Solutions to Exercise 3E Foundation questions

1a $2i = 2e^{i\frac{\pi}{2}}$ (note that $i = e^{i\frac{\pi}{2}}$)

1b $2e^{i\frac{\pi}{2}} = 2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$

1c $z = re^{i\theta}$

$z^2 = r^2e^{2i\theta}$

$2e^{i\left(\frac{\pi}{2}+2k\pi\right)} = 2i = z^2 = r^2e^{2i\theta}$

Hence $r^2e^{2i\theta} = 2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$, thus $r^2 = 2$ and $2\theta = \frac{\pi}{2} + 2k\pi = \frac{\pi+4k\pi}{2}$ and so

$r = \sqrt{2}, \theta = \frac{(4k+1)\pi}{4}$

1d $z = \sqrt{2}e^{\frac{(4k+1)i\pi}{4}} = \sqrt{2}e^{i\frac{\pi}{4}}, \sqrt{2}e^{\frac{-3i\pi}{4}}$

1e $z = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

$z = \sqrt{2}e^{-i\frac{3\pi}{4}} = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$

2a $-1 = e^{i\pi}$

2b $e^{i(\pi+2k\pi)}$

2c $z^4 = \left(re^{i\theta}\right)^4 = r^4e^{4i\theta}$

Hence $r^4e^{4i\theta} = e^{i(\pi+2k\pi)}$

$r = 1$ and

$4\theta = (\pi + 2k\pi)$

$\theta = \dfrac{\pi + 2k\pi}{4}$

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Chapter 3 worked solutions – Complex numbers II: de Moivre and Euler

2d

$$z = e^{-i\frac{3\pi}{4}}, e^{-i\frac{\pi}{4}}, \quad e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}$$

2e

$$z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

3a

$$-i = e^{-\frac{i\pi}{2}}$$

3b

$$-i = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$$

3c $z = re^{i\theta}$

$$\left(re^{i\theta}\right)^3 = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$$

$$r^3e^{3i\theta} = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$$

$$r = 1$$

$$3\theta = -\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\theta = -\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\theta = -\frac{1}{3}\left(\frac{\pi + 4k\pi}{2}\right)$$

$$\theta = -\frac{(4k+1)\pi}{6}$$

3d

$$z = e^{\frac{i\pi}{2}}, e^{-\frac{i\pi}{6}}, e^{-\frac{5i\pi}{6}}$$

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Chapter 3 worked solutions – Complex numbers II: de Moivre and Euler

4a $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i \sin n\theta + \cos(-n\theta) + i \sin(-n\theta)$

$= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$

$= 2 \cos n\theta$

4b $\left(e^{i\theta} + e^{-i\theta}\right)^3$

$= e^{3i\theta} + 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} + e^{-3i\theta}$

$= e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}$

$= e^{3i\theta} + 3\left(e^{i\theta} + e^{-i\theta}\right) + e^{-3i\theta}$

$= \left(e^{3i\theta} + e^{-3i\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)$

4c $\cos^3 \theta$

$= \left(\frac{1}{2} \times 2 \cos \theta\right)^3$

$= \left(\frac{1}{2} \times \left(e^{i\theta} + e^{-i\theta}\right)\right)^3$

$= \frac{1}{2^3}\left(e^{i\theta} + e^{-i\theta}\right)^3$

$= \frac{1}{8}\left(\left(e^{3i\theta} + e^{-3i\theta}\right) + 3\left(e^{i\theta} + e^{-i\theta}\right)\right)$

$= \frac{1}{8}(2 \cos 3\theta + 6 \cos \theta)$

$= \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$

5

$e^{in\theta} - e^{-in\theta}$

$= \cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta)$

$= 2i \sin n\theta$

Hence

$\sin n\theta = \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)$

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$$\sin^3\theta$$

$$= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3$$

$$= \frac{1}{(2i)^3}\left(e^{i\theta} - e^{-i\theta}\right)^3$$

$$= \frac{1}{(2i)^3}\left(e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta}\right)$$

$$= \frac{1}{(2i)^3}\left(e^{3i\theta} - e^{-3i\theta} - 3e^{i\theta} + 3e^{-i\theta}\right)$$

$$= \frac{1}{(2i)^3}\left(e^{3i\theta} - e^{-3i\theta} - 3\left(e^{i\theta} - 3e^{-i\theta}\right)\right)$$

$$= \frac{1}{(2i)^3}(2i\sin 3\theta - 6i\sin\theta)$$

$$= -\frac{1}{8}(2\sin 3\theta - 6\sin\theta)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

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Solutions to Exercise 3E Development questions

6a Consider the equation

$z^4 + 16 = 0$

Let $z = re^{i\theta}$ be a root of the equation.

$\left(re^{i\theta}\right)^4 + 16 = 0$

$r^4 e^{4i\theta} = -16$

$r^4 = 16$ and $e^{4i\theta} = -1$

Hence $r = 2$ and $4\theta = \pi + 2n\pi$ so $\theta = \frac{(2n+1)\pi}{4}$ where n is an integer. Thus

$z = 2e^{\pm\frac{i\pi}{4}}, 2e^{\pm\frac{3i\pi}{4}}$

Hence writing $z^4 + 16$ as a product of factors gives

$$z^4 + 16 = \left(z - 2e^{\frac{i\pi}{4}}\right)\left(z - 2e^{-\frac{i\pi}{4}}\right)\left(z - 2e^{\frac{3i\pi}{4}}\right)\left(z - 2e^{-\frac{3i\pi}{4}}\right)$$

6b $z^4 + 16$

$$= \left(z - 2e^{\frac{i\pi}{4}}\right)\left(z - 2e^{-\frac{i\pi}{4}}\right)\left(z - 2e^{\frac{3i\pi}{4}}\right)\left(z - 2e^{-\frac{3i\pi}{4}}\right)$$

$$= \left(z^2 - 2ze^{\frac{i\pi}{4}} - 2ze^{-\frac{i\pi}{4}} + 4e^0\right)\left(z^2 - 2ze^{\frac{3i\pi}{4}} - 2ze^{-\frac{3i\pi}{4}} + 4e^0\right)$$

$$= \left(z^2 - 2z\left(e^{\frac{i\pi}{4}} + e^{-\frac{i\pi}{4}}\right) + 4\right) + \left(z^2 - 2z\left(e^{\frac{3i\pi}{4}} + e^{-\frac{3i\pi}{4}}\right) + 4\right)$$

$$= \left(z^2 - 2z\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) + 4\right)$$

$$+ \left(z^2 - 2z\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + 4\right)$$

$$= \left(z^2 - 2z\left(2\cos\frac{\pi}{4}\right) + 4\right) + \left(z^2 - 2z\left(2\cos\frac{3\pi}{4}\right) + 4\right)$$

$$= \left(z^2 - 2z\left(\frac{2}{\sqrt{2}}\right) + 4\right) + \left(z^2 - 2z\left(-\frac{2}{\sqrt{2}}\right) + 4\right)$$

$$= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$$

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6c $z^4 + 16$

$= (z^4 + 8z^2 + 16) - 8z^2$

$= (z^2 + 4)^2 - \left(2\sqrt{2}z\right)^2$

$= \left(z^2 + 4 - 2\sqrt{2}z\right)\left(z^2 + 4 + 2\sqrt{2}z\right)$

$= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$

7a Consider the equation

$z^5 + 1 = 0$

Let $z = r\,\text{cis}(\theta)$. It follows that

$r^5\text{cis}(5\theta) + 1 = 0$

$r^5\text{cis}(5\theta) = -1$

Hence $r = 1$ and $\text{cis}(5\theta) = -1$ so $5\theta = \pi + 2n\pi$ and thus $\theta = \frac{(2n+1)\pi}{5}$. From this it follows that

$z = e^{\pm\frac{i\pi}{5}}, e^{\pm\frac{3i\pi}{5}}, e^{\pm\frac{5i\pi}{5}}$

$z = e^{\pm\frac{i\pi}{5}}, e^{\pm\frac{3i\pi}{5}}, -1$

Hence writing $z^5 + 1$ as a product of roots gives

$z^5 + 1 = (z - (-1))\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$

$= (z + 1)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$

7b $z^5 + 1$

$= (z + 1)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$

$= (z + 1)\left(z - \left(e^{\frac{i\pi}{5}} + e^{-\frac{i\pi}{5}}\right) + \left(e^{\frac{i\pi}{5}}e^{-\frac{i\pi}{5}}\right)\right)\left(z - \left(e^{3\frac{i\pi}{5}} + e^{-3\frac{i\pi}{5}}\right) + \left(e^{3\frac{i\pi}{5}}e^{-3\frac{i\pi}{5}}\right)\right)$

$= (z + 1)\left(z - \left(e^{\frac{i\pi}{5}} + e^{-\frac{i\pi}{5}}\right) + e^0\right)\left(z - \left(e^{3\frac{i\pi}{5}} + e^{-3\frac{i\pi}{5}}\right) + e^0\right)$

$= (z + 1)\left(z - \left(e^{\frac{i\pi}{5}} + e^{-\frac{i\pi}{5}}\right) + 1\right)\left(z - \left(e^{3\frac{i\pi}{5}} + e^{-3\frac{i\pi}{5}}\right) + 1\right)$

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$$= (z+1)\left(z - \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right) + 1\right)$$
$$\left(z - \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} + \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right) + 1\right)$$

$$= (z+1)(z - 2\cos\frac{\pi}{5} + 1)(z - 2\cos\frac{3\pi}{5} + 1)$$

$$= (z+1)(z - 2\cos\frac{\pi}{5} + 1)(z + 2\cos\frac{2\pi}{5} + 1)$$

7c The sum of the roots of the equation $z^5 + 1$ is

$$\text{cis}\left(\frac{3\pi}{5}\right) + \text{cis}\left(-\frac{3\pi}{5}\right) + \text{cis}\left(\frac{\pi}{5}\right) + \text{cis}\left(-\frac{\pi}{5}\right) - 1 = 0$$

$$2\cos\left(-\frac{3\pi}{5}\right) + 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$$

$$-2\cos\left(\frac{2\pi}{5}\right) + 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$$

$$2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5} + 1 = 0$$

$$\cos\tfrac{2\pi}{5} = 2\left(\cos\tfrac{\pi}{5}\right)^2 - 1$$

Hence,

$$4\left(\cos\frac{\pi}{5}\right)^2 - 2 - 2\cos\frac{\pi}{5} + 1 = 0$$

$$4\left(\cos\frac{\pi}{5}\right)^2 - 2\cos\frac{\pi}{5} - 1 = 0$$

Hence, roots are

$$\cos\frac{\pi}{5} = \frac{2 \pm \sqrt{(4+16)}}{8}$$

$$= \frac{1+\sqrt{5}}{4} \quad \text{(remove the other solution as its value is less than 0)}$$

Hence,

$$\cos\frac{2\pi}{5} = 2 \times \frac{\left(1+\sqrt{5}\right)^2}{16} - 1$$

$$= \frac{2\sqrt{5} - 2}{8}$$

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$$= \frac{-1 + \sqrt{5}}{4}$$

$$= \frac{\sqrt{5} - 1}{4}$$

8a Noting that $\text{cis}(\theta) = e^{i\theta}$

it follows that

$\cos\theta + i\sin\theta = e^{i\theta}$ (1)

And hence that

$\cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

$\cos\theta - i\sin\theta = e^{-i\theta}$ (2)

(1) + (2):

$2\cos\theta = e^{i\theta} + e^{-i\theta}$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

(1) − (2):

$2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

8b i $\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)$

$$= \frac{1}{4}\left(e^{2i\theta} + e^{0} + e^{0} + e^{-2i\theta}\right) + \frac{1}{4}\left(e^{2i\theta} - e^{0} - e^{0} + e^{-2i\theta}\right)$$

$$= \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^2$$

$$= \cos^2\theta - \sin^2\theta$$

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8b ii $\sin 2\theta$

$$= \frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right)$$

$$= \frac{1}{2i}\left(e^{2i\theta} - e^{0} + e^{0} - e^{-2i\theta}\right)$$

$$= \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= 2\left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= 2\cos\theta\sin\theta$$

8b iii $\cos(\alpha + \beta)$

$$= \frac{1}{2}\left(e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{4}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right)$$

$$+ \frac{1}{4}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{2}\left(e^{i(\alpha)} + e^{-i(\alpha)}\right)\frac{1}{2}\left(e^{i(\beta)} + e^{-i(\beta)}\right) - \frac{1}{2i}\left(e^{i(\alpha)} - e^{-i(\alpha)}\right)\frac{1}{2i}\left(e^{i(\beta)} - e^{-i(\beta)}\right)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

8b iv $\sin(\alpha + \beta)$

$$= \frac{1}{2i}\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{4i}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)}\right)$$

$$+ \frac{1}{4i}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{2i}\left(e^{i(\alpha)} - e^{-i(\alpha)}\right)\frac{1}{2}\left(e^{i(\beta)} + e^{-i(\beta)}\right) + \frac{1}{2}\left(e^{i(\alpha)} + e^{-i(\alpha)}\right)\frac{1}{2i}\left(e^{i(\beta)} - e^{-i(\beta)}\right)$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

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9a $\cos^6\theta$

$$= \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^6$$

$$= \frac{1}{2^6}\left(e^{6i\theta} + 6e^{4i\theta} + 15e^{2i\theta} + 20 + 15e^{-2i\theta} + 6e^{-4i\theta} + e^{-6i\theta}\right)$$

$$= \frac{1}{2^6}\left(\left(e^{6i\theta} + e^{-6i\theta}\right) + 6\left(e^{4i\theta} + e^{-4i\theta}\right) + 15\left(e^{2i\theta} + e^{-2i\theta}\right) + 20\right)$$

$$= \frac{1}{2^6}(2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20)$$

$$= \frac{1}{2^5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)$$

9b

$$\int_0^{\frac{\pi}{4}} \cos^6\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2^5}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$$

$$= \frac{1}{2^5}\int_0^{\frac{\pi}{4}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$$

$$= \frac{1}{2^5}\left[\frac{1}{6}\sin 6\theta + \frac{6}{4}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2^5}\left(\frac{1}{6}\sin\frac{6\pi}{4} + \frac{6}{4}\sin\pi + \frac{16}{2}\sin\frac{\pi}{2} + \frac{10\pi}{4} - 0\right)$$

$$= \frac{15\pi + 44}{192}$$

10a $\sin^3\theta$

$$= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3$$

$$= \frac{1}{8i^3}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right)$$

$$= -\frac{1}{8i}\left(\left(e^{3i\theta} - e^{-3i\theta}\right) - 3\left(e^{i\theta} + e^{-i\theta}\right)\right)$$

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$$= -\frac{1}{8i}(2i\sin 3\theta - 6i\sin\theta)$$

$$= -\frac{1}{4}(\sin 3\theta - 3\sin\theta)$$

$\sin^5\theta$

$$= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^5$$

$$= \frac{1}{32i^5}\left(e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}\right)$$

$$= \frac{1}{32i}\left(\left(e^{5i\theta} - e^{-5i\theta}\right) - 5\left(e^{3i\theta} - e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= \frac{1}{32i}(2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta)$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

10b $\sin^3\theta\cos^2\theta$

$$= \sin^3\theta\,(1 - \sin^2\theta)$$

$$= \sin^3\theta - \sin^5\theta$$

$$= -\frac{1}{4}(\sin 3\theta - 3\sin\theta) - \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$= \frac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)$$

10c

$$\int_0^{\frac{\pi}{3}} \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)\,d\theta$$

$$= \frac{1}{16}\left[-2\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta\right]_0^{\frac{\pi}{3}}$$

Uncorrected proofs

$$= \frac{1}{16}\left(-2\cos\frac{\pi}{3} - \frac{1}{3}\cos\pi + \frac{1}{5}\cos\frac{5\pi}{3} - \left(-2\cos 0 - \frac{1}{3}\cos 0 + \frac{1}{5}\cos 0\right)\right)$$

$$= \frac{1}{16}\left(-1 + \frac{1}{3} + \frac{1}{10} - \left(-2 - \frac{1}{3} + \frac{1}{5}\right)\right)$$

$$= \frac{47}{480}$$

11a $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

11b Since

$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$z^2(5z^2 - 11z + 16 - 11z^{-1} + 5z^{-2}) = 0$

$z^2(5(z^2 + z^{-2}) - 11(z + z^{-1}) + 16) = 0$

$5(2\cos 2\theta) - 11(2\cos\theta) + 16 = 0$

$5\cos 2\theta - 11\cos\theta + 8 = 0$

11c $5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 5 - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$

$(5\cos\theta - 3)(2\cos\theta - 1) = 0$

$\cos\theta = \frac{3}{5}$ or $\frac{1}{2}$

When $\cos\theta = \frac{3}{5}$, $\quad \sin\theta = \pm\frac{\sqrt{5^2 - 3^2}}{5} = \pm\frac{4}{5}$

When $\cos\theta = \frac{1}{2}$, $\quad \sin\theta = \pm\frac{\sqrt{2^1 - 1}}{2} = \pm\frac{\sqrt{3}}{2}$

Hence the roots are $z = \frac{3}{5} \pm \frac{4}{5}i, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

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12 $1 - i$

$= \sqrt{2}e^{-i\frac{\pi}{4}}$

$= e^{\ln\sqrt{2}}e^{-i\frac{\pi}{4}}$

$= e^{\ln\sqrt{2}-\frac{i\pi}{4}}$

So $a = \ln\sqrt{2} = \frac{1}{2}\ln 2$ and $b = -\frac{\pi}{4}$.

13a $\cos(A+B) + \cos(A-B)$

$= \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B$

$= 2\cos A\cos B$

13b Let $A = \frac{\alpha+\beta}{2}$ and $B = \frac{\alpha-\beta}{2}$, thus

$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= \cos\left(\left(\frac{\alpha+\beta}{2}\right)+\left(\frac{\alpha-\beta}{2}\right)\right) + \cos\left(\left(\frac{\alpha+\beta}{2}\right)-\left(\frac{\alpha-\beta}{2}\right)\right)$

$= \cos\alpha + \cos\beta$

13c $\sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B$

Let $A = \frac{\alpha+\beta}{2}$ and $B = \frac{\alpha-\beta}{2}$, thus

$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= \sin\left(\left(\frac{\alpha+\beta}{2}\right)+\left(\frac{\alpha-\beta}{2}\right)\right) + \sin\left(\left(\frac{\alpha+\beta}{2}\right)-\left(\frac{\alpha-\beta}{2}\right)\right)$

$= \sin\alpha + \sin\beta$

as required

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13d $e^{i\alpha} + e^{i\beta}$

$= \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta$

$= (\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + i\left(2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right)$

$= 2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{\frac{i}{2}(\alpha+\beta)}$

14ai $\cos\alpha + \cos\beta$

$= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right)$

$= \frac{1}{2}\left(e^{i\alpha} + e^{i\beta} + e^{-i\alpha} + e^{-i\beta}\right)$

$= \frac{1}{2}\left(e^{\frac{i}{2}(\alpha+\beta)} + e^{-\frac{i}{2}(\alpha+\beta)}\right)\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)$

$= 2\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha+\beta)} + e^{-\frac{i}{2}(\alpha+\beta)}\right)\right)\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

14aii $\sin\alpha + \sin\beta$

$= \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) + \frac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$

$= \frac{1}{2i}\left(e^{i\alpha} + e^{i\beta} - e^{-i\alpha} - e^{-i\beta}\right)$

$= \frac{1}{2i}\left(e^{\frac{i}{2}(\alpha+\beta)} - e^{-\frac{i}{2}(\alpha+\beta)}\right)\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)$

$= 2\left(\frac{1}{2i}\left(e^{\frac{i}{2}(\alpha+\beta)} - e^{-\frac{i}{2}(\alpha+\beta)}\right)\right)\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)\right)$

$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

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14b $\tan\theta$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}{\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

$$= \frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

14c $\tan 2\theta$

$$= \frac{\left(e^{2i\theta} - e^{-2i\theta}\right)}{i\left(e^{2i\theta} + e^{-2i\theta}\right)}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} + e^{-2i\theta}\right)}{2\left(e^{2i\theta} + e^{-2i\theta}\right)}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} + e^{-2i\theta}\right)}{e^{2i\theta} + e^{-2i\theta} + e^{2i\theta} + e^{-2i\theta}}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} + e^{-2i\theta}\right)}{e^{2i\theta} + 2 + e^{-2i\theta} + e^{2i\theta} - 2 + e^{-2i\theta}}$$

$$= \frac{\frac{2}{i}\left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)}{\left(e^{i\theta} + e^{-i\theta}\right)^2 + \left(e^{i\theta} - e^{-i\theta}\right)^2}$$

$$= \frac{2\left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}\right)}{1 + \frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{\left(e^{i\theta} + e^{-i\theta}\right)^2}}$$

$$= \frac{2\left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}\right)}{1 - \frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{-\left(e^{i\theta} + e^{-i\theta}\right)^2}}$$

$$= \frac{2\left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}\right)}{1 - \left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}\right)^2}$$

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$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

15a $z + z^2 + z^3 + \cdots + z^n$

This is a geometric series with $a = z$ and $r = z$. Now, assuming that $|z| < 1$, we have that the sum of the geometric series is

$$S_n = \frac{a(1-r^n)}{1-r}$$
$$= \frac{z(1-z^n)}{1-z}$$
$$= \frac{z^{n+1}-z}{z-1}$$

15b $z + z^2 + \cdots + z^n = \frac{z^{n+1}-z}{z-1}$

Putting $z = e^{i\theta}$

$$e^{i\theta} + \left(e^{i\theta}\right)^2 + \cdots + \left(e^{i\theta}\right)^n = \frac{\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{i\theta}-1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{i\theta}-1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{i\theta}-1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{i(n+1)\theta} + e^{i(\theta+\pi)}\right)}{e^{i\theta}+e^{i\pi}}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{i\theta}\left(e^{in\theta}-1\right)}{e^{i\theta}-1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{i\theta}\left(e^{in\theta}+e^{i\pi}\right)}{e^{i\theta}+e^{i\pi}}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{in\theta}-1\right)}{1-e^{-i\theta}}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}in\theta}\left(e^{\frac{1}{2}in\theta} - e^{-\frac{1}{2}in\theta}\right)}{e^{-\frac{1}{2}i\theta}\left(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}\right)}$$

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$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(e^{\frac{1}{2}in\theta} - e^{-\frac{1}{2}in\theta}\right)}{(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta})}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(2i\sin\frac{1}{2}n\theta\right)}{2i\sin\frac{1}{2}\theta}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

$$\cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \cdots + \cos n\theta + i\sin n\theta$$

$$= \frac{\left(\cos\frac{1}{2}(n+1)\theta + i\sin\frac{1}{2}(n+1)\theta\right)\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

Equating the imaginary component in the above equation

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin\frac{1}{2}(n+1)\theta\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

15c Let $\theta = \frac{\pi}{n}$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\frac{n\pi}{n} = \frac{\sin\frac{1}{2}n\left(\frac{\pi}{n}\right)\sin\frac{\frac{1}{2}(n+1)\pi}{n}}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\pi = \frac{\sin\frac{1}{2}\pi\sin\frac{1}{2}\left(1+\frac{1}{n}\right)\pi}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + 0 = \frac{\sin\frac{1}{2}\pi\sin\frac{1}{2}\left(1+\frac{1}{n}\right)\pi}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \frac{(1)\sin\left(\frac{\pi}{2} + \frac{\pi}{2n}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

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$$\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}=\frac{\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}=\cot\frac{\pi}{2n}$$

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Ex 2

Ex 3 f

Q 16

$$\text{LHS} = \frac{e^{(\alpha+\beta)i} - e^{-(\alpha+\beta)i}}{i\left(e^{(\alpha+\beta)i} + e^{-(\alpha+\beta)i}\right)}$$

$$\text{RHS} = \frac{\left(\frac{e^{\alpha i} - e^{-i\alpha}}{i(e^{i\alpha} + e^{-i\alpha})} + \frac{e^{i\beta} - e^{-i\beta}}{i(e^{i\beta} + e^{-i\beta})}\right)}{\left(1 + \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}} \cdot \frac{e^{i\beta} - e^{-i\beta}}{e^{i\beta} + e^{-i\beta}}\right)}$$

$$= \frac{(e^{i\alpha} - e^{-i\alpha})(e^{i\beta} + e^{-i\beta}) + (e^{i\beta} - e^{-i\beta})(e^{i\alpha} + e^{-i\alpha})}{i\left[(e^{i\alpha} + e^{-i\alpha})(e^{i\beta} + e^{-i\beta}) + (e^{i\alpha} - e^{-i\alpha})(e^{i\beta} - e^{-i\beta})\right]}$$

$$= \frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)} + e^{i(\alpha+\beta)} + e^{i(\beta-\alpha)} - e^{i(\alpha-\beta)} - e^{-i(\alpha+\beta)}}{i\left[e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)} + e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right]}$$

$$= \frac{2\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right)}{2i\left(e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}\right)}$$

$$= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{i\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right)}$$

#

$$= \text{LHS}.$$

Ext 2

Ex 3F2

Q17 a) $z^{2n+1} = 1$

so $e^{i(2n+1)\theta} = e^{i2k\pi}$

where $z = e^{i\theta}$.

thus $(2n+1)\theta = 2k\pi$

$$\theta = \frac{2k\pi}{(2n+1)}$$

and for prin. values $-n \le k \le n$.

note: the answers in the text book are equivalent when converted to prin. value. The proof is left as an exercise.

b) $z^{2n+1} - 1 = 0$

so by the factor theorem & part (a)

$$z^{2n+1} - 1 = \prod_{k=-n}^{n} (z - e^{i\theta}) \quad ; \quad \theta = \frac{2k\pi}{2n+1}$$

$$= (z-1) \prod_{k=1}^{n} (z - e^{i\theta})(z - e^{-i\theta})$$

$$= (z-1) \prod_{k=1}^{n} (z^2 - (e^{i\theta} + e^{-i\theta})z + 1)$$

$$= (z-1) \prod_{k=1}^{n} \left(z^2 - 2\left(\cos\frac{2k\pi}{2n+1}\right)z + 1\right)$$

but $z^{2n+1} - 1 = (z-1)(z^{2n} + z^{2n-1} + \dots + z + 1)$ by GP theory

Hence $z^{2n} + z^{2n-1} + \dots + z + 1 = \prod_{k=1}^{n} \left(z^2 - 2\left(\cos\frac{2k\pi}{2n+1}\right)z + 1\right)$

c) When $z = 1$ $\quad LHS = 1^{2n} + 1^{2n-1} + \dots + 1^2 + 1^1 + 1$

$$= 2n+1$$

$$RHS = \prod_{k=1}^{n} \left(2 - 2\cos\frac{2k\pi}{2n+1}\right)$$

$$= \prod_{k=1}^{n} 2 \cdot 2 \cdot \sin^2\frac{k\pi}{2n+1}$$

(double angle)

Ex 2

Ex 3E

Q17 (c) hence $2n+1 = \prod_{k=1}^{n} \left(2 \sin \frac{k\pi}{2n+1}\right)^2$

(cont'd)

$$= \left(\prod_{k=1}^{n} 2 \sin \frac{k\pi}{2n+1} \right)^2$$

thus $\prod_{k=1}^{n} 2 \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}$

viz $2^n \sin \frac{\pi}{2n+1} \sin \frac{2\pi}{2n+1} \sin \frac{3\pi}{2n+1} \cdots \sin \frac{n\pi}{2n+1} = \sqrt{2n+1}$