Chapter 3 worked solutions – Complex numbers II: de Moivre and Euler

Solutions to Exercise 3D Foundation questions

1a $\left(e^{i\theta}\right)^3 = e^{i\theta\times 3} = e^{3i\theta}$

1b $\left(e^{-i\theta}\right)^6 = e^{-i\theta\times 6} = e^{-6i\theta}$

1c $\left(e^{2i\theta}\right)^4 = e^{2i\theta\times 4} = e^{8i\theta}$

1d $\left(e^{-5i\theta}\right)^{-2} = e^{-5i\theta\times -2} = e^{10i\theta}$

2a $e^{i\theta} \times e^{-2i\theta} = e^{i\theta-2i\theta} = e^{-i\theta}$

2b

$$\frac{e^{6i\theta}}{e^{3i\theta}} = e^{6i\theta-3i\theta} = e^{3i\theta}$$

2c $\left(e^{4i\theta}\right)^{-2} \times \left(e^{-2i\theta}\right)^{-5}$

$= e^{-8i\theta} \times e^{10i\theta}$

$= e^{-8i\theta+10i\theta} = e^{2i\theta}$

2d

$$\frac{\left(e^{2i\theta}\right)^3 \times \left(e^{-3i\theta}\right)^{-4}}{(e^{-i\theta})^2}$$

$$= \frac{e^{6i\theta} \times e^{12i\theta}}{e^{-2i\theta}}$$

$$= \frac{e^{6i\theta+12i\theta}}{e^{-2i\theta}}$$

$$= \frac{e^{18i\theta}}{e^{-2i\theta}}$$

$$= e^{18i\theta-(-2i\theta)}$$

© Cambridge University Press 2019

$= e^{20i\theta}$

3a $2i = 2e^{\frac{\pi}{2}i}$ (note that $i = e^{\frac{\pi}{2}i}$)

3b $1 + i$

$= \sqrt{1^2 + 1^2}\, e^{i \times \tan^{-1}\frac{1}{1}}$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$

3c -6

$= \sqrt{0^2 + (-6)^2}\, e^{i\left(\pi - \tan^{-1}\frac{0}{6}\right)}$

$= 6e^{i\pi}$

3d $-1 + \sqrt{3}i$

$= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}\, e^{i\left(\pi - \tan^{-1}\frac{\sqrt{3}}{1}\right)}$

$= 2e^{\frac{2i\pi}{3}}$

3e $-3 - 3i$

$= \sqrt{(-3)^2 + (-3)^2} e^{i\left(-\pi + \tan^{-1}\frac{3}{3}\right)}$

$= \sqrt{18} e^{-\frac{3i\pi}{4}}$

$= 3\sqrt{2} e^{-\frac{3i\pi}{4}}$

3f $2\sqrt{3} - 2i$

$= \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2} e^{i\tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right)}$

$= \sqrt{16} e^{-\frac{i\pi}{6}}$

© Cambridge University Press 2019

$= 4e^{-\frac{i\pi}{6}}$

4a $5e^{i\pi}$

$= 5(\cos\pi + i\sin\pi)$

$= 5(-1 + 0i)$

$= -5$

4b

$e^{\frac{i\pi}{3}}$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

4c

$4e^{-\frac{i\pi}{2}}$

$= 4\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

$= 4(0 - i)$

$= -4i$

4d

$2e^{\frac{5i\pi}{6}}$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -\sqrt{3} + i$

© Cambridge University Press 2019

4e $2\sqrt{2}e^{-\frac{i\pi}{4}}$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= 2 - 2i$

4f $4\sqrt{3}e^{-\frac{2i\pi}{3}}$

$= 4\sqrt{3}\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= 4\sqrt{3}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$= -2\sqrt{3} - 6i$

© Cambridge University Press 2019

Solutions to Exercise 3D Development questions

5a $zw = (1+\sqrt{3}i)(1-i)$

$= \left(2e^{i\frac{\pi}{3}}\right)\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$

$= 2\sqrt{2}e^{i\frac{\pi}{12}}$

5b

$\frac{w}{z} = \frac{(1-i)}{(1+\sqrt{3}i)}$

$= \frac{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}{\left(2e^{i\frac{\pi}{3}}\right)}$

$= \frac{1}{\sqrt{2}}e^{-i\frac{7\pi}{12}}$

5c $z^3w = \left(1+\sqrt{3}i\right)^3(1-i)$

$= \left(2e^{i\frac{\pi}{3}}\right)^3\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$

$= 8e^{i\pi}\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$

$= 8\sqrt{2}e^{i\frac{3\pi}{4}}$

5d

$\frac{z^2}{w} = \frac{\left(1+\sqrt{3}i\right)^2}{(1-i)}$

$= \frac{\left(2e^{i\frac{\pi}{3}}\right)^2}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}$

$= \frac{4e^{\frac{i2\pi}{3}}}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}$

Uncorrected proofs

© Cambridge University Press 2019

$= 2\sqrt{2}e^{\frac{i11\pi}{12}}$

6a $\left(\sqrt{3}+i\right)^6$

$= \left(2e^{i\frac{\pi}{6}}\right)^6$

$= (2)^6 e^{i\pi}$

$= 2^6(-1)$

$= -64$

6b $(-1+i)^5$

$= \left(\sqrt{2}e^{\frac{3\pi}{4}}\right)^5$

$= \left(\sqrt{2}\right)^5 e^{\frac{15\pi}{4}}$

$= \left(\sqrt{2}\right)^4 \sqrt{2}e^{-\frac{\pi}{4}}$

$= \left(\sqrt{2}\right)^4 (1-i)$

$= 4-4i$

6c

$\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-8}$

$= \left(e^{-\frac{\pi}{3}}\right)^{-8}$

$= e^{\frac{8\pi}{3}}$

$= e^{\frac{2\pi}{3}}$

$= -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

Uncorrected proofs

© Cambridge University Press 2019

Chapter 3 worked solutions – Complex numbers II

6d $\left(-3-3\sqrt{3}i\right)^4$

$= \left(6e^{-\frac{2\pi}{3}}\right)^4$

$= 1296e^{-\frac{8\pi}{3}}$

$= 1296e^{-\frac{2\pi}{3}}$

$= -648-648\sqrt{3}i$

7a $z^{10}-w^{10}=2i$

$\left(\frac{1+i}{\sqrt{2}}\right)^{10}-\left(\frac{1-i}{\sqrt{2}}\right)^{10}$

$= \left(e^{\frac{\pi}{4}}\right)^{10}-\left(e^{-\frac{\pi}{4}}\right)^{10}$

$= e^{\frac{10\pi}{4}}-e^{-\frac{10\pi}{4}}$

$= e^{\frac{2\pi}{4}}-e^{-\frac{2\pi}{4}}$

$= e^{\frac{\pi}{2}}-e^{-\frac{\pi}{2}}$

$= i-(-i)$

$= 2i$

7b $1+z+z^2+z^3+z^4$

$= 1+\left(e^{\frac{\pi}{4}}\right)+\left(e^{\frac{\pi}{4}}\right)^2+\left(e^{\frac{\pi}{4}}\right)^3+\left(e^{\frac{\pi}{4}}\right)^4$

$= 1+e^{\frac{\pi}{4}}+e^{\frac{\pi}{2}}+e^{\frac{3\pi}{4}}+e^{\pi}$

$= 1+\frac{1}{\sqrt{2}}(1+i)+i+\frac{1}{\sqrt{2}}(1-i)-1$

$= \frac{2}{\sqrt{2}}i+i$

$= \left(\sqrt{2}+1\right)i$

Uncorrected proofs

© Cambridge University Press 2019

8a $\left(1+\sqrt{3}i\right)^5(1-i)^4+\left(1-\sqrt{3}i\right)^5(1+i)^4$

$=32e^{-i\frac{\pi}{3}}(-4)+32e^{i\frac{\pi}{3}}(-4)$

$=-128e^{-i\frac{\pi}{3}}-128e^{i\frac{\pi}{3}}$

$=-128\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)$

$=-128$

8b

$\frac{\left(1+\sqrt{3}i\right)^5}{(1-i)^4}+\frac{\left(1-\sqrt{3}i\right)^5}{(1+i)^4}$

$=\frac{32e^{-i\frac{\pi}{3}}}{(-4)}+\frac{32e^{i\frac{\pi}{3}}}{(-4)}$

$=-2^3e^{-i\frac{\pi}{3}}-2^3e^{i\frac{\pi}{3}}$

$=-2^3\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)$

$=-2^3(1)$

$=-8$

9a $1+z^4$

$=1+\left(e^{i\theta}\right)^4$

$=1+e^{4i\theta}$

$=1+\cos 4\theta+i\sin 4\theta$

$=1+(\cos^2 2\theta-\sin^2 2\theta)+i(2\sin\theta\cos\theta)$

$=1-\sin^2 2\theta+\cos^2 2\theta+2i\sin\theta\cos\theta$

$=\cos^2 2\theta+\cos^2 2\theta+2i\sin\theta\cos\theta$

$=2\cos^2 2\theta+2i\sin\theta\cos\theta$

$=2\cos 2\theta\,(\cos 2\theta+i\sin 2\theta)$

Uncorrected proofs

© Cambridge University Press 2019

$= 2\cos 2\theta \operatorname{cis} 2\theta$

9b

$$\frac{1+z^4}{1+z^{-4}}$$

$$= \frac{z^4(1+z^4)}{z^4+1}$$

$$= z^4$$

$$= \left(\operatorname{cis}(\theta)\right)^4$$

$$= \operatorname{cis}(4\theta)$$

11a $(1+i)^n = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n$

$$= \left(\sqrt{2}\right)^n e^{i\frac{n\pi}{4}}$$

This is real when the imaginary part of the exponent is a multiple of $2\lambda\pi$ or $2\lambda\pi \pm \pi$, that is, when $\frac{n\pi}{4} = 2\pi\lambda \pm \pi$ or $2\lambda\pi$ where λ is an integer.

So $n = 8\lambda \pm 4$ or 8λ. Hence $n = 0, 4, 8$ ....

Therefore $(1+i)^n$ is real when n is a multiple of 4.

11b $(1-i)^n = \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n$

$$= \left(\sqrt{2}\right)^n e^{-i\frac{n\pi}{4}}$$

This is purely imaginary when the imaginary part of the exponent is $2\lambda\pi \pm \frac{\pi}{2}$ where λ is an integer, that is, when $-\frac{n\pi}{4} = 2\pi\lambda \pm \pi$ where λ is an integer.

$$n\pi = -4\left(2\lambda\pi \pm \frac{\pi}{2}\right)$$

$n = 8\lambda \pm 2$ where λ is an integer

Hence $n = 2, 6, 10$ ...

Uncorrected proofs

11c $\left(\sqrt{3}-i\right)^n$

$= \left(2e^{-i\frac{\pi}{6}}\right)^n$

$= 2^n e^{-\frac{in\pi}{6}}$

$-\frac{n\pi}{6} = 2\lambda\pi \pm \pi$ or $2\pi\lambda$

$n = -6(2\lambda \pm 1)$ or $n = -12\lambda$

So $n = 6\lambda$ where λ is an integer

This means n is divisible by 6.

11d $\left(1+\sqrt{3}i\right)^n$

$= \left(2e^{\frac{i\pi}{3}}\right)^n$

$= 2^n e^{\frac{in\pi}{3}}$

$\frac{n\pi}{3} = 2\lambda\pi \pm \frac{\pi}{2}$

$n = 3\left(2\lambda \pm \frac{1}{2}\right)$

$n = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}$

12a i $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

12a ii $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

Uncorrected proofs

© Cambridge University Press 2019

12b i Using part a i with $n = 3$

$e^{3i\theta} - e^{-3i\theta} = 2i \sin 3\theta$

12b ii $\left(e^{i\theta} + e^{-i\theta}\right)^2$

$= e^{2i\theta} + 2e^{i\theta - i\theta} + e^{-2i\theta}$

$= e^{2i\theta} + 2e^0 + e^{-2i\theta}$

$= e^{2i\theta} + 2 + e^{-2i\theta}$

$= \left(e^{2i\theta} + e^{-2i\theta}\right) + 2$

$= 2\cos 2\theta + 2$

$= 4\cos^2 \theta$

12biii $\left(e^{i\theta} - e^{-i\theta}\right)^3$

$= e^{-3i\theta}(e^{2i\theta} - 1)^3$

$= e^{-3i\theta}\left(e^{i\theta} - 1\right)^3\left(e^{i\theta} + 1\right)^3$

$= 3e^{-i\theta} - 3e^{i\theta}3 - e^{-3i\theta} + e^{3i\theta}$

$= -8i \sin^3 \theta$

12b iv $e^{2i\theta} + e^{i\theta} + 2 + e^{-i\theta} + e^{-2i\theta}$

$= \left(e^{2i\theta} + e^{-2i\theta}\right) + \left(e^{i\theta} + e^{-i\theta}\right) + 2$

$= 2\cos 2\theta + 2\cos \theta + 2$

12v $e^{3i\theta} - e^{i\theta} + e^{-i\theta} - e^{-3i\theta}$

$= \left(e^{3i\theta} - e^{-3i\theta}\right) - (e^{i\theta} - e^{-i\theta})$

$= 2i \sin 3\theta - 2i \sin \theta$

13a Using question 12a with $n = 1$,

$e^{i\theta} + e^{-i\theta} = 2\cos \theta$

Uncorrected proofs

© Cambridge University Press 2019

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

13b $\cos(-\theta)$

$$= \frac{1}{2}\left(e^{-i\theta} + e^{-(-i\theta)}\right)$$

$$= \frac{1}{2}\left(e^{-i\theta} + e^{i\theta}\right)$$

$$= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$= \cos\theta$$

Hence $\cos\theta$ is an even function.

13c

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Hence

$\sin(-\theta)$

$$= \frac{1}{2i}\left(e^{-i\theta} - e^{-(-i\theta)}\right)$$

$$= \frac{1}{2i}\left(e^{-i\theta} - e^{i\theta}\right)$$

$$= -\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= -\sin\theta$$

Hence $\sin\theta$ is an odd function.

13d

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Hence $\tan\theta$ is an odd function.

Uncorrected proofs

© Cambridge University Press 2019

$$\cot(-\theta) = \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta$$

Hence $\cot\theta$ is an odd function.

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta$$

Hence $\sec\theta$ is an even function.

$$\operatorname{cosec}(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\operatorname{cosec}\theta$$

Hence $\operatorname{cosec}\theta$ is an odd function.

14a $\left(z + 2e^{\frac{i\pi}{2}}\right)\left(z - 2e^{\frac{i\pi}{2}}\right)$

$= z^2 - \left(2e^{\frac{i\pi}{2}}\right)^2$

$= z^2 - 4e^{i\pi}$

$= z^2 + 4$

14b $\left(z - e^{\frac{i\pi}{3}}\right)\left(z - e^{-\frac{i\pi}{3}}\right)$

$= z^2 - ze^{-\frac{i\pi}{3}} - ze^{\frac{i\pi}{3}} + \left(e^{\frac{i\pi}{3}}\right)\left(e^{-\frac{i\pi}{3}}\right)$

$= z^2 - z\left(e^{-\frac{i\pi}{3}} + e^{\frac{i\pi}{3}}\right) + e^0$

$= z^2 - z\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + 1$

$= z^2 - 2z\cos\frac{\pi}{3} + 1$

$= z^2 - z + 1$

14c $(z + 2)\left(z - 2e^{\frac{i\pi}{3}}\right)\left(z - 2e^{-\frac{i\pi}{3}}\right)$

$= (z + 2)(z^2 - 2z + 4)$

$= z^3 + 8$

Uncorrected proofs

© Cambridge University Press 2019

14d $\left(z-\sqrt{2}e^{\frac{i\pi}{4}}\right)\left(z-\sqrt{2}e^{-\frac{i\pi}{4}}\right)\left(z-\sqrt{2}e^{\frac{3i\pi}{4}}\right)\left(z-\sqrt{2}e^{-\frac{3i\pi}{4}}\right)$

$=\left(z^2-z\sqrt{2}\left(e^{\frac{i\pi}{4}}+e^{-\frac{i\pi}{4}}\right)+\left(-\sqrt{2}e^{\frac{i\pi}{4}}\right)\left(-\sqrt{2}e^{-\frac{i\pi}{4}}\right)\right)$

$\times\left(z^2-z\sqrt{2}\left(e^{\frac{3i\pi}{4}}+e^{-\frac{3i\pi}{4}}\right)+\left(-\sqrt{2}e^{\frac{3i\pi}{4}}\right)\left(-\sqrt{2}e^{-\frac{3i\pi}{4}}\right)\right)$

$=\left(z^2-2\sqrt{2}z\cos\frac{\pi}{4}+2\right)\left(z^2-2\sqrt{2}z\cos\frac{3\pi}{4}+2\right)$

$=\left(z^2-z2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)+2\right)\left(z^2-z2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)+2\right)$

$=(z^2-2z+2)(z^2+2z+2)$

$=z^4+2z^3+2z^2-2z^3-4z^2-4z+2z^2+4z+4$

$=z^4+4$

15a $re^{i\theta}=se^{i\phi}$

$\left|re^{i\theta}\right|=\left|se^{i\phi}\right|$

$|r|\left|e^{i\theta}\right|=|s|\left|e^{i\phi}\right|$

$|r|(1)=|s|(1)$

$|r|=|s|$

And since $r>0$ and $s>0$

$r=s$

15b $re^{i\theta}=se^{i\phi}$

Since $r=s$,

$e^{i\theta}=e^{i\phi}$

$\frac{e^{i\theta}}{e^{i\phi}}=1$

$e^{i(\theta-\phi)}=1$

Hence $-2\pi<\phi-\theta<2\pi$ within this range, $\theta-\phi=0$ and so $\theta=\phi$.

Uncorrected proofs

© Cambridge University Press 2019

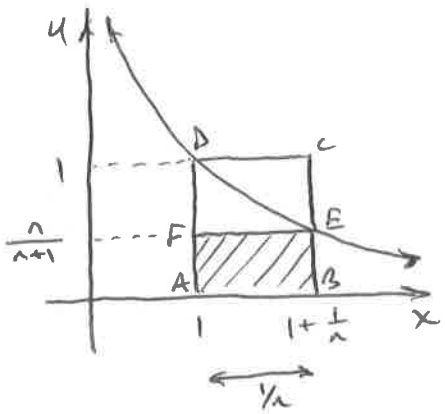
15c If two complex numbers are equal, then they represent the same point in the Argand diagram. Hence the moduli are equal and the principal arguments are equal.

Uncorrected proofs

Ext 2

Ex 3D These enrichment Qs provide another way of arriving at Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

Q16



a) $\text{area } ABEF \leq \left\{\begin{matrix}\text{area}\\ \text{under}\\ \text{curve}\end{matrix}\right\} \leq \text{area } ABCD$

$$\frac{n}{1+n} \times \frac{1}{n} \leq \int_1^{1+\frac{1}{n}} \frac{1}{x}\,dx \leq 1 \times \frac{1}{n}$$

$$\frac{1}{1+n} \leq \left[\log x\right]_1^{1+\frac{1}{n}} \leq \frac{1}{n}$$

$$\frac{1}{1+n} \leq \log\left(1+\tfrac{1}{n}\right) - \cancel{\log 1}^{=0} \leq \frac{1}{n}$$

so $\frac{1}{1+n} \leq \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$

b) $\frac{n}{1+n} \leq \log\left(1+\frac{1}{n}\right)^n \leq 1$ (by the log laws)

taking exponentials of each part.

$$e^{\frac{n}{1+n}} \leq \left(1+\tfrac{1}{n}\right)^n \leq e \qquad (\star)$$

then take the limit as $n \to \infty$

$$\lim_{n\to\infty} e^{\frac{n}{1+n}} \leq \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n \leq e$$

$$e \leq \lim_{n\to\infty} \left(1+\tfrac{1}{n}\right)^n \leq e$$

hence $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

c) In $(\star)$ of (b) replace n with n/x then

$$e^{\frac{n}{x+n}} \leq \left(1+\tfrac{x}{n}\right)^{n/x} \leq e$$

raise to power x

$$\left(e^{\frac{n}{x+n}}\right)^x \leq \left(1+\tfrac{x}{n}\right)^n \leq e^x$$

once again take the limit as $n \to \infty$

$$e^x \leq \lim_{n\to\infty} \left(1+\tfrac{x}{n}\right)^n \leq e^x$$

that is $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

Text 2
Ex 3b

(Q17 a) $e^t \geq 1$ for $0 \leq t \leq x$

so $\int_0^x e^t\,dt \geq \int_0^x 1\,dt$

$[e^t]_0^x \geq [t]_0^x$

$e^x - 1 \geq x$

so $e^x \geq 1 + x$

b) from a) $e^t \geq 1 + t$, $0 \leq t \leq x$

so $\int_0^x e^t\,dt \geq \int_0^x 1 + t\,dt$

$e^x - 1 \geq x + \frac{x^2}{2}$

so $e^x \geq 1 + x + \frac{x^2}{2}$

c) Induction is useful here, with step A done above.

B Assume that the result is true for $n = k$, that is,

assume that $e^x \geq \sum_{n=0}^{k} \frac{x^n}{n!}$ (1)

Now prove the result true for $n = k+1$, that is,

prove that $e^x \geq \sum_{n=0}^{k+1} \frac{x^n}{n!}$

put $x = t$ in (1) and integrate from 0 to x to get

$\int_0^x e^t\,dt \geq \int_0^x \left(\sum_{n=0}^{k} \frac{t^n}{n!} \right) dt$

$\left[e^t \right]_0^x \geq \left[\sum_{n=0}^{k} \frac{t^{n+1}}{(n+1)!} \right]_0^x$

$e^x - 1 \geq \sum_{n=0}^{k} \frac{x^{n+1}}{(n+1)!} - 0$

$e^x \geq 1 + \sum_{n=1}^{k+1} \frac{x^n}{n!}$ (replacing $n+1$ with n)

$\geq \sum_{n=0}^{k+1} \frac{x^n}{n!}$ (since $1 = \frac{(x)^0}{0!}$; $x \neq 0$)

C From parts A & B, by mathematical induction, the result is true for all n; viz

$e^x \geq \sum_{n=0}^{n} \frac{x^n}{n!}$

and taking the limit as $n \to \infty$

$e^x \geq \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Ex 6.2
Ex 3D
Q17 (ctd)

d) $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Let $h(x) = e^{-x} \cdot F(x)$

$$h'(x) = -e^{-x} \cdot F(x) + e^{-x} \cdot F'(x)$$

now put $F_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!}$

(*) see special note at end of Question.

then $F_m'(x) = \sum_{n=0}^{m} \frac{n x^{n-1}}{n!}$

$$= 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{m-1}}{(m-1)!}$$

$$= \sum_{n=0}^{m-1} \frac{x^n}{n!}$$

So $F'(x) = \lim_{m \to \infty} \sum_{n=0}^{m-1} \frac{x^n}{n!}$

$$= \sum_{n=0}^{m} \frac{x^n}{n!}$$

$$= F(x)$$

hence $h'(x) = -e^{-x} F(x) + e^{-x} \cdot F(x)$

$$= 0.$$

and this is true for all x.

e) Thus $h(x)$ must be constant

But $h(0) = e^{-0} F(0)$

$$= 1 \times 1$$

$$= 1$$

Thus $e^{-x} F(x) = 1$

that is $F(x) = e^x$

i.e. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ✱

Ex 2 Ch 3

Ex 3D

Q17 Special Note:

Some readers may find it strange that the infinite series for $F'(x)$ is determined using the finite series $F_n(x)$ and $F'_n(x)$.

This is done because, in general, it is not valid to simply differentiate the terms of a series expansion.

That is, if $f(x) = \sum_{n=0}^{\infty} a_n$ then it does not necessarily follow that

$$f'(x) = \sum_{n=0}^{\infty} \frac{da_n}{dx}.$$

For example $f(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(4^n x)$

is a Fourier series that converges, but

$$\frac{d}{dx}\text{RHS} = \sum_{n=0}^{\infty} 2^n \cos(4^n x)$$

does not converge, so cannot be equal to $f'(x)$.

The first series converges because the coefficients 2^{-n} form a GP with $|\text{ratio}| < 1$

The 2nd series does not converge because the coefficients 2^n form a GP with $|\text{ratio}| > 1$.

In Q17(d), by differentiating the finite sum & taking the limit, the above problem is avoided.

Ext 2

Ex 3D

Q18 a) $C(x) = a_0 + a_2x^2 + a_4x^4 + \dots$

(i) at $x=0$ $\cos 0 = 1$ $C(0) = a_0$

so $a_0 = 1$

(ii) $C'(x) = 2a_2x + 4a_4x^3 + 6a_6x^5 + \dots$

$C''(x) = 2a_2 + 12a_4x^2 + 30a_6x^4 + \dots$

$-C(x) = -a_0 - a_2x^2 - a_4x^4 - \dots$

so if $C''(x) = -C(x)$ then, equating coefficients of like powers of x

x^0: $2a_2 = -a_0 = -1$

so $a_2 = -\frac{1}{2} = \frac{-1}{2!}$

x^2: $12a_4 = -a_2 = \frac{1}{2}$

$a_4 = \frac{1}{24} = \frac{1}{4!}$

(iii) like Q17 this can be done by induction.

Here is just an outline of the proof:

assuming $a_{2k} = \frac{(-1)^k}{(2k)!}$

$(2k+2)(2k+1)a_{2k+2} = -a_{2k}$

so $a_{2k+2} = \frac{(-1)^{k+1}}{(2k+2)(2k+1)(2k)!}$

$= \frac{(-1)^{k+1}}{(2k+2)!}$

hence $C(x) = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

b) $S(x) = a_1x + a_3x^3 + a_5x^5 + \dots$

(i) at $x=0$ $\sin 0 = 0$ $S(0) = 0$ no information.

differentiating at $x=0$ $\frac{d}{dx}\sin x\Big|_{x=0} = \cos x\Big|_{x=0}$

$= 1$

Ext 2
Ex 3J
Q18(b)
(ctd)

and at $x=0$ $\left.\frac{d}{dx} S(x)\right|_{x=0} = \left. a_1 + 3a_3x^2 + 5a_5x^4 + \dots \right|_{x=0}$

$= a_1$

so $a_1 = 1$.

(ii) differentiating again $S''(x) = 6a_3x + 20a_5x^3 + 42a_7x^5 + \dots$

$-S(x) = -a_1x - a_3x^3 - a_5x^5 - \dots$

so if $S''(x) = -S(x)$ then, equating coefficients of like powers of x

x^1: $\quad 6a_3 = -a_1 = -1$

$a_3 = \frac{-1}{6} = \frac{-1}{3!}$

x^3: $\quad 20a_5 = -a_3 = +\frac{1}{3!}$

$a_5 = \frac{+1}{5\times 3!} = \frac{+1}{5!}$

(iii) once again, induction may be used. Here is an outline:

assuming $a_{2k+1} = \frac{(-1)^k}{(2k+1)!}$

$(2k+3)(2k+2)\,a_{2k+3} = -a_{2k+1}$

$= \frac{(-1)^{k+1}}{(2k+1)!}$

$a_{2k+3} = \frac{(-1)^{k+1}}{(2k+3)(2k+2)(2k+1)!}$

$= \frac{(-1)^{k+1}}{(2k+3)!}$

hence $S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

C) $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

assuming we can differentiate term by term (which like Q17 can be shown by taking the limit of a partial sum)

Ext 2
Ex 3D
Q18 (c)
(cont'd)

$$c'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n\, x^{(2n-1)}}{(2n)!}$$

$$= \sum_{n=\phi}^{\infty} \frac{(-1)^n 2n\, x^{2n-1}}{(2n)!} \quad \left(\text{since when } n=0 \text{ the 1st term is } 0\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} \quad (\text{cancelling } 2n)$$

$$= (-1) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$= (-1) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (\text{replacing } n \text{ with } n+1)$$

$$= -s(x)$$

$$s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+0)} \cdot (2n+1)}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)\, x^{2n}}{(2n)!}$$

$$= c(x)$$

(ii) $h = (c(x) - \cos x)^2 + (s(x) - \sin x)^2$

so $h' = 2(c(x) - \cos x)(c'(x) + \sin x) + 2(s(x) - \sin x)(s'(x) - \cos x)$

$= 2(c(x) - \cos x)(-s(x) + \sin x) + 2(s(x) - \sin x)(c(x) - \cos x)$

$= -2(c(x) - \cos x)(s(x) - \sin x) + 2(s(x) - \sin x)(c(x) - \cos x)$

$= 0$

hence $h(x)$ is constant.

$h(0) = (c(0) - \cos 0)^2 + (s(0) - \sin 0)^2$

$= 0^2 + 0^2$

$= 0$

hence $h(x) = 0$ for all real x.

Ext 2

Ex 3D

Q18 c)(iii)

(cont'd)

Since $h(x) = 0$

$$(c(x) - \cos x)^2 + (s(x) - \sin x)^2 = 0$$

The only time the sum of two square reals is zero is if each is zero. Hence

$$c(x) - \cos x = 0 \quad \text{and} \quad s(x) - \sin x = 0$$

thus $c(x) = \cos x$ and $s(x) = \sin x$,

for all real x

Ext 2

Ex 3I

Q19 a) Let $e^{i\theta} = E(i\theta)$

$$= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \quad \text{(by Question 17)}$$

$$= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + \dots$$

$$+ \frac{i\theta}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \dots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$+ i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right)$$

$$= \cos\theta + i\sin\theta \quad \text{(by Q18)}$$