dx$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

2e

$$\int \frac{4x+2}{x^2+x+1}dx$$

Let $f(x) = x^2 + x + 1, f'(x) = 2x + 1$

Hence,

$$\int \frac{4x+2}{x^2+x+1}dx$$

$$= 2\int \frac{2x+1}{x^2+x+1}dx$$

$$= 2\ln|x^2+x+1| + C$$

$$= 2\ln(x^2+x+1) + C \qquad (\text{since } x^2+x+1 > 0)$$

2f

$$\int 2x(x^2+1)^4 dx$$

Let $f(x) = x^2 + 1, f'(x) = 2x$

Hence,

$$\int 2x(x^2+1)^4 dx$$

$$= \frac{1}{5}(x^2+1)^5 + C$$

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3a

$$\int_0^4 e^{\frac{x}{2}}\,dx$$

Let $f(x) = \frac{x}{2}, f'(x) = \frac{1}{2}$

Hence,

$$\int_0^4 e^{\frac{x}{2}}\,dx$$

$$= 2\int_0^4 \frac{1}{2}e^{\frac{x}{2}}\,dx$$

$$= 2\left[e^{\frac{x}{2}}\right]_0^4$$

$$= 2(e^2 - e^0)$$

$$= 2(e^2 - 1)$$

3b

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\ dx$$

Let $f(x) = 2x, f'(x) = 2$

Hence,

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{8}} 2\sec^2 2x\ dx$$

$$= \frac{1}{2}[\tan 2x]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= \frac{1}{2}$$

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3c

$$\int_{-4}^{4} \frac{1}{16+x^2}\,dx$$

$$= \int_{-4}^{4} \frac{1}{4^2+x^2}\,dx$$

Let $f(x) = x, f'(x) = 1$

Hence,

$$\int_{-4}^{4} \frac{1}{16+x^2}\,dx$$

$$= \left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= \frac{1}{4}\tan^{-1}1 - \frac{1}{4}\tan^{-1}(-1)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{8}$$

3d

$$\int_{0}^{1} \frac{1}{\sqrt{2-x^2}}\,dx$$

$$= \int_{0}^{1} \frac{1}{\sqrt{(\sqrt{2})^2-x^2}}\,dx$$

Let $f(x) = x, f'(x) = 1$

Hence,

$$\int_{0}^{1} \frac{1}{\sqrt{(\sqrt{2})^2-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{0}^{1}$$

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$$= \left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0\right)$$

$$= \frac{\pi}{4}$$

3e

$$\int_{-2}^{-1} \frac{3}{2-3x}\, dx$$

Let $f(x) = 2 - 3x, f'(x) = -3$

Hence,

$$\int_{-2}^{-1} \frac{3}{2-3x}\, dx$$

$$= -\int_{-2}^{-1} \frac{-3}{2-3x}\, dx$$

$$= -[\ln|2-3x|]_{-2}^{-1}$$

$$= -(\ln 5 - \ln 8)$$

$$= \ln 8 - \ln 5$$

$$= \frac{\ln 8}{\ln 5}$$

3f

$$\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\, dx$$

Let $f(x) = \sin x, f'(x) = \cos x$

Hence,

$$\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\, dx$$

$$= \left[\frac{1}{4}\sin^4 x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\sin^4 \frac{\pi}{4} - \sin^4 0\right)$$

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$$= \frac{1}{4}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right)$$

$$= \frac{1}{16}$$

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Solutions to Exercise 4A Development questions

4a

$$\int -\frac{1}{x^2} e^{\frac{1}{x}}\, dx$$

Let $u = \frac{1}{x}$

$du = -\frac{1}{x^2} dx$

Hence

$$\int -\frac{1}{x^2} e^{\frac{1}{x}}\, dx$$

$$= \int e^u\, du$$

$$= e^u + C$$

$$= e^{\frac{1}{x}} + C$$

4b

$$\int \frac{\cos 3x}{1 + \sin 3x} dx$$

Let $u = 1 + \sin 3x$

$du = 3\cos 3x\, dx$

Hence

$$\int \frac{\cos 3x}{1 + \sin 3x} dx$$

$$= \frac{1}{3}\int \frac{1}{u} du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|1 + \sin 3x| + C$$

Uncorrected proofs

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4c

$$\int x \sec^2(x^2)\, dx$$

Let $u = x^2$

$du = 2x\, dx$

Hence

$$\int x \sec^2(x^2)\, dx$$

$$= \frac{1}{2}\int 2x \sec^2(x^2)\, dx$$

$$= \frac{1}{2}\int \sec^2 u\, du$$

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan(x^2) + C$$

4d

$$\int 5^{2x}\, dx$$

$$= \int \left(e^{\ln 5}\right)^{2x} dx$$

$$= \int e^{2\ln(5)x}\, dx$$

$$= \frac{1}{2\ln 5} e^{2\ln(5)x} + C$$

$$= \frac{1}{2\ln 5} 5^{2x} + C$$

4e

$$\int \frac{1 + \sec^2 x}{x + \tan x} dx$$

Let $u = x + \tan x$

$du = 1 + \sec^2 x\, dx$

Uncorrected proofs

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Hence

$$\int \frac{1+\sec^2 x}{x+\tan x}dx$$

$$= \int \frac{1}{u}du$$

$$= \ln|u| + C$$

$$= \ln|x+\tan x| + C$$

4f

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$$

Let $u = e^x$

$du = e^x dx$

Hence

$$\int \frac{e^x}{\sqrt{1-(e^x)^2}}dx$$

$$= \int \frac{1}{\sqrt{1-u^2}}du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(e^x) + C$$

5a

$$\int_0^4 (1-x)^3\,dx$$

Let $u = 1 - x$

$du = -1\,dx$

$x = 4, u = -3$

$x = 0, u = 1$

Hence

$$\int_0^4 (1-x)^3\,dx$$

Uncorrected proofs

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$$= -\int_1^{-3} u^3\, du$$

$$= \int_{-3}^{1} u^3\, du$$

$$= \left[\frac{1}{4}u^4\right]_{-3}^{1}$$

$$= \frac{1}{4} - \frac{1}{4} \times (-3)^4$$

$$= \frac{1}{4} - \frac{1}{4} \times 81$$

$$= -20$$

5b

$$\int_0^1 \frac{x^2}{1+x^3}\, dx$$

Let $u = 1 + x^3$

$du = 3x^2\, dx$

$x = 1, u = 2$

$x = 0, u = 1$

Hence

$$\int_0^1 \frac{x^2}{1+x^3}\, dx$$

$$= \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\, dx$$

$$= \frac{1}{3}\int_1^2 \frac{1}{u}\, du$$

$$= \frac{1}{3}[\ln|u|]_1^2$$

$$= \frac{1}{3}(\ln 2 - \ln 1)$$

$$= \frac{1}{3}\ln 2$$

Uncorrected proofs

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5c

$$\int_0^1 \frac{dx}{1+3x^2}$$

$$= \frac{1}{3}\int_0^1 \frac{dx}{\frac{1}{3}+x^2}$$

$$= \frac{1}{3}\times\sqrt{3}\int_0^1 \frac{\frac{1}{\sqrt{3}}}{\left(\frac{1}{\sqrt{3}}\right)^2+x^2}dx$$

$$= \frac{\sqrt{3}}{3}\left[\tan^{-1}\left(\sqrt{3}x\right)\right]_0^1$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}-0\right)$$

$$= \frac{\sqrt{3}\pi}{9}$$

Note: This is the rationalised answer. $\frac{\pi}{3\sqrt{3}}$ is also acceptable.

5d

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}dx$$

Let $u = e^{2x}+1$

$du = 2e^{2x}dx$

$x = 1, u = e^2+1$

$x = 0, u = 2$

Hence

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}dx$$

$$= \frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}dx$$

$$= \frac{1}{2}\int_2^{e^2+1}\frac{1}{u}du$$

Uncorrected proofs

$$= \frac{1}{2}[\ln|u|]_2^{e^2+1}$$

$$= \frac{1}{2}(\ln(e^2+1) - \ln 2)$$

$$= \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$$

5f

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$$

Let $u = 1 + \tan x$

$du = \sec^2 x\,dx$

$x = \frac{\pi}{4}, u = 2$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [\ln|u|]_1^2$$

$$= \ln|2| - \ln|1|$$

$$= \ln(2) - 0$$

$$= \ln 2$$

Uncorrected proofs

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Ext 2

Ex 4A

Q6 $$\int_e^{e^2} \frac{1}{x \ln x}\,dx = \int_e^{e^2} \frac{1/x}{\ln x}\,dx$$

$$= \left[\ln(\ln x)\right]_e^{e^2}$$

$$= \ln 2 - \ln 1$$

$$= \ln 2.$$

Q7 Let $y = \frac{\ln x}{x} = \frac{1}{x} \cdot \ln x$

then $y' = \frac{1}{x^2} - \frac{\ln x}{x^2}$ (product rule)

so $\frac{\ln x}{x^2} = \frac{1}{x^2} - y'$

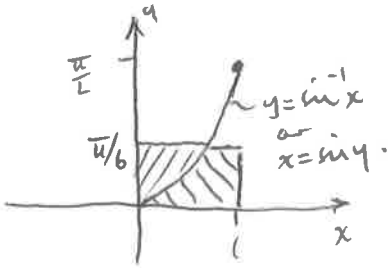
hence $\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x} - y + C$

$$= -\frac{1+\ln x}{x} + C.$$

Q8 Let $y = x \sin^{-1} x$

then $y' = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

so $\sin^{-1} x = y' - \frac{x}{\sqrt{1-x^2}}$



hence $\int_0^{1/2} \sin^{-1} x\,dx = \left[y - \sqrt{1-x^2}\right]_0^{1/2}$

$$= \left[x \sin^{-1} x - \sqrt{1-x^2}\right]_0^{1/2}$$

$$= \left(\frac{1}{2} \cdot \frac{\pi}{6} - \sqrt{\frac{3}{4}}\right) - (0 - 1)$$

$$= \frac{\pi}{12} - \frac{\sqrt{3}}{2} + 1$$

Better still, use inverse functions & subtraction of areas, as indicated in the diagram above.

Ex 2

Ex 4A

Q9 Let $y = \tan^2 x$

$y' = 3\tan^2 x \sec^2 x$

$= 3\tan^2 x + 3\tan^4 x$

So $\tan^4 x = \frac{1}{3}y' - \tan^2 x$

$$\int_0^{\pi/4} \tan^4 x \, dx = \int_0^{\pi/4} \tfrac{1}{2}y' - \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \tfrac{1}{3}y' + 1 - \sec^2 x \, dx \qquad \text{(by Pythagoras) (viz } \tan^2 x = \sec^2 x - 1\text{)}$$

$$= \left[\tfrac{1}{3}y + x - \tan x\right]_0^{\pi/4}$$

$$= \left[\tfrac{1}{3}\tan^3 x + x - \tan x\right]_0^{\pi/4}$$

$$= \left(\tfrac{1}{3} + \tfrac{\pi}{4} - 1\right) - (0)$$

$$= \tfrac{\pi}{4} - \tfrac{2}{3}.$$

Alternatively,

$$\int_0^{\pi/4} \tan^4 x \, dx = \int_0^{\pi/4} \tan^2 x \sec^2 x - \tan^2 x \, dx \quad \text{(by Pyth.)}$$

$$= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x + 1 - \sec^2 x \, dx \quad \text{(by Pyth. again)}$$

$$= \left[\tfrac{1}{3}\tan^3 x + x - \tan x\right]_0^{\pi/4}$$

$$= \left(\tfrac{1}{3} + \tfrac{\pi}{4} - 1\right) - (0)$$

$$= \tfrac{\pi}{4} - \tfrac{2}{3} \quad \text{as before.}$$