Solutions to Exercise 3F Chapter review

1a $(\cos\theta + i\sin\theta)^3(\cos 2\theta + i\sin 2\theta)^2$

$= (\text{cis}\,\theta)^3(\text{cis}\,2\theta)^2$

$= \text{cis}\,3\theta\ \text{cis}\,4\theta$

$= \text{cis}\,7\theta$

1b

$$\frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^2}$$

$$= \frac{(\text{cis}\,\theta)^4}{(\text{cis}(-\theta))^2}$$

$$= \frac{\text{cis}\,4\theta}{\text{cis}(-2\theta)}$$

$= \text{cis}\,6\theta$

2

$$\frac{\left(e^{-\frac{i\pi}{7}}\right)^3}{\left(e^{i\frac{\pi}{7}}\right)^4}$$

$$= \frac{e^{-i\frac{3\pi}{7}}}{e^{i\frac{4\pi}{7}}}$$

$$= e^{-i\frac{7\pi}{7}}$$

$= e^{-i\pi}$

$= -1$

3a $1 - i$

$= \sqrt{1+1}\text{cis}\left(\tan^{-1}-\frac{1}{1}\right)$

$= \sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$

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3b $(1-i)^{13}$

$= \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^{13}$

$= \left(\sqrt{2}\right)^{13}\text{cis}\left(\frac{3\pi}{4}\times 13\right)$

$= 2^6\sqrt{2}\,\text{cis}\left(\frac{39\pi}{4}\right)$

$= 2^6\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$= 2^6\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$

$= -64+64i$

4a $\left(\sqrt{3}+i\right)^{12}+\left(\sqrt{3}-i\right)^{12}$

$= \left(2e^{\frac{\pi}{6}}\right)^{12}+\left(2e^{-\frac{\pi}{6}}\right)^{12}$

$= 2^{12}e^{2\pi}+2^{12}e^{-2\pi}$

$= 2^{12}+2^{12}$

$= 2\times 2^{12}$

$= 2^{13}$

4b i $\left(\sqrt{3}+i\right)^{n}+\left(\sqrt{3}-i\right)^{n}$

$= \left(2e^{\frac{\pi}{6}}\right)^{n}+\left(2e^{-\frac{\pi}{6}}\right)^{n}$

$= 2^{12}e^{n\pi}+2^{12}e^{-n\pi}$

$= 2^{12}e^{n\pi}+2^{12}e^{-n\pi}$

$= 2^{12}(e^{n\pi}+e^{-n\pi})$

$= 2^{12}(\cos n\pi+i\sin n\pi+\cos n\pi-i\sin n\pi)$

$= 2^{12}(2\cos n\pi)$

$= 2^{13}\cos n\pi$

which is real

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4b ii $\left(\sqrt{3}+i\right)^n+\left(\sqrt{3}-i\right)^n$

is rational when $2^{13}\cos n\pi$ is rational and hence when $\cos n\pi$ is rational. This is when n is even or a multiple of 3.

5a $\cos 6\theta + i\sin 6\theta$

$= \text{cis}\, 6\theta$

$= (\text{cis}\, \theta)^6$

$= (\cos\theta + i\sin\theta)^6$

$= \cos^6\theta + 6i\cos^5\theta\sin\theta + 15i^2\cos^4\theta\sin^2\theta + 20i^3\cos^3\theta\sin^3\theta$
$+15i^4\cos^2\theta\sin^4\theta + 6i^5\cos\theta\sin^5\theta + i^6\sin^6\theta$

$= \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20i\cos^3\theta\sin^3\theta$
$+15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta$

$= (\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta)$
$+ i(6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta)$

Equating the real components of the above equation gives

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

Equating the imaginary components of the above equation gives

$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$

5b $\tan 6\theta$

$= \dfrac{\sin 6\theta}{\cos 6\theta}$

$= \dfrac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}$

$= \dfrac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta} \div \dfrac{\cos^6\theta}{\cos^6\theta}$

$= \dfrac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$

$= \dfrac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}$

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$$= \frac{2t(3 - 10t^2 + 3t^4)}{1 - 15t^2 + 15t^4 - t^6}$$

6a

$$\left(z + \frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$$

$$\left(z - \frac{1}{z}\right)^4$$

$$= z^4 - 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) - 4z\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$

6b Adding the above two results from part a,

$$\left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4 = 2(z^4 + 6 + z^{-4})$$

$$\left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4 = 2(z^4 + z^{-4} + 6)$$

$$(2\cos\theta)^4 + (2i\sin\theta)^4 = 2(\cos 4\theta + 6)$$

$$16\cos^4\theta + 16\sin^4\theta = 2(\cos 4\theta + 6)$$

$$\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$$

7a If ω is a cube root of -1 it follows that

$$\omega^3 = -1$$

Now

$$(-\omega^2)^3$$

$$= -\omega^6$$

$$= -(\omega^3)^2$$

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$= -(-1)^2$

$= -1$

Hence ω^2 is a cube root of -1.

7b $(6\omega + 1)(6\omega^2 - 1)$

$= 36\omega^3 + 6(\omega^2 - \omega) - 1$

$= 36(-1) - 6(\omega - \omega^2) - 1$

$= -36 - 6(\omega - \omega^2 - 1) - 6 - 1$

Since $\omega - \omega^2 - 1$ is the sum of roots of -1, but note there is no coefficient of ω^2 in $\omega^3 - 1 = 0$ and hence it follows that $\omega - \omega^2 - 1 = 0$

$(6\omega + 1)(6\omega^2 - 1)$

$= -36 - 6(\omega - \omega^2 - 1) - 6 - 1$

$= -36 - 6(0) - 6 - 1$

$= -36 - 6 - 1$

$= -43$

8 $z^3 - 8i = 0$

$z^3 = 8i$

Let $z = re^{i\theta}$, hence $z^3 = r^3 e^{3i\theta}$.

Thus $r^3 = 8$ and so $r = 2$ whilst $3\theta = \frac{\pi}{2}$ and thus $\theta = \frac{\pi}{6}$. Hence

$z = 2 \operatorname{cis}\left(\frac{\pi}{6}\right)$

9a $2 + 2i$

$= \sqrt{2^2 + 2^2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

Let $z = r \operatorname{cis}(\theta)$ be a cube root of $2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$. It follows that

$z^3 = 2\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

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$r^3 \text{ cis}(3\theta) = 2\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)$

$r^3 = 2\sqrt{2}$ and hence $r = \sqrt{2}, 3\theta = 2n\pi + \frac{\pi}{4} = \frac{(8n+1)\pi}{4}$

$\theta = \frac{(8n \pm 1)\pi}{12}$ where n is an integer

$z = \sqrt{2} \text{ cis}\left(\frac{(8n \pm 1)\pi}{12}\right)$

$z = \sqrt{2} \text{ cis}\left(\frac{k\pi}{12}\right)$ for $k = -7, 1, 9$

9b Let $z = r \text{ cis } \theta$ be a sixth root of i, it follows that

$z^6 = r^6 \text{ cis } 6\theta$

$= i$

$r^6 \text{ cis } 6\theta = i$

$r^6 \text{ cis } 6\theta = \frac{\pi}{2}$

Hence $r = 1$ and thus

$6\theta = 2n\pi + \frac{\pi}{2}$

$6\theta = \frac{(4n+1)\pi}{2}$

$\theta = \frac{(4n+1)\pi}{12}$

$z = \text{cis}\left(\frac{(4n+1)\pi}{12}\right)$ where n is an integer

$z = \text{cis}\left(\frac{k\pi}{12}\right)$ for $k = -11, -7, -3, 1, 5, 9$

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10a $z = 4\sqrt{3}e^{\frac{i\pi}{3}} - 4e^{\frac{5i\pi}{6}}$

$= 4\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 4\sqrt{3}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= 4\sqrt{3} + 4i$

$= 8\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= 8e^{\frac{i\pi}{6}}$

10b $\frac{z}{8} + i\left(\frac{z}{8}\right)^2 + \left(\frac{z}{8}\right)^3$

$= \frac{8e^{\frac{i\pi}{6}}}{8} + i\left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^2 + \left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^3$

$= e^{\frac{i\pi}{6}} + i\left(e^{\frac{i\pi}{6}}\right)^2 + \left(e^{\frac{i\pi}{6}}\right)^3$

$= e^{\frac{i\pi}{6}} + ie^{\frac{i\pi}{3}} + e^{\frac{i\pi}{2}}$

$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + i\cos\frac{\pi}{3} + i^2\sin\frac{\pi}{3} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{i}{2} - \frac{\sqrt{3}}{2}i + 0 + i$

$= 2i$

10c Let $\lambda = re^{i\theta}$ be a cube root of z. It follows that

$\lambda^3 = r^3e^{3i\theta}$

$= 8e^{\frac{i\pi}{6}}$

Hence $r^3 = 8$ and so $r = 2$

$3\theta = 2n\pi + \frac{\pi}{6}$

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$$3\theta = \frac{(12n+1)\pi}{6}$$

$$\theta = \frac{(12n+1)\pi}{18}$$

Hence

$\lambda = 2e^{\frac{(12n+1)\pi}{18}}$ where n is an integer

$\lambda = 2e^{-\frac{11i\pi}{18}}, 2e^{\frac{i\pi}{18}}, 2e^{\frac{13i\pi}{18}}$

11a $(z - z^{-1})^7$

$= z^7 - 7(z^6)(z^{-1}) + 21(z^5)(z^{-2}) - 35(z^4)(z^{-3}) + 35(z^3)(z^{-4}) - 21(z^2)(z^{-5})$
$+7(z)(z^{-6}) - z^{-7}$

$= z^7 - 7z^5 + 21z^3 - 35z + 35z^{-1} - 21z^{-3} + 7z^{-5} - z^{-7}$

$= (z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})$

11b $z = \cos\theta + i\sin\theta = \text{cis}\,\theta$

$z - z^{-1} = \text{cis}\,\theta - \text{cis}(-\theta)$

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

$z = \cos n\theta + i\sin n\theta = \text{cis}\,n\theta$

$z - z^{-1} = \text{cis}\,n\theta - \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

11c $\sin^7\theta$

$= i^8\sin^7\theta$

$= i(i^7\sin^7\theta)$

$= \frac{i}{128}(128i^7\sin^7\theta)$

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$$= \frac{i}{128}(2^7 i^7 \sin^7\theta)$$

$$= \frac{i}{128}(2i\sin\theta)^7$$

$$= \frac{i}{128}(z - z^{-1})^7$$

$$= \frac{i}{128}\left((z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})\right)$$

$$= \frac{i}{128}\left(2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)\right)$$

$$= -\frac{1}{64}\left(2\sin 7\theta - 7(2\sin 5\theta) + 21(2\sin 3\theta) - 35(2\sin\theta)\right)$$

$$= \frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$$

11d

$$\int (35\sin\theta - 64\sin^7\theta)\, d\theta$$

$$= \int \left(35\sin\theta - 64\left(\frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)\right)\right) d\theta$$

$$= \int \left(35\sin\theta - (35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)\right)\, d\theta$$

$$= \int (21\sin 3\theta - 7\sin 5\theta + \sin 7\theta)\, d\theta$$

$$= -7\cos 3\theta + \frac{7}{5}\cos 5\theta - \frac{1}{7}\cos 7\theta + C$$

12a $\cos 5\theta + i\sin 5\theta$

$$= \text{cis}\, 5\theta$$

$$= (\text{cis}\,\theta)^5$$

$$= (\cos\theta + i\sin\theta)$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$$
$$+5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Equating real components in the above equation

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$\cos 5\theta$

$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta (1 - \cos^2\theta) + 5\cos\theta (1 - \cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta (1 - \cos^2\theta) + 5\cos\theta (1 - 2\cos^2\theta + \cos^4\theta)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

12b $16x^4 - 20x^2 + 5 = 0$

Let $x = \cos\theta$

$16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$\cos\theta (16\cos^4\theta - 20\cos^2\theta + 5) = 0 \times \cos\theta$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$\cos 5\theta = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$

But we omit $\frac{\pi}{2}$ as that solution was introduced when we multiplied the equation by $\theta = \frac{\pi}{2}$.

So the solutions are $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Hence the solutions are $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$

12c The product of the roots is

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10} = 5$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{3\pi}{10}\cos\frac{\pi}{10} = \frac{5}{16}$

$\left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)^2 = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \pm\frac{\sqrt{5}}{4}$

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But $\cos\frac{\pi}{10}\cos\frac{3\pi}{10} > 0$ and hence

$$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$$

12d Let $u = 2x^2 - 1$

$$4u^2 - 2u - 1$$
$$= 4(2x^2 - 1)^2 - 2(2x^2 - 1) - 1$$
$$= 4(4x^4 - 4x^2 + 1) - (4x^2 - 2) - 1$$
$$= 16x^4 - 20x^2 + 5$$
$$= 0$$

12e $x = \cos\frac{\pi}{10}$ is a solution to $16x^4 - 20x^2 + 5 = 0$

$$\cos\frac{\pi}{5} = \left(\cos^2\frac{\pi}{10} - \sin^2\frac{\pi}{10}\right)$$
$$= 2\cos^2\frac{\pi}{10} - 1$$
$$= 2x^2 - 1$$

From part d, $u = 2x^2 - 1$ is a solution to the equation $4u^2 - 2u - 1 = 0$.

The solutions to the equation are

$$u = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$$
$$= \frac{2 \pm \sqrt{4 + 16}}{8}$$
$$= \frac{2 \pm 2\sqrt{5}}{8}$$
$$= \frac{1 \pm \sqrt{5}}{4}$$

But since $\cos\frac{\pi}{5} > 0, \cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$

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13a Begin by noting that

$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$

and that

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

Hence

$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$

$(1) + (2)$:

$e^{i\theta} + e^{-i\theta}$

$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

Hence

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$(1) - (2)$:

$e^{i\theta} - e^{-i\theta}$

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

Hence

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

13b i $2\cos^2\theta$

$= 2\left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^2$

$= 2\left(\frac{1}{4}\left(e^{2i\theta} + 2e^0 + e^{-2i\theta}\right)\right)$

$= \frac{1}{2}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right)$

$= \frac{1}{2}(\cos 2\theta + i\sin 2\theta + 2 + \cos 2\theta - i\sin 2\theta)$

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$$= \frac{1}{2}(2 + 2\cos 2\theta)$$

$$= 1 + \cos 2\theta$$

13b ii $2\sin^2\theta$

$$= 2\left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^2$$

$$= 2\left(\frac{1}{-4}\left(e^{2i\theta} - 2e^0 + e^{-2i\theta}\right)\right)$$

$$= -\frac{1}{2}\left(e^{2i\theta} - 2 + e^{-2i\theta}\right)$$

$$= -\frac{1}{2}(\cos 2\theta + i\sin 2\theta - 2 + \cos 2\theta - i\sin 2\theta)$$

$$= -\frac{1}{2}(-2 + 2\cos 2\theta)$$

$$= 1 - \cos 2\theta$$

13b iii $\cos(\alpha - \beta)$

$$= \frac{1}{2}\left(e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}\right)$$

$$= \frac{1}{4}\left(2e^{i(\alpha-\beta)} + 2e^{i(-\alpha+\beta)}\right)$$

$$= \frac{1}{4}\left(\begin{matrix} e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)} \\ -e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)} \end{matrix}\right)$$

$$= \frac{1}{4}\left(\begin{matrix} e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)} \\ -\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right) \end{matrix}\right)$$

$$= \frac{1}{4}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right)$$

$$-\frac{1}{4}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right)$$

$$= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \times \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right) + \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) \times \frac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

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13b iv $\sin(\alpha-\beta)$

$$=\frac{1}{2i}\left(e^{i(\alpha-\beta)}-e^{-i(\alpha-\beta)}\right)$$

$$=\frac{1}{2i}\left(2e^{i(\alpha-\beta)}-2e^{-i(\alpha-\beta)}\right)$$

$$=\frac{1}{4i}\left(\begin{matrix}e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{i(-\alpha+\beta)}-e^{i(-\alpha-\beta)}-e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{i(-\alpha+\beta)}\\+e^{i(-\alpha-\beta)}\end{matrix}\right)$$

$$=\frac{1}{4i}\left(\begin{matrix}e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{i(-\alpha+\beta)}-e^{i(-\alpha-\beta)}-e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{i(-\alpha+\beta)}\\+e^{i(-\alpha-\beta)}\end{matrix}\right)$$

$$=\frac{1}{4i}\left(\begin{matrix}e^{i(\alpha+\beta)}+e^{i(\alpha-\beta)}-e^{i(-\alpha+\beta)}-e^{i(-\alpha-\beta)}\\-\left(e^{i(\alpha+\beta)}-e^{i(\alpha-\beta)}+e^{i(-\alpha+\beta)}-e^{i(-\alpha-\beta)}\right)\end{matrix}\right)$$

$$=\frac{1}{4i}\left(\left(e^{i\alpha}-e^{-i\alpha}\right)\left(e^{i\beta}+e^{-i\beta}\right)-\left(e^{i\alpha}+e^{-i\alpha}\right)\left(e^{i\beta}-e^{-i\beta}\right)\right)$$

$$=\left(\frac{1}{2i}\left(e^{i\alpha}-e^{-i\alpha}\right)\right)\left(\frac{1}{2}\left(e^{i\beta}+e^{-i\beta}\right)\right)-\left(\frac{1}{2i}\left(e^{i\alpha}+e^{-i\alpha}\right)\right)\left(\frac{1}{2}\left(e^{i\beta}-e^{-i\beta}\right)\right)$$

$$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

14a Let $z=r\text{cis}\,\theta$. It follows that

$$z^7=r^7(\text{cis}\,\theta)^7$$

$$=r^7\text{cis}\,7\theta$$

$$=-1$$

It follows that $r=1$ and hence $\text{cis}\,7\theta=-1$

Hence $\cos 7\theta=-1$ and so $7\theta=\pm\pi,\pm3\pi,\pm5\pi,7\pi$

Thus $\theta=\pm\frac{\pi}{7},\pm\frac{3\pi}{7},\pm\frac{5\pi}{7},\pi$ and so the roots are

$$z=\text{cis}\left(\pm\frac{\pi}{7}\right),\text{cis}\left(\pm\frac{3\pi}{7}\right),\text{cis}\left(\pm\frac{5\pi}{7}\right),\text{cis}(\pi)$$

Hence

$$z=\text{cis}\left(\pm\frac{\pi}{7}\right),\text{cis}\left(\pm\frac{3\pi}{7}\right),\text{cis}\left(\pm\frac{5\pi}{7}\right),-1$$

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14b i The roots of the equation $z^7 = -1$ are the same as the roots of the equation $z^7 + 1 = 0$. Since there is no coefficient of z^6, it follows that the sum of the roots of the equation is equal to zero. Hence

$$\text{cis}\left(\frac{\pi}{7}\right) + \text{cis}\left(-\frac{\pi}{7}\right) + \text{cis}\left(\frac{3\pi}{7}\right) + \text{cis}\left(-\frac{3\pi}{7}\right) + \text{cis}\left(\frac{5\pi}{7}\right) + \text{cis}\left(-\frac{5\pi}{7}\right) + (-1) = 0$$

$$\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right) + \left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right) + \left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right) + \left(\cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}\right) + \left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right) + \left(\cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}\right) + (-1) = 0$$

$$2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} + 2\cos\frac{5\pi}{7} - 1 = 0$$

$$2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} + 2\cos\frac{5\pi}{7} = 1$$

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$$

14b ii Writing $z^7 + 1$ as a product of factors (using part a) gives

$$z^7 + 1$$

$$= (z - (-1))\left(z - \text{cis}\left(\frac{\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{\pi}{7}\right)\right)\left(z - \text{cis}\left(\frac{3\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{3\pi}{7}\right)\right)\left(z - \text{cis}\left(\frac{5\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{5\pi}{7}\right)\right)$$

$$= (z+1)\left(z^2 - z\left(\text{cis}\left(\frac{\pi}{7}\right) + \text{cis}\left(-\frac{\pi}{7}\right)\right) + \text{cis}\left(\frac{\pi}{7}\right)\text{cis}\left(-\frac{\pi}{7}\right)\right)\left(z^2 - z\left(\text{cis}\left(\frac{3\pi}{7}\right) + \text{cis}\left(-\frac{3\pi}{7}\right)\right) + \text{cis}\left(\frac{3\pi}{7}\right)\text{cis}\left(-\frac{3\pi}{7}\right)\right)\left(z^2 - z\left(\text{cis}\left(\frac{5\pi}{7}\right) + \text{cis}\left(-\frac{5\pi}{7}\right)\right) + \text{cis}\left(\frac{5\pi}{7}\right)\text{cis}\left(-\frac{5\pi}{7}\right)\right)$$

$$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + \cos^2\frac{\pi}{7} + \sin^2\frac{\pi}{7}\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + \cos^2\frac{3\pi}{7} + \sin^2\frac{3\pi}{7}\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + \cos^2\frac{5\pi}{7} + \sin^2\frac{5\pi}{7}\right)$$

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$$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

14b iii $(z+1)(z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1)$

$= z^7 - 7^6 + z^5 - z^4 + z^3 - z^2 + z + z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1$

$= z^7 + 1$

Hence

$(z+1)(z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1)$

$= z^7 + 1$

$$= (z+1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

Thus

$z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1$

$$= \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

14c Dividing both sides of the identity in part b iii by z^3 gives

$z^3 - z^2 + z - 1 + z^{-1} - z^{-2} + z^{-3}$

$$= \frac{\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)}{z}\frac{\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)}{z}\frac{\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)}{z}$$

$(z^3 + z^{-3}) - (z^2 + z^{-2}) + (z + z^{-1}) - 1$

$$= (z - 2\cos\frac{\pi}{7} + z^{-1})(z - 2\cos\frac{3\pi}{7} + z^{-1})(z - 2\cos\frac{5\pi}{7} + z^{-1})$$

$$= \left((z + z^{-1}) - 2\cos\frac{\pi}{7}\right)\left((z + z^{-1}) - 2\cos\frac{3\pi}{7}\right)\left((z + z^{-1}) - 2\cos\frac{5\pi}{7}\right)$$

We have already seen that $z^n + z^{-n} = 2\cos n\theta$

$2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1$

$$= \left(2\cos\theta - 2\cos\frac{\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{5\pi}{7}\right)$$

$$= 8\left(\cos\theta - \cos\frac{\pi}{7}\right)\left(\cos\theta - \cos\frac{3\pi}{7}\right)\left(\cos\theta - \cos\frac{5\pi}{7}\right)$$

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15a Let $z = re^{i\theta}$. If it is a fifth root of unity, then $z^5 = 1$

$r^5 e^{5i\theta} = 1$

$r^5 = 1$

Hence $r = 1$

$5i\theta = 0, \pm 2\pi, \pm 4\pi$

$i\theta = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$

Thus

$z = e^0, e^{\pm\frac{2\pi}{5}}, e^{\pm\frac{4\pi}{5}}$

$z = 1, e^{\pm\frac{2\pi}{5}}, e^{\pm\frac{4\pi}{5}}$

15b i $u + v$

$= \alpha + \alpha^4 + \alpha^2 + \alpha^3$

$= \alpha + \alpha^2 + \alpha^3 + \alpha^4$

$= e^{\frac{2\pi}{5}} + \left(e^{\frac{2\pi}{5}}\right)^2 + \left(e^{\frac{2\pi}{5}}\right)^3 + \left(e^{\frac{2\pi}{5}}\right)^4$

$= e^{\frac{2\pi}{5}} + e^{\frac{4\pi}{5}} + e^{\frac{6\pi}{5}} + e^{\frac{8\pi}{5}}$

$= e^{\frac{2\pi}{5}} + e^{\frac{4\pi}{5}} + e^{\frac{-4\pi}{5}} + e^{\frac{-2\pi}{5}}$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$= 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$

$u - v$

$= \alpha + \alpha^4 - \alpha^2 - \alpha^3$

$= \alpha - \alpha^2 - \alpha^3 + \alpha^4$

$= e^{\frac{2\pi}{5}} - \left(e^{\frac{2\pi}{5}}\right)^2 - \left(e^{\frac{2\pi}{5}}\right)^3 + \left(e^{\frac{2\pi}{5}}\right)^4$

$= e^{\frac{2\pi}{5}} - e^{\frac{4\pi}{5}} - e^{\frac{6\pi}{5}} + e^{\frac{8\pi}{5}}$

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$= e^{\frac{2\pi}{5}} - e^{\frac{4\pi}{5}} - e^{\frac{-4\pi}{5}} + e^{\frac{-2\pi}{5}}$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} - \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} - \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$= 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5}$

15b ii From question 12e,

$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$

Now

$\cos\frac{2\pi}{5} = 2\cos^2\left(\frac{\pi}{5}\right) - 1$

Hence

$\cos\frac{2\pi}{5} = 2 \times \frac{\left(1+\sqrt{5}\right)^2}{16} - 1$

$= \frac{6+2\sqrt{5}-8}{8}$

$= \frac{\sqrt{5}-1}{4}$

16a $z^n + z^{-n}$

$= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}$

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

16b $\sin(A+B) - \sin(A-B)$

$= \sin A\cos B + \cos A\sin B - (\sin A\cos B - \cos A\sin B)$

$= 2\cos A\sin B$

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16c $(z^{2n} + z^{2n-2} + z^{2n-4} + \cdots + z^{-2n}) \sin\theta$

$= \left((z^{2n} + z^{-2n}) + (z^{2n-2} + z^{-2n+2}) + \cdots + (z^2 + z^{-2}) + z^0\right) \sin\theta$

$= (2\cos 2n\theta + 2\cos(2n-2)\theta + \cdots + \cos 0)\sin\theta$

$= 2\cos 2n\theta \sin\theta + 2\cos(2n-2)\theta \sin\theta + \cdots + \cos 0 \sin\theta$

$= (\sin(2n\theta + \theta) - \sin(2n\theta - \theta)) + (\sin(2(n-1)\theta + \theta) - \sin(2(n-1)\theta - \theta))$

$+ \cdots + \sin\theta$

$= \sin(2n\theta + \theta)$

$= \sin(2n+1)\theta$

16d Using the result in part c with $n = 3$

$(z^6 + z^4 + z^2 + z^0 + z^2 + z^4 + z^6)\sin\theta = \sin 7\theta$

$z^6 + z^4 + z^2 + z^0 + z^{-2} + z^{-4} + z^{-6} = \dfrac{\sin 7\theta}{\sin\theta}$

$(z^6 + z^{-6}) + (z^4 + z^{-4}) + (z^2 + z^{-2}) + 1 = \dfrac{\sin 7\theta}{\sin\theta}$

$2\cos 6\theta + 2\cos 4\theta + 2\cos 2\theta + 1 = \dfrac{\sin 7\theta}{\sin\theta}$

$2(4\cos^3 2\theta - 3\cos 2\theta) + 2(2\cos^2 2\theta - 1) + 2\cos 2\theta + 1 = \dfrac{\sin 7\theta}{\sin\theta}$

$8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = \dfrac{\sin 7\theta}{\sin\theta}$

17 $\sin\alpha - \sin\beta$

$= \dfrac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) - \dfrac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$

$= \dfrac{1}{2i}\left(e^{i\alpha} - e^{i\beta} - \left(e^{-i\alpha} - e^{-i\beta}\right)\right)$

$= \dfrac{1}{2i}\left(e^{i\alpha} - e^{i\beta} + e^{-i\beta} - e^{-i\alpha}\right)$

$= \dfrac{1}{2i}\left(e^{i\left(\frac{\alpha+\beta}{2}\right)} + e^{-i\left(\frac{\alpha+\beta}{2}\right)}\right)\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} - e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right)$

$= 2\left(\left[\dfrac{1}{2}\left(e^{i\left(\frac{\alpha+\beta}{2}\right)} + e^{-i\left(\frac{\alpha+\beta}{2}\right)}\right)\right]\left[\dfrac{1}{2i}\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} - e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right)\right]\right)$

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$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

18a $(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1)$

$= \omega - 1 + 2\omega^2 - 2\omega + 3\omega^3 - 3\omega^2 + 4\omega^4 - 4\omega^3 + \cdots + n\omega^n - n\omega^{n-1}$

$= -1 - \omega - \omega^2 - \omega^3 - \cdots - \omega^{n-1} + n\omega^n$

$= -(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) + n\omega^n$

Since ω is an nth root of unity, $\omega^n = 1$ and $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$

Hence

$(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1)$

$= -0 + n(1)$

$= n$

18b

$$\frac{1}{z^2 - 1} = \frac{z^{-1}}{z - z^{-1}}$$

Let $z = \text{cis}\,\theta$

$$\frac{1}{(\text{cis}\,\theta)^2 - 1} = \frac{(\text{cis}\,\theta)^{-1}}{\text{cis}\,\theta - (\text{cis}\,\theta)^{-1}}$$

$$\frac{1}{\text{cis}\,2\theta - 1} = \frac{\text{cis}(-\theta)}{\text{cis}\,\theta - \text{cis}(-\theta)}$$

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}$$

$$\frac{1}{\cos 2\theta + i\sin 2\theta - 1} = \frac{\cos\theta - i\sin\theta}{2i\sin\theta}$$

18c Considering the above equation, let $\theta = \frac{\pi}{n}$. It follows that

$$\frac{1}{\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} - 1} = \frac{\cos\frac{\pi}{n} - i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}$$

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$$\frac{1}{\omega - 1} = \frac{\cos\frac{\pi}{n}}{2i\sin\frac{\pi}{n}} - \frac{i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}$$

$$= -i\frac{\cos\frac{\pi}{n}}{2\sin\frac{\pi}{n}} - \frac{\sin\frac{\pi}{n}}{2\sin\frac{\pi}{n}}$$

$$= -\frac{1}{2} - i\frac{\cos\frac{\pi}{n}}{2\sin\frac{\pi}{n}}$$

Hence the real part of $\frac{1}{\omega - 1}$ is $-\frac{1}{2}$.

18d Consider

$$1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5}$$

Begin by noting that all components of the expression are real and thus the result must be real.

$$1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5}$$

$$= 1 + 2\cos\frac{2\pi}{5} + 3\cos\left(\pi - \frac{\pi}{5}\right) + 4\cos\left(\pi + \frac{\pi}{5}\right) + 5\cos\left(\pi + \frac{3\pi}{5}\right)$$

$$= 1 + 2\cos\frac{2\pi}{5} - 3\cos\frac{\pi}{5} - 4\cos\frac{\pi}{5} + 5\cos(2\pi - \frac{2\pi}{5})$$

$$= 1 + 2\cos\frac{2\pi}{5} - 7\cos\frac{\pi}{5} + 5\cos\frac{2\pi}{5}$$

$$= 1 + 7\cos\frac{2\pi}{5} - 7\cos\frac{\pi}{5}$$

$$= 1 + 7 \times \left[\frac{\sqrt{5} - 1}{4} - \frac{\sqrt{5} + 1}{4}\right]$$

$$= 1 - \frac{7}{2}$$

$$= -\frac{5}{2}$$

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18e From part d,

$$1 + 7\cos\frac{2\pi}{5} - 7\cos\frac{\pi}{5} = -\frac{5}{2}$$

$$\cos\frac{2\pi}{5} - \cos\frac{\pi}{5} = -\frac{1}{2}$$

$$2\left(\cos\frac{\pi}{5}\right)^2 - 1 - \cos\frac{\pi}{5} = -\frac{1}{2}$$

$$2\left(\cos\frac{\pi}{5}\right)^2 - \cos\frac{\pi}{5} - \frac{1}{2} = 0$$

$$4\left(\cos\frac{\pi}{5}\right)^2 - 2\cos\frac{\pi}{5} - 1 = 0$$

$$\cos\frac{\pi}{5} = \frac{2 \pm \sqrt{4 - 4(4)(-1)}}{8}$$

$$= \frac{2 \pm \sqrt{20}}{8}$$

$$= \frac{1 \pm \sqrt{5}}{4}$$

Since $\cos\frac{\pi}{5} > 0, \cos\frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}$

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