Chapter 3 worked solutions – Complex numbers II: de Moivre and Euler

## Solutions to Exercise 3C Foundation questions

1a $z^3 = 1$

$(r\text{cis}\,\theta)^3 = 1$

$r^3\text{cis}\,3\theta = 1$

$r^3\text{cis}\,3\theta = 1 \times \text{cis}\,2k\pi$

$r = \sqrt[3]{1} = 1$ and $3\theta = 2k\pi$ so $\theta = \frac{2k\pi}{3}$

$z = \text{cis}\left(\frac{\pi + 2k\pi}{3}\right)$

$= \text{cis}\frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis}\,0$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1$

1b

$\left|\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right|$

$= |\sqrt{3}i|$

$= \sqrt{3}$

$\left|\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) - 1\right|$

$= \left|-\frac{3}{2} + \frac{\sqrt{3}}{2}i\right|$

$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{9}{4} + \frac{3}{4}}$

$= \sqrt{3}$

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$$\left|\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)-1\right|$$
$$=\left|-\frac{3}{2}-\frac{\sqrt{3}}{2}i\right|$$
$$=\sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}$$
$$=\sqrt{\frac{9}{4}+\frac{3}{4}}$$
$$=\sqrt{3}$$

This shows that all sides of the triangle have the same length and thus it is equilateral.

1c In the case that $\text{cis}\,\frac{2\pi}{3}$ is the root,

$\left(\text{cis}\,\frac{2\pi}{3}\right)^2=\text{cis}\,\frac{4\pi}{3}=\text{cis}\left(\frac{4\pi}{3}-2\pi\right)=\text{cis}\left(-\frac{2\pi}{3}\right)$ is the other root.

In the case that $\text{cis}\left(-\frac{2\pi}{3}\right)$ is the root,

$\left(\text{cis}\left(-\frac{2\pi}{3}\right)\right)^2=\text{cis}\left(-\frac{4\pi}{3}\right)=\text{cis}\left(2\pi-\frac{4\pi}{3}\right)=\text{cis}\,\frac{2\pi}{3}$ is the other root.

1d i

$$\left(\text{cis}\,\frac{2\pi}{3}\right)^3=\text{cis}\,\frac{6\pi}{3}=\text{cis}\,2\pi=1$$
$$\left(\text{cis}\,\frac{4\pi}{3}\right)^3=\text{cis}\,\frac{12\pi}{3}=\text{cis}\,4\pi=1$$

so in either case the answer is one.

Alternately, covering both cases at once:

$$\omega=\text{cis}\,\frac{2k\pi}{3}$$
$$\omega^3=\text{cis}\left(3\times\frac{2k\pi}{3}\right)$$
$$=\text{cis}\,2k\pi$$
$$=1$$

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1d ii If $\omega = \text{cis}\,\frac{2\pi}{3}$,

$$1 + \omega + \omega^2$$

$$= 1 + \text{cis}\,\frac{2\pi}{3} + \left(\text{cis}\,\frac{2\pi}{3}\right)^2$$
$$= 1 + \text{cis}\,\frac{2\pi}{3} + \text{cis}\,\frac{4\pi}{3}$$
$$= 1 + \text{cis}\,\frac{2\pi}{3} + \text{cis}\left(-\frac{2\pi}{3}\right)$$
$$= 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$
$$= 1 + 2\cos\frac{2\pi}{3}$$
$$= 1 + 2\left(-\frac{1}{2}\right)$$
$$= 0$$

If $\omega = \text{cis}\,\frac{4\pi}{3}$,

$$1 + \omega + \omega^2$$

$$= 1 + \text{cis}\,\frac{4\pi}{3} + \left(\text{cis}\,\frac{4\pi}{3}\right)^2$$
$$= 1 + \text{cis}\,\frac{4\pi}{3} + \text{cis}\,\frac{8\pi}{3}$$
$$= 1 + \text{cis}\left(-\frac{2\pi}{3}\right) + \text{cis}\,\frac{2\pi}{3}$$
$$= 1 + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= 1 + 2\cos\frac{2\pi}{3}$$
$$= 1 + 2\left(-\frac{1}{2}\right)$$
$$= 0$$

1e i $(1 + \omega^2)^3$
$= (-\omega)^3$
$= -\omega^3$
$= -1$

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1e ii $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)$
$= \big(1-(\omega+\omega^2)\big)(1+\omega^2-\omega)(1+\omega-\omega^2)$
$= \big(1-(-1)\big)(-\omega-\omega)(-\omega^2-\omega^2)$
$= 2(-2\omega)(-2\omega^2)$
$= 8\omega^3$
$= 8(1)$
$= 8$

1e iii $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$
$= (1-\omega)(1-\omega^2)(1-\omega^3\omega)(1-\omega^3\omega^2)$
$= (1-\omega)(1-\omega^2)(1-(1)\omega)(1-(1)\omega^2)$
$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$
$= \big((1-\omega)(1-\omega^2)\big)^2$
$= (1-\omega-\omega^2+\omega^3)^2$
$= (1-\omega-\omega^2+1)^2$
$= (1+1+1)^2$
$= 3^2$
$= 9$

2a $z^6 = 1$

Let $z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^6 = 1$

$r^6\,\text{cis}\,6\theta = 1$

$r = 1$

$\text{cis}\,6\theta = \text{cis}\,2k\pi$

$6\theta = 2k\pi$

$\theta = \dfrac{k\pi}{3}$

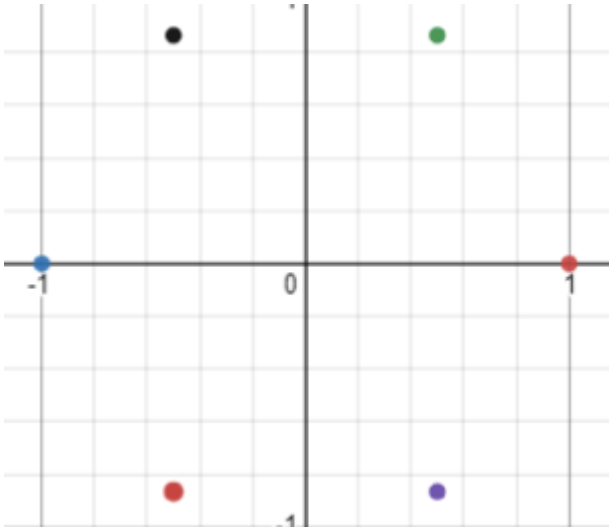
$z = \text{cis}\dfrac{k\pi}{3}$

$z = \text{cis}\,0, \text{cis}\left(\pm\dfrac{\pi}{3}\right), \text{cis}\left(\pm\dfrac{2\pi}{3}\right), \text{cis}\,\pi$

$z = 1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i, -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i, -1$

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2b




All points are the same distance from the origin as,
$|1| = 1$

$$\left|\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$$\left|-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$|-1| = 1$

Since $z = \text{cis}\ \frac{k\pi}{3}$, each root has an argument of $\frac{\pi}{3}$ between it and the adjacent roots, hence all roots are the same distance from the origin with the same argument between them relative to the origin so they form the corners of a regular hexagon.

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2c

$\alpha = \text{cis}\,\frac{\pi}{3}$

$\alpha^2 = \left(\text{cis}\,\frac{\pi}{3}\right)^2 = \text{cis}\left(\frac{2\pi}{3}\right)$ which is a root

$\alpha^{-2} = \left(\text{cis}\,\frac{\pi}{3}\right)^{-2} = \text{cis}\left(-\frac{2\pi}{3}\right)$ which is a root

$\alpha^{-1} = \left(\text{cis}\,\frac{\pi}{3}\right)^{-1} = \text{cis}\left(-\frac{\pi}{3}\right)$ which is a root

2d $(z^4 + z^2 + 1)(z^2 - 1)$
$= z^6 + z^4 + z^2 - (z^4 + z^2 + 1)$
$= z^6 - 1$

2e The roots of $z^2 - 1$ are $z = \pm 1$, which are the real roots of $z^6 - 1$. So the roots of $z^4 + z^2 + 1$ must be the complex roots of $z^6 - 1$. Thus

$z^4 + z^2 + 1$

$= \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)$

$= \left(z^2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) \times$

$\left(z^2 - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$

$= \left(z^2 - z + \left(\frac{1}{4} + \frac{3}{4}\right)\right)\left(z^2 - (-1)z + \left(\frac{1}{4} + \frac{3}{4}\right)\right)$

$= (z^2 - z + 1)(z^2 + z + 1)$

3a $z^4 = -1$

Let $z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^4 = -1$

$r^4\text{cis}\,4\theta = -1$

$r = 1$

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$\text{cis}\, 4\theta = -1$

$4\theta = \pi \pm 2k\pi$

$\theta = \dfrac{\pi \pm 2k\pi}{4}$

$z = \text{cis}\left(\dfrac{\pi \pm 2k\pi}{4}\right)$

$z = \text{cis}\left(\pm\dfrac{\pi}{4}\right), \text{cis}\left(\pm\dfrac{3\pi}{4}\right)$

$z = \dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}}i, -\dfrac{1}{\sqrt{2}} \pm \dfrac{1}{\sqrt{2}}i$

3b $\left(z - \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right)\left(z - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)\left(z - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right)\left(z - \left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)$
$= \left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right)$

4a $z^6 + 1 = 0$

$z^6 = -1$

$z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^6 = -1$

$r^6\text{cis}\, 6\theta = -1$

$r = 1$

$\text{cis}\, 6\theta = -1$

$6\theta = \pi \pm 2k\pi$

$\theta = \dfrac{\pi \pm 2k\pi}{6}$

$z = \text{cis}\left(\dfrac{\pi \pm 2k\pi}{6}\right)$

$z = \text{cis}\left(\pm\dfrac{\pi}{6}\right), \text{cis}\left(\pm\dfrac{3\pi}{6}\right), \text{cis}\left(\pm\dfrac{5\pi}{6}\right)$

$z = \dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i, \pm i, -\dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i$

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4b $(z^6+1)$

$$= (z-i)(z+i)\left(z-\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right)\left(z-\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right)\left(z-\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\right)\times$$

$$\left(z-\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)\right)$$

$$= (z^2+1)\left(z^2-\sqrt{3}z+1\right)\left(z^2+\sqrt{3}z+1\right)$$

4c $z^6+1 = (z^2+1)\left(z^2-\sqrt{3}z+1\right)\left(z^2+\sqrt{3}z+1\right)$

$$\frac{z^6+1}{z^3} = \frac{(z^2+1)\left(z^2-\sqrt{3}z+1\right)\left(z^2+\sqrt{3}z+1\right)}{z^3}$$

$$z^3+z^{-3} = (z+z^{-1})\left(z-\sqrt{3}+z^{-1}\right)\left(z+\sqrt{3}+z^{-1}\right)$$

Using the result from question 3 in Exercise 3B:

$$2\cos 3\theta = (2\cos\theta)\left(2\cos\theta-\sqrt{3}\right)\left(2\cos\theta+\sqrt{3}\right)$$

$$\cos 3\theta = (\cos\theta)\left(2\cos\theta-\sqrt{3}\right)\left(2\cos\theta+\sqrt{3}\right)$$

$$\cos 3\theta = 4\cos\theta\left(\cos\theta-\frac{\sqrt{3}}{2}\right)\left(\cos\theta+\frac{\sqrt{3}}{2}\right)$$

$$\cos 3\theta = 4\cos\theta\left(\cos\theta-\cos\frac{\pi}{6}\right)\left(\cos\theta-\cos\frac{5\pi}{6}\right)$$

5a $z^5 = i$

$$(r\,\mathrm{cis}\,\theta)^5 = \mathrm{cis}\,\frac{\pi}{2}$$

$$r^5\mathrm{cis}\,5\theta = \mathrm{cis}\left(\tfrac{\pi}{2}+2k\pi\right)$$

$r^5 = 1$ and hence $r = 1$

$$5\theta = \frac{\pi}{2}+2k\pi$$

$$\theta = \frac{1}{5}\left(\frac{\pi}{2}+2k\pi\right)$$

$$z = \mathrm{cis}\left(\frac{1}{5}\left(\frac{\pi}{2}+2k\pi\right)\right)$$

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$= \text{cis}\left(-\frac{7\pi}{10}\right), \text{cis}\left(-\frac{3\pi}{10}\right), \text{cis}\frac{\pi}{10}, \text{cis}\frac{\pi}{2}, \text{cis}\frac{9\pi}{10}$

5b $z^4 = -i$

Let $z = (r\text{cis}\,\theta)^4$
$= r^4\text{cis}\,4\theta$

$r^4\text{cis}\,4\theta = -i$

$r = 1$

$\text{cis}\,4\theta = -i$

$4\theta = \frac{3\pi}{2} \pm 2k\pi$

$\theta = \frac{3\pi}{8} \pm \frac{k\pi}{2}$

$z = \text{cis}\left(\frac{3\pi}{8} \pm \frac{k\pi}{2}\right)$

$z = \text{cis}\left(-\frac{5\pi}{8}\right), \text{cis}\left(-\frac{\pi}{8}\right), \text{cis}\frac{3\pi}{8}, \text{cis}\frac{7\pi}{8}$

5c $z^4 = -8 - 8\sqrt{3}i$

$(r\text{cis}\,\theta)^4 = -8 - 8\sqrt{3}i$

$r^4\text{cis}\,4\theta = -8 - 8\sqrt{3}i$

$r^4 = \sqrt{8^2 + \left(8\sqrt{3}\right)^2}$
$= 16$

$r = 2$

$4\theta = 2k\pi + \left(-\pi + \tan^{-1}\left(\frac{8\sqrt{3}}{8}\right)\right)$

$4\theta = 2k\pi - \frac{2\pi}{3}$

$\theta = \frac{1}{2}\left(k\pi - \frac{\pi}{3}\right)$

$z = 2\text{cis}\left(\frac{1}{2}\left(k\pi - \frac{\pi}{3}\right)\right)$

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$= 2\text{cis}\left(-\frac{\pi}{6}\right), 2\text{cis}\left(-\frac{2\pi}{3}\right), 2\text{cis}\frac{\pi}{3}, 2\text{cis}\frac{5\pi}{6}$
$= \sqrt{3} - i, -1 - i\sqrt{3}, 1 + i\sqrt{3}, -\sqrt{3} + i$

5d $z^5 = 16\sqrt{2} - 16\sqrt{2}i$

$(r\text{cis}\,\theta)^5 = 16\sqrt{2} - 16\sqrt{2}i$

$r^5\text{cis}\,5\theta = 16\sqrt{2} - 16\sqrt{2}i$

$r^5 = \sqrt{\left(16\sqrt{2}\right)^2 + \left(16\sqrt{2}\right)^2}$
$= 32$

$r = 2$

$5\theta = 2k\pi + \left(\tan^{-1}\left(\frac{-16\sqrt{2}}{16\sqrt{2}}\right)\right)$

$5\theta = 2k\pi - \frac{\pi}{4}$

$\theta = \frac{2k\pi}{5} - \frac{\pi}{20}$

$z = 2\,\text{cis}\left(\frac{2k\pi}{5} - \frac{\pi}{20}\right)$

$= 2\,\text{cis} - \frac{17\pi}{20}, 2\,\text{cis} - \frac{9\pi}{20}, 2\,\text{cis} - \frac{\pi}{20}, 2\,\text{cis}\frac{7\pi}{20}, 2\,\text{cis}\frac{3\pi}{4}$

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## Solutions to Exercise 3C Development questions

6a Let $z = r\text{cis}(\theta)$ be a fifth root of $-1$.

$r^5\text{cis}^5(\theta) = -1$

$r^5\text{cis}(5\theta) = \cos(\pi)$

Hence $r = 1$ and $5\theta = 2\lambda\pi + \pi$ where $\lambda$ is an integer.

This means that $\theta = \left(\frac{2\lambda+1}{5}\right)\pi$ and so

$\theta = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pi$

$z = \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right), \text{cis}(\pi)$

$z = \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right), -1$

6b The root with least principle argument is $\alpha = \text{cis}\left(\frac{\pi}{5}\right)$

$\alpha^3 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^3 = \text{cis}\left(\frac{3\pi}{5}\right)$

$\alpha^7 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^7 = \text{cis}\left(\frac{7\pi}{5}\right) = \text{cis}\left(\frac{7\pi}{5} - 2\pi\right) = \text{cis}\left(-\frac{3\pi}{5}\right)$

$\alpha^9 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^9 = \text{cis}\left(\frac{9\pi}{5}\right) = \text{cis}\left(\frac{9\pi}{5} - 2\pi\right) = \text{cis}\left(-\frac{\pi}{5}\right)$

Hence these are the three other complex roots.

6c $\alpha^7$

$= \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^7$

$= \text{cis}\left(\frac{7\pi}{5}\right)$

$= \text{cis}(\pi)\text{cis}(2\pi)$

$= \text{cis}(\pi)\left(\text{cis}\left(\frac{\pi}{5}\right)\right)^2$

Uncorrected proofs

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$= -\alpha^2$

$\alpha^9$

$= \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^9$

$= \text{cis}\left(\frac{9\pi}{5}\right)$

$= \text{cis}(\pi)\text{cis}\left(\frac{4\pi}{5}\right)$

$= -\text{cis}\left(\frac{4\pi}{5}\right)$

$= -\alpha^4$

6d $(1 + \alpha^2 + \alpha^4)$

$= 1 - \alpha^7 - \alpha^9$ (from part c)

$= -(-1 + \alpha^7 + \alpha^9)$

$= -(-1 + \alpha + \alpha^3 + \alpha^7 + \alpha^9) + (\alpha + \alpha^3)$

$= -0 + \alpha + \alpha^3$ (as the sum of roots is 0)

$= \alpha + \alpha^3$

7a $z = r\,\text{cis}(\theta)$

$z^7 = r^7\text{cis}(7\theta)$

$r^7\text{cis}(7\theta) = 1 = \text{cis}(0)$

Hence $r = 1$ and

$7\theta = \pm 2n\pi$

$\theta = \pm\frac{2n\pi}{7}$

$z = \text{cis}(0), \text{cis}\left(\pm\frac{2\pi}{7}\right), \text{cis}\left(\pm\frac{4\pi}{7}\right), \text{cis}\left(\pm\frac{6\pi}{7}\right)$

$z = 1, \text{cis}\left(\pm\frac{2\pi}{7}\right), \text{cis}\left(\pm\frac{4\pi}{7}\right), \text{cis}\left(\pm\frac{6\pi}{7}\right)$

Uncorrected proofs

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7b The sum of the roots is

$$1 + \text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{2\pi}{7}\right) + \text{cis}\left(\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right) + \text{cis}\left(\frac{6\pi}{7}\right) + \text{cis}\left(-\frac{6\pi}{7}\right) = 0$$

(as the coefficient of $z^6$ in the equation $z^7 - 1 = 0$ is 0)

$$1 + \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$$
$$+ \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} + \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7} = 0$$

$$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$$

$$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$$

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

7c Writing the equation as a product of factors gives

$$(z-1)\left(z - \text{cis}\left(\frac{2\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)\left(z - \text{cis}\left(\frac{4\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{4\pi}{7}\right)\right)$$
$$\left(z - \text{cis}\left(\frac{6\pi}{7}\right)\right)\left(z - \text{cis}\left(-\frac{6\pi}{7}\right)\right)$$

$$= (z-1)\left(z^2 - z\left(\text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{2\pi}{7}\right)\right) + \left(\text{cis}\left(\frac{2\pi}{7}\right)\text{cis}\left(-\frac{2\pi}{7}\right)\right)\right)$$
$$\left(z^2 - z\left(\text{cis}\left(\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right)\right) + \left(\text{cis}\left(\frac{4\pi}{7}\right)\text{cis}\left(-\frac{4\pi}{7}\right)\right)\right)$$
$$\left(z^2 - z\left(\text{cis}\left(\frac{6\pi}{7}\right) + \text{cis}\left(-\frac{6\pi}{7}\right)\right) + \left(\text{cis}\left(\frac{6\pi}{7}\right)\text{cis}\left(-\frac{6\pi}{7}\right)\right)\right)$$

$$= (z-1)\left(z^2 - z\left(\text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{2\pi}{7}\right)\right) + (\text{cis}(0))\right)$$
$$\left(z^2 - z\left(\cos\left(\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right)\right) + (\text{cis}(0))\right)$$
$$\left(z^2 - z\left(\text{cis}\left(\frac{6\pi}{7}\right) + \text{cis}\left(-\frac{6\pi}{7}\right)\right) + (\text{cis}(0))\right)$$

Uncorrected proofs

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$$= (z-1)\left(z^2 - z\left(2\cos\frac{2\pi}{7}\right) + (\text{cis}(0))\right)\left(z^2 - z\left(2\cos\frac{4\pi}{7}\right) + (\text{cis}(0))\right)$$

(Noting that $\text{cis}\, x + \text{cis}(-x) = \cos x + i\sin x + \cos x - i\sin x = 2\cos x$)

$$\left(z^2 - z\left(2\cos\frac{6\pi}{7}\right) + (\text{cis}(0))\right)$$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$$

7d

$$\alpha = \text{cis}\left(\frac{2\pi}{7}\right)$$

$$\alpha^2 = \left(\text{cis}\left(\frac{2\pi}{7}\right)\right)^2$$

$$= \text{cis}\left(\frac{4\pi}{7}\right)$$

$$\alpha^3 = \left(\text{cis}\left(\frac{2\pi}{7}\right)\right)^3$$

$$= \text{cis}\left(\frac{6\pi}{7}\right)$$

$$\alpha^4 = \left(\text{cis}\left(\frac{2\pi}{7}\right)\right)^4$$

$$= \text{cis}\left(\frac{8\pi}{7}\right)$$

$$= \text{cis}\left(-\frac{6\pi}{7}\right)$$

$$\alpha^5 = \left(\text{cis}\left(\frac{2\pi}{7}\right)\right)^5$$

$$= \text{cis}\left(\frac{10\pi}{7}\right)$$

$$= \text{cis}\left(-\frac{3\pi}{7}\right)$$

Uncorrected proofs

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$$\alpha^6 = \left(\text{cis}\left(\frac{2\pi}{7}\right)\right)^6$$

$$= \text{cis}\left(\frac{12\pi}{7}\right)$$

$$= \text{cis}\left(-\frac{2\pi}{7}\right)$$

These are the other complex roots that we have previously found.

7e For this question note that

$$\text{cis}\, x + \text{cis}(-x) = \cos x + i\sin x + \cos x - i\sin x = 2\cos x$$

$$\alpha + \alpha^6$$

$$= \text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{2\pi}{7}\right)$$

$$= 2\cos\frac{2\pi}{7}$$

$$\alpha^2 + \alpha^5$$

$$= \text{cis}\left(\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right)$$

$$= 2\cos\frac{4\pi}{7}$$

$$\alpha^3 + \alpha^4$$

$$= \text{cis}\left(\frac{6\pi}{7}\right) + \text{cis}\left(-\frac{6\pi}{7}\right)$$

$$= 2\cos\frac{6\pi}{7}$$

Hence, writing the equation as a product of linear factors gives

$$\left(x - 2\cos\frac{2\pi}{7}\right)\left(x - 2\cos\frac{4\pi}{7}\right)\left(x - 2\cos\frac{6\pi}{7}\right)x^3$$

$$-\left(2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7}\right)x^2$$

$$+\left(2\cos\frac{2\pi}{7} \times 2\cos\frac{4\pi}{7} + 2\cos\frac{2\pi}{7} \times 2\cos\frac{6\pi}{7} + 2\cos\frac{4\pi}{7} \times 2\cos\frac{4\pi}{7}\right)x$$

$$-\left(2\cos\frac{2\pi}{7}\right)\left(2\cos\frac{4\pi}{7}\right)\left(2\cos\frac{6\pi}{7}\right)$$

Uncorrected proofs

$= x^3 + x^2 - 2x - 1$

8a i Let $z = r\text{cis}(\theta)$ be a fifth root of 1.

$r^5\text{cis}^5(\theta) = 1$

$r^5\text{cis}(5\theta) = \cos(\pi)$

Hence $r = 1$ and $5\theta = 2\lambda\pi$ where $\lambda$ is an integer.

This means that $\theta = \left(\frac{2\lambda}{5}\right)\pi$ and so

$\theta = \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}, \pi$

$z = \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right), \text{cis}(\pi)$

$z = \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right), 1$

8a ii Note that all roots have a modulus of 1, and that all roots are evenly spaced as they have an argument of $\frac{2\pi}{5}$ between them. For example, $\frac{\text{cis}\left(\frac{4\pi}{5}\right)}{\text{cis}\left(\frac{2\pi}{5}\right)} = \text{cis}\left(\frac{2\pi}{5}\right)$.

Hence, they form the vertices of a regular polygon. As there are five vertices they must form the vertices of a regular pentagon.

8a iii The sum of the roots is

$\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) + 1 = 0$

(as the coefficient of $z^4$ in the equation $z^5 - 1 = 0$ is 0)

$\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)$

$-i\sin\left(\frac{4\pi}{5}\right) + 1 = 0$

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1 = 0$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

Uncorrected proofs

8b i $(z-1)(z^4+z^3+z^2+z+1)$

$= z^5+z^4+z^3+z^2+z-(z^4+z^3+z^2+z+1)$

$= z^5-1$

8b ii Since the roots of $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$ are
$z=\text{cis}\left(\pm\frac{2\pi}{5}\right),\text{cis}\left(\pm\frac{4\pi}{5}\right),1$

it follows that the roots of $(z^4+z^3+z^2+z+1)$ are

$z=\text{cis}\left(\pm\frac{2\pi}{5}\right),\text{cis}\left(\pm\frac{4\pi}{5}\right)$

Writing $(z^4+z^3+z^2+z+1)$

$$\left(z-\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{5}\right)+\text{cis}\left(-\frac{2\pi}{5}\right)\right)+\text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{4\pi}{5}\right)+\text{cis}\left(-\frac{4\pi}{5}\right)\right)+\text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{5}\right)+\text{cis}\left(-\frac{2\pi}{5}\right)\right)+\text{cis}(0)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{4\pi}{5}\right)+\text{cis}\left(-\frac{4\pi}{5}\right)\right)+\text{cis}(0)\right)$$

$$=(z^2-2z\cos\frac{2\pi}{5}+1)(z^2-2z\cos\frac{4\pi}{5}+1)$$

as required

8b iii Equating the coefficients of $z$ gives

$$1=-2\cos\frac{2\pi}{5}-2\cos\frac{4\pi}{5}$$

$$1=-2\cos\frac{2\pi}{5}+2\cos\frac{\pi}{5}$$

$$1+2\cos\frac{2\pi}{5}=2\cos\frac{\pi}{5}$$

Uncorrected proofs

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$$1 + 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\cos\frac{\pi}{5}$$

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

8c i $x^2 + x - 1 = 0$

$$\left(u + \frac{1}{u}\right)^2 + \left(u + \frac{1}{u}\right) - 1 = 0$$

$$u^2 + 2 + \frac{1}{u^2} + u + \frac{1}{u} - 1 = 0$$

$$u^2 + \frac{1}{u^2} + u + \frac{1}{u} + 1 = 0$$

$$u^4 + 1 + u^3 + u + u^2 = 0$$

$$u^4 + u^3 + u^2 + u + 1 = 0$$

This has roots $u = \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$

Hence

For $u = \text{cis}\left(\frac{2\pi}{5}\right)$,

$$x = \text{cis}\left(\frac{2\pi}{5}\right) + \frac{1}{\text{cis}\left(\frac{2\pi}{5}\right)}$$

$$= \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$= 2\cos\frac{2\pi}{5}$$

For $u = \text{cis}\left(-\frac{2\pi}{5}\right)$,

$$x = \text{cis}\left(-\frac{2\pi}{5}\right) + \frac{1}{\text{cis}\left(-\frac{2\pi}{5}\right)}$$

$$= \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right)$$

Uncorrected proofs

$$= \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$= 2\cos\frac{2\pi}{5}$$

For $u = \text{cis}\left(\frac{4\pi}{5}\right)$,

$$x = \text{cis}\left(\frac{4\pi}{5}\right) + \frac{1}{\text{cis}\left(\frac{4\pi}{5}\right)}$$

$$= \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$$

$$= 2\cos\frac{4\pi}{5}$$

For $u = \text{cis}\left(-\frac{4\pi}{5}\right)$,

$$x = \text{cis}\left(-\frac{4\pi}{5}\right) + \frac{1}{\text{cis}\left(-\frac{4\pi}{5}\right)}$$

$$= \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right)$$

$$= \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$$

$$= 2\cos\frac{4\pi}{5}$$

Uncorrected proofs

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8c ii

$$\left(x - 2\cos\frac{2\pi}{5}\right)\left(x - 2\cos\frac{4\pi}{5}\right) = 0$$

$$x^2 - 2x\cos\frac{4\pi}{5} - 2x\cos\frac{2\pi}{5} + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = 0$$

$$x^2 + 2x\cos\frac{\pi}{5} - 2x\cos\frac{2\pi}{5} - 4\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = 0$$

Comparing with $x^2 + x - 1 = 0$

$$4\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = 1$$

$$\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = \frac{1}{4}$$

9a $z = r\,\text{cis}(\theta)$

$z^9 = r^9\,\text{cis}(9\theta)$

$z^9 = 1$

$r^9\,\text{cis}(9\theta) = 1$

$r^9\,\text{cis}(9\theta) = \text{cis}(0)$

$r = 1$ and so $9\theta = 2n\pi$ which means that

$\theta = \dfrac{2n\pi}{9}$ so

$$z = \text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{6\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right), 1$$

9b $z^9 - 1$

$= (z^9 + z^6 + z^3) - (z^6 + z^3 + 1)$

$= (z^3 - 1)(z^6 + z^3 + 1)$

Uncorrected proofs

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Writing $z^9 - 1$ as a product of factors gives

$$(z-1)\left(z-\text{cis}\left(-\frac{2\pi}{9}\right)\right)\left(z-\text{cis}\left(\frac{2\pi}{9}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{9}\right)\right)\left(z-\text{cis}\left(\frac{4\pi}{9}\right)\right)$$
$$\left(z-\text{cis}\left(-\frac{6\pi}{9}\right)\right)\left(z-\text{cis}\left(\frac{6\pi}{9}\right)\right)\left(z-\text{cis}\left(-\frac{8\pi}{9}\right)\right)\left(z-\text{cis}\left(\frac{8\pi}{9}\right)\right)$$

Note that

$$\big(z-\text{cis}(x)\big)\big(z-\text{cis}(-x)\big)$$
$$= \left(z^2 - z\big(\text{cis}(x)+\text{cis}(-x)\big)\right) + (\text{cis}(0))$$
$$= z^2 - z(\cos x + i\sin x + \cos x - i\sin x) + 1$$
$$= z^2 - 2z\cos x + 1$$

Thus

$$z^9 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$
$$\left(z^2 - 2z\cos\frac{6\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$
$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$
$$\left(z^2 - 2z\cos\frac{6\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$
$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$
$$(z^2 + z + 1)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$
$$= (z-1)(z^2 + z + 1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)$$
$$\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$
$$= (z^3 - 1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

Hence

$$z^6 + z^3 + 1 = \left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

Uncorrected proofs

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9c $z^6 + z^3 + 1 = \left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$

Dividing both sides by $z^3$

$$z^3 + 1 + z^{-3} = \left(z - 2\cos\frac{2\pi}{9} + z^{-1}\right)\left(z - 2\cos\frac{4\pi}{9} + z^{-1}\right)\left(z - 2\cos\frac{8\pi}{9} + z^{-1}\right)$$

$$z^3 + z^{-3} + 1 = \left(z + z^{-1} - 2\cos\frac{2\pi}{9}\right)\left(z + z^{-1} - 2\cos\frac{4\pi}{9}\right)\left(z + z^{-1} - 2\cos\frac{8\pi}{9}\right)$$

$$z^3 + z^{-3} + 1 = \left(z + z^{-1} - 2\cos\frac{2\pi}{9}\right)\left(z + z^{-1} - 2\cos\frac{4\pi}{9}\right)\left(z + z^{-1} - 2\cos\frac{8\pi}{9}\right)$$

Now

$z^n + z^{-n}$

$= \text{cis}(n\theta) + \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

Hence

$$2\cos 3\theta + 1 = \left(\cos\theta - 2\cos\frac{2\pi}{9}\right)\left(\cos\theta - 2\cos\frac{4\pi}{9}\right)\left(\cos\theta - 2\cos\frac{8\pi}{9}\right)$$

10a $\omega = \text{cis}\left(\frac{2\pi}{9}\right)$

Let $z = \omega^k$

$z^9 = (\omega^k)^9$

$= (\omega^9)^k$

$= \left(\left(\text{cis}\left(\frac{2\pi}{9}\right)\right)^9\right)^k$

$= \left(\text{cis}(2\pi)\right)^k$

$= (1)^k$

$= 1$

Uncorrected proofs

10b $(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8)$

$= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9$

$-(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8)$

$= \omega^9 - 1$

Hence the equation
$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8) = 0$

has the same roots as $\omega^9 - 1 = 0$ which are the ninth roots of unity.

Hence the roots of

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$

will be the complex ninth roots of unity.

So, provided that $\omega \neq 1$,

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$

and so

$\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$

10c

$$\left(\text{cis}\frac{2\pi}{9}\right) + \left(\text{cis}\frac{2\pi}{9}\right)^2 + \left(\text{cis}\frac{2\pi}{9}\right)^3 + \left(\text{cis}\frac{2\pi}{9}\right)^4 + \left(\text{cis}\frac{2\pi}{9}\right)^5 + \left(\text{cis}\frac{2\pi}{9}\right)^6 + \left(\text{cis}\frac{2\pi}{9}\right)^7 + \left(\text{cis}\frac{2\pi}{9}\right)^8 = -1$$

$$\text{cis}\frac{2\pi}{9} + \text{cis}\frac{4\pi}{9} + \text{cis}\frac{6\pi}{9} + \text{cis}\frac{8\pi}{9} + \text{cis}\frac{10\pi}{9} + \text{cis}\frac{12\pi}{9} + \text{cis}\frac{14\pi}{9} + \text{cis}\frac{16\pi}{9} = -1$$

$$\text{cis}\frac{2\pi}{9} + \text{cis}\frac{4\pi}{9} + \text{cis}\frac{6\pi}{9} + \text{cis}\frac{8\pi}{9} + \text{cis}\left(\frac{-8\pi}{9}\right) + \text{cis}\left(-\frac{6\pi}{9}\right) + \text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\left(-\frac{2\pi}{9}\right) = -1$$

$$\text{cis}\frac{2\pi}{9} + \text{cis}\left(-\frac{2\pi}{9}\right) + \text{cis}\frac{4\pi}{9} + \text{cis}\left(-\frac{4\pi}{9}\right) + \text{cis}\frac{6\pi}{9} + \text{cis}\left(-\frac{6\pi}{9}\right) + \text{cis}\frac{8\pi}{9} + \text{cis}\left(\frac{-8\pi}{9}\right) = -1$$

Uncorrected proofs

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}-1+2\cos\frac{8\pi}{9}=-1$$

$$2\cos\frac{2\pi}{9}+2\cos\frac{4\pi}{9}+2\cos\frac{8\pi}{9}=0$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}=0$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}=0$$

$$\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}=\cos\frac{\pi}{9}$$

10d <Needs adjusting>

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$=\left(\cos\frac{2\pi}{9}+\cos\frac{4\pi}{9}\right)\left(\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\right)$$

$$=(\omega+\omega^2)(\omega\times\omega^2)$$

$$=\omega^4+\omega^5$$

$$=\text{cis}\frac{8\pi}{9}+\text{cis}\frac{10\pi}{9}$$

$$=-\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}+\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}$$

$$=2i\sin\frac{\pi}{9}$$

11a $\rho^7=\left(\cos\frac{2\pi}{7}+i\cos\frac{2\pi}{7}\right)^7$

$$=\left(\text{cis}\frac{2\pi}{7}\right)^7$$

$$=\text{cis}(2\pi)$$

$$=1$$

Hence $\rho^7-1=0$

$(\rho-1)(1+\rho+\rho^2+\cdots+\rho^6)=0$

Since $\rho\neq 1$,

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$(1 + \rho + \rho^2 + \cdots + \rho^6) = 0$

11b Since the equation has real coefficients, and $\alpha$ is complex, the complex conjugate must also be a root. Hence

$\beta = \overline{\alpha}$

$= \overline{\rho + \rho^2 + \rho^4}$

$= \overline{\rho} + \overline{\rho^2} + \overline{\rho^4}$

$= \overline{\text{cis}\left(\frac{2\pi}{7}\right)} + \overline{\text{cis}\left(\frac{2\pi}{7}\right)^2} + \overline{\text{cis}\left(\frac{2\pi}{7}\right)^4}$

$= \overline{\text{cis}\left(\frac{2\pi}{7}\right)} + \overline{\text{cis}\left(\frac{4\pi}{7}\right)} + \overline{\text{cis}\left(\frac{8\pi}{7}\right)}$

$= \text{cis}\left(-\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{8\pi}{7}\right)$

$= \text{cis}\left(\frac{12\pi}{7}\right) + \text{cis}\left(\frac{10\pi}{7}\right) + \text{cis}\left(\frac{6\pi}{7}\right)$

$= \text{cis}\left(\frac{2\pi}{7}\right)^6 + \text{cis}\left(\frac{2\pi}{7}\right)^5 + \text{cis}\left(\frac{2\pi}{7}\right)^3$

$= \rho^6 + \rho^5 + \rho^3$

So $\beta = \rho^3 + \rho^5 + \rho^6$

11c $\alpha + \beta = -a$

Hence

$a = -(\alpha + \beta)$

$= -(\rho + \rho^2 + \rho^4 + \rho^3 + \rho^5 + \rho^6)$

$= -(-1)$

$= 1$

$\alpha\beta = b$

$b = \alpha\beta$

$= (\rho + \rho^2 + \rho^4)(\rho^3 + \rho^5 + \rho^6)$

Uncorrected proofs

$$= \rho^4 + \rho^6 + \rho^7 + \rho^5 + \rho^7 + \rho^8 + \rho^7 + \rho^8 + \rho^{10}$$

$$= \rho^4 + \rho^6 + 1 + \rho^5 + 1 + \rho + 1 + \rho^2 + \rho^3 \quad (\text{since } \rho^7 = 1)$$

$$= 3 + \rho + \rho^2 + \rho^4 + \rho^3 + \rho^5 + \rho^6$$

$$= 3 + (-1)$$

$$= 2$$

11d <Solution to come>

Uncorrected proofs

Ext 2

Ex 3C

Q12 a) (i) $\cos 4\theta = (\cos\theta)^4$ by de Moivre

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

equating real & imaginary parts

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(ii) so for $\cos 4\theta \neq 0$

$$\tan 4\theta = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \times \frac{\left(\frac{1}{\cos^4\theta}\right)}{\left(\frac{1}{\cos^4\theta}\right)} \quad ; \cos\theta \neq 0$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(iii) put $\theta = \tan^{-1}\frac{1}{3}$ then $\tan\theta = \frac{1}{3}$ and

$\tan(RHS) = \tan 4\theta$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$$= \frac{4\cdot\frac{1}{3} - 4\cdot\frac{1}{27}}{1 - 6\cdot\frac{1}{9} + \frac{1}{81}} \times \frac{81}{81}$$

$$= \frac{108 - 12}{81 - 54 + 1}$$

$$= \frac{96}{28}$$

$$= \frac{24}{7}$$

hence $RHS = \tan^{-1}\left(\frac{24}{7}\right)$

$= LHS.$

Note: $\theta < \frac{\pi}{4}$ $(\tan\theta = \frac{1}{3})$ and $\tan 4\theta > 0$ so all angles in 1st quadrant.

b) let $z$ be a fourth root then $iz$, $-z$ and $-iz$ are also fourth roots. Now let $z = r\cis\theta$, $\theta$ acute then

$$r^4\cis 4\theta = 7 + 24i$$
$$= 25\left(\frac{7}{25} + \frac{24}{25}i\right)$$

so $r^4 = 25$ i.e. $r = \sqrt{5}$

and $\tan 4\theta = \frac{24}{7}$

i.e. $4\theta = \tan^{-1}\frac{24}{7}$

Ext 2

Ex 3C

Q 12 b (ctd) It follows from (a) that $\theta = \tan^{-1}\left(\frac{1}{3}\right)$

hence $z = \sqrt{5}(\operatorname{cis}\theta)$ where $\theta$ acute and $\tan\theta = \frac{1}{3}$




thus $z = \sqrt{5}\left(\frac{3}{\sqrt{10}} + i\,\frac{1}{\sqrt{10}}\right)$

$= \frac{3}{\sqrt{2}} + \frac{i}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}(3+i)$

and the other roots are

$iz = \frac{1}{\sqrt{2}}(-1+3i)$

$-z = \frac{-1}{\sqrt{2}}(3+i)$

$-iz = \frac{1}{\sqrt{2}}(1-3i)$.

Ext 2

Ex 3C

(Q13 a) The LHS is a 7th degree polynomial so there should be 7 roots.

Clearly $\zeta = 0$ is not a solution of $(\zeta+1)^8 - \zeta^8 = 0$,

hence $\left(1 + \frac{1}{\zeta}\right)^8 - 1 = 0$

or $\omega^8 - 1 = 0$

where $\omega = 1 + \frac{1}{\zeta}$, excluding $\omega = 1$ since $\zeta$ is undefined there.

Put $\omega = \text{cis}\,\theta$ and $1 = \text{cis}\,2k\pi$ then

$$\text{cis}\,8\theta = \text{cis}\,2k\pi$$

so $\theta = \frac{k\pi}{4}$ $\quad (k \neq 0$, since $\omega \neq 1)$

$k = \pm 1, \pm 2, \pm 3, 4$

thus $1 + \frac{1}{\zeta} = \text{cis}\,\frac{k\pi}{4}$

$$\zeta = \frac{1}{\text{cis}\,\frac{k\pi}{4} - 1} \times \frac{\left(\overline{\text{cis}\,\frac{k\pi}{8}}\right)}{\left(\overline{\text{cis}\,\frac{k\pi}{8}}\right)}$$

(noting the half angle in the given roots) (see also 3A Q18)

$$= \frac{\overline{\text{cis}\,\frac{k\pi}{8}}}{\text{cis}\,\frac{k\pi}{8} - \overline{\text{cis}\,\frac{k\pi}{8}}}$$

$$= \frac{\cos\frac{k\pi}{8} - i\sin\frac{k\pi}{8}}{2i\sin\frac{k\pi}{8}}$$

$$= -\frac{1}{2}\left(i\cot\frac{k\pi}{8} + 1\right) \qquad k = \pm 1, \pm 2, \pm 3, 4$$

$$= -\frac{1}{2} \qquad (k = 4)$$

or $-\frac{1}{2}\left(1 \pm i\cot\frac{k\pi}{8}\right)$ $\quad k = 1, 2, 3$ (since cot is odd)

b) By the factor theorem for polynomials, & the roots of (a)

$$(\zeta+1)^8 - \zeta^8 = 8\left(\zeta + \tfrac{1}{2}\right)\left(\zeta + \tfrac{1}{2}\left(1 + i\cot\tfrac{\pi}{8}\right)\right)\left(\zeta + \tfrac{1}{2}\left(1 - i\cot\tfrac{\pi}{8}\right)\right)$$
$$\times \left(\zeta + \tfrac{1}{2}\left(1 + i\cot\tfrac{\pi}{4}\right)\right)\left(\zeta + \tfrac{1}{2}\left(1 - i\cot\tfrac{\pi}{4}\right)\right)$$
$$\times \left(\zeta + \tfrac{1}{2}\left(1 + i\cot\tfrac{3\pi}{8}\right)\right)\left(\zeta + \tfrac{1}{2}\left(1 - i\cot\tfrac{3\pi}{8}\right)\right)$$

Ext 2

Ex 3C

Q13 (b) (ctd)

$$(\zeta+1)^8-\zeta^8 = 4(2\zeta+1)\left(\zeta^2+\zeta+\tfrac{1}{4}\left(1+\cot^2\tfrac{\pi}{8}\right)\right) \times \left(\zeta^2+\zeta+\tfrac{1}{4}\left(1+\cot^2\tfrac{\pi}{4}\right)\right) \times \left(\zeta^2+\zeta+\tfrac{1}{4}\left(1+\cot^2\tfrac{3\pi}{8}\right)\right) \quad (\dagger)$$

$$= 4(2\zeta+1)\left(\zeta^2+\zeta+\tfrac{1}{4}\operatorname{cosec}^2\tfrac{\pi}{8}\right) \times \left(\zeta^2+\zeta+\tfrac{1}{2}\right) \times \left(\zeta^2+\zeta+\tfrac{1}{4}\operatorname{cosec}^2\tfrac{3\pi}{8}\right)$$

$$= \tfrac{1}{8}(2\zeta+1)\left(4\zeta^2+4\zeta+\operatorname{cosec}^2\tfrac{\pi}{8}\right) \times \left(2\zeta^2+2\zeta+1\right) \times \left(4\zeta^2+4\zeta+\operatorname{cosec}^2\tfrac{3\pi}{8}\right)$$

c) put $\left(\zeta+\frac{1}{2}\right) = \frac{\cos 2\theta}{2}$ into $(\dagger)$ above to get

$$\text{RHS}(\dagger) = 4\cos 2\theta\left(\frac{\cos^2 2\theta}{4}+\tfrac{1}{4}\cot^2\tfrac{\pi}{8}\right) \times \left(\frac{\cos^2 2\theta}{4}+\tfrac{1}{4}\cot^2\tfrac{\pi}{4}\right) \times \left(\frac{\cos^2 2\theta}{4}+\tfrac{1}{4}\cot^2\tfrac{3\pi}{8}\right)$$

$$= \tfrac{1}{16}\cos 2\theta\left(\cos^2 2\theta+\cot^2\tfrac{\pi}{8}\right)\left(\cos^2 2\theta+1\right)\left(\cos^2 2\theta+\cot^2\tfrac{3\pi}{8}\right)$$

$$\text{LHS}(\dagger) = \left(\frac{\cos 2\theta}{2}+\tfrac{1}{2}\right)^8-\left(\frac{\cos 2\theta}{2}-\tfrac{1}{2}\right)^8$$

$$= (\cos^2\theta)^8-(\sin^2\theta)^8 \qquad \text{(double angle formulae)}$$

$$= \cos^{16}\theta-\sin^{16}\theta \qquad \neq$$

Ext 2 Ch 3

Ex 3C

Q 14 $\omega^3 = 1$, $\omega \neq 1$ so $\omega = \text{cis}\,\frac{2\pi}{3}$ or $\overline{\text{cis}\,\frac{2\pi}{3}}$

and the proof is left as an exercise.

Note the conjugate roots since the polynomial equation has real roots.

For $\omega = \text{cis}\,\frac{2\pi}{3}$, suppose $\omega^k = 1$, then

$\text{cis}\,\frac{2k\pi}{3} = 1$ by de Moivre

so $\frac{2k\pi}{3}$ is a multiple of $2\pi$

hence $k$ is a multiple of 3.

Likewise for $\omega = \overline{\text{cis}\,\frac{2\pi}{3}}$.

a) since $\omega^3 = 1$ it follows that $(\omega^3)^k = 1$

so $(\omega^k)^3 - 1 = 0$

or $(\omega^k - 1)(\omega^{2k} + \omega^k + 1) = 0$

<u>either</u> $\omega^k - 1 = 0$ in which case $k$ is a multiple of 3

and $\omega^{2k} + \omega^k + 1 = (\omega^k)^2 + (\omega^k) + 1$

$= 1 + 1 + 1$

$= 3$

<u>or</u> $\omega^{2k} + \omega^k + 1 = 0$

in which case $k$ is not a multiple of 3

<u>but not both</u>. (The proof of this is left as an exercise)

Thus $\omega^{2k} + \omega^k + 1 = \begin{cases} 3 & \text{if } k \text{ is a multiple of } 3 \\ 0 & \text{otherwise} \end{cases}$

b) $(1+\omega)^n = \sum_{r=0}^{n} {}^nC_r\, \omega^r$ and $(1+\omega^2)^n = \sum_{r=0}^{n} {}^nC_r\, \omega^{2r}$

c) $\text{RHS} = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right)$

$= \frac{1}{3}\left((1+1)^n + (1+\omega)^n + (1+\omega^2)^n\right)$

$= \frac{1}{3}\sum_{r=0}^{n}\left({}^nC_r + {}^nC_r\,\omega^r + {}^nC_r\,\omega^{2r}\right)$

$= \frac{1}{3}\sum_{r=0}^{n} {}^nC_r\left(1 + \omega^r + \omega^{2r}\right)$

$= \frac{1}{3}\left({}^nC_0 \cdot 3 + {}^nC_3 \cdot 3 + {}^nC_6 \cdot 3 + \cdots + {}^nC_{3L} \cdot 3\right)$ (all other terms zero by part (a))

$= {}^nC_0 + {}^nC_3 + {}^nC_6 + \cdots + {}^nC_{3L} = \text{LHS}.$

Ex 12

Ex 13C

Q14 (ctd)

d) since $n$ is a multiple of 6, $3k = n$ with $k$ even

so from (c)

$$^nC_0 + {}^nC_3 + {}^nC_6 + \dots + {}^nC_n = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right) \quad (*)$$

now in $1 + \omega^k + \omega^{2k}$ put $k = 1$ to get

$$1 + \omega + \omega^2 = 0 \quad \text{by (a)}$$

So $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$

thus RHS $(*) = \frac{1}{3}\left(2^n + (-\omega^2)^n + (-\omega)^n\right)$

$= \frac{1}{3}\left(2^n + (\omega^n)^2 + \omega^n\right)$ since $n$ even

$= \frac{1}{3}\left(2^n + (\omega^{2n} + \omega^n + 1) - 1\right)$

$= \frac{1}{3}\left(2^n + 3 - 1\right)$ by (a), since $n$ a multiple of 3

$= \frac{1}{3}\left(2^n + 2\right)$

Ext 2 Ch 3

Ex 3C

Q15 a) $(z+1)^{2n} + (z-1)^{2n} = 0$

so $z \neq 1, -1$. Hence, rearranging

$$\left(\frac{(z+1)}{(z-1)}\right)^{2n} = -1$$
$$= \text{cis}\,(2k+1)\pi$$

Thus by de Moivre

$$\frac{z+1}{z-1} = \text{cis}\,\frac{(2k+1)\pi}{2n}$$
$$= \text{cis}\,2\alpha \quad , \text{ where } \alpha = \frac{(2k+1)\pi}{4n}$$

Also, for princ. values $-\pi < \frac{(2k+1)\pi}{2n} \leq \pi$

$$-2n < 2k+1 \leq 2n$$
$$-2n-1 < 2k \leq 2n-1$$
so $$-n-\tfrac{1}{2} < k \leq n-\tfrac{1}{2}$$
or $$-n \leq k \leq n-1$$

Now $z+1 = (z-1)\,\text{cis}\,2\alpha$

$$z(\text{cis}\,2\alpha - 1) = \text{cis}\,2\alpha + 1$$
$$z = \frac{\text{cis}\,2\alpha + 1}{\text{cis}\,2\alpha - 1}$$
$$= \frac{\text{cis}\,2\alpha + 1}{\text{cis}\,2\alpha - 1} \times \frac{\overline{\text{cis}\,\alpha}}{\overline{\text{cis}\,\alpha}}$$

(using the half angle result)

(see also Ex 3A Q18 and Ex 3C Q13)

$$= \frac{\text{cis}\,\alpha + \overline{\text{cis}\,\alpha}}{\text{cis}\,\alpha - \overline{\text{cis}\,\alpha}}$$
$$= \frac{2\cos\alpha}{2i\sin\alpha}$$
$$= -i\cot\alpha$$
$$= -i\cot\frac{(2k+1)\pi}{4n} \quad ; \quad -n \leq k \leq n-1$$

Writing these in conjugate pairs (a polynomial equation with real coeffs.)

$z = -i\cot\frac{\pi}{4n}$, $i\cot\frac{\pi}{4n}$, $-i\cot\frac{3\pi}{4n}$, $i\cot\frac{3\pi}{4n}$, ---

$(k=0)$ $(k=-1)$ $(k=1)$ $(k=-2)$

$-i\cot\frac{(2n-1)\pi}{4n}$, $i\cot\frac{(2n-1)\pi}{4n}$

$(k=n-1)$ $(k=-n)$

Ext 2

Ex 3C

Q15 (b) (ctd)

$$OP_1^2 + OP_2^2 + \dots + OP_{2n}^2$$

$$= |z_1|^2 + |z_2|^2 + \dots + |z_{2n}|^2$$

$$= |z_1^2| + |z_2^2| + \dots + |z_{2n}^2|$$

$$= -z_1^2 - z_2^2 + \dots - z_{2n}^2 \qquad \left(\begin{matrix}\text{since each root}\\ \text{is imaginary}\end{matrix}\right)$$

$$= -\left(z_1^2 + z_2^2 + \dots + z_{2n}^2\right)$$

which is the opposite of the sum of squares of roots.

now $\left(\begin{matrix}\text{sum of squares}\\ \text{of roots}\end{matrix}\right) = \left(\begin{matrix}\text{sum of}\\ \text{roots}\end{matrix}\right)^2 - 2\left(\begin{matrix}\text{sum of roots}\\ \text{in pairs}\end{matrix}\right)$

Now $(z+1)^{2n} + (z-1)^{2n} = 0$

so leading terms are

$$\left.\begin{matrix} z^{2n} + {}^{2n}C_1 z^{2n-1} + {}^{2n}C_2 z^{2n-2} + \dots + 1 \\ + z^{2n} - {}^{2n}C_1 z^{2n-1} + {}^{2n}C_2 z^{2n-2} + \dots + 1 \end{matrix}\right\} = 0$$

cancelling opposite terms & dividing by 2

$$z^{2n} + {}^{2n}C_2 z^{2n-2} + \dots + 1 = 0$$

Hence sum of roots $= 0$

$$\text{sum of roots in pairs} = {}^{2n}C_2$$

Hence $OP_1^2 + OP_2^2 + \dots + OP_{2n}^2 = -\left(0^2 - 2 \cdot {}^{2n}C_2\right)$

$$= 2 \cdot {}^{2n}C_2$$

$$= 2 \cdot \frac{(2n)!}{(2n-2)! \cdot 2!}$$

$$= 2n(2n-1).$$

* Q15 a) Note: the answers in the text book are not written in conjugate pairs but $-i\cot\frac{(2k-1)\pi}{4n}$ for $k = 1, 2, 3, \dots, 2n$ gives the same roots. The proof is left as an exercise. [Hint: consider the quadrants of $\frac{(2k-1)\pi}{4n}$.]