Solutions to Exercise 3B Foundation questions

1a $\cos 3\theta + i \sin 3\theta$

$= (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

1a i Equating the real components:

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta\,(1 - \cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

1aii Equating imaginary components:

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

1b $\tan 3\theta$

$= \dfrac{\sin 3\theta}{\cos 3\theta}$

$= \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}$

$= \dfrac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \div \dfrac{\cos^3\theta}{\cos^3\theta}$

$= \dfrac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta}$

$= \dfrac{3\tan\theta\left(\tan^2\theta + 1\right) - 4\tan^3\theta}{4 - 3\left(\tan^2\theta + 1\right)}$

$= \dfrac{3\tan^3\theta + 3\tan\theta - 4\tan^3\theta}{4 - 3\tan^2\theta - 3}$

$= \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

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2 $\cos 4\theta + i \sin 4\theta$

$= (\cos\theta + i\sin\theta)^4$
$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4$
$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

2a Equating the real components of the above equation:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

2b Equating the imaginary components of the above equation:

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

2c $\tan 4\theta$

$$= \frac{\sin 4\theta}{\cos 4\theta}$$
$$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$
$$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \div \frac{\cos^4\theta}{\cos^4\theta}$$
$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

3a $z^n + z^{-n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$
$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$
$= 2\cos n\theta$

3b $(z + z^{-1})^4$

$= z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4z^1z^{-3} + z^{-4}$
$= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$
$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

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3c $\cos^4\theta$

$$= \left(\frac{1}{2} \times 2\cos\theta\right)^4$$
$$= \left(\frac{1}{2}(z + z^{-1})\right)^4$$
$$= \frac{1}{16}(z + z^{-1})^4$$
$$= \frac{1}{16}\left((z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 16\right)$$
$$= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 16)$$
$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + 1$$

4 $z^n - z^{-n}$

$$= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$
$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$
$$= 2i\sin n\theta$$

$\sin^4\theta$

$$= \left(\frac{1}{2} \times 2\sin\theta\right)^4$$
$$= \left(\frac{1}{2i}(z - z^{-1})\right)^4$$
$$= \frac{1}{(2i)^4}(z - z^{-1})^4$$
$$= \frac{1}{2^4 i^4}(z^4 - 4z^3 z^{-1} + 6z^2 z^{-2} - 4zz^{-3} + z^{-4})$$
$$= \frac{1}{16}\left((z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6\right)$$
$$= \frac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6)$$
$$= \frac{\cos 4\theta}{8} - \frac{\cos 2\theta}{2} + \frac{3}{8}$$

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Solutions to Exercise 3B Development questions

5a Let $z = \cos\theta + i\sin\theta$

From question 3,

$z^n + z^{-n}$

$= 2\cos n\theta$

$= 2^5\cos^5\theta$

$= (2\cos\theta)^5$

$= (\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)^5$

$= \big(\text{cis}(\theta) + \text{cis}(-\theta)\big)^5$

$= (z + z^{-1})^5$

$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

$= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

$= 2\cos(5\theta) + 5(2\cos(3\theta)) + 10(2\cos(\theta))$

$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

Hence

$2^5\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$\cos^5\theta = \frac{1}{2^5}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta)$

$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

5b

$\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\, d\theta$

$= \left[\frac{1}{16}\left(\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right)\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{16}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2}\right)$

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$$= \frac{1}{16}\left(\frac{1}{5}(1) + \frac{5}{3}(-1) + 10(1)\right)$$

$$= \frac{8}{15}$$

6a $\cos 6\alpha + \mathrm{i}\sin 6\alpha$

$= \mathrm{cis}(6\alpha)$

$= \left(\mathrm{cis}(\alpha)\right)^6$

$= (\cos\alpha + i\sin\alpha)^6$

$= \cos^6\alpha + 6i\cos^5\alpha\sin\alpha + 15i^2\cos^4\alpha\sin^2\alpha + 20i^3\cos^3\alpha\sin^3\alpha$
$+15i^4\cos^2\alpha\sin^4\alpha + 6i^5\sin\alpha\cos^5\alpha + i^6\sin^6\alpha$

$= \cos^6\alpha + 6i\cos^5\alpha\sin\alpha - 15\cos^4\alpha\sin^2\alpha - 20i\cos^3\alpha\sin^3\alpha$
$+15\cos^2\alpha\sin^4\alpha + 6i\sin\alpha\cos^5\alpha - \sin^6\alpha$

$= (\cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha)$
$+i(6\cos^5\alpha\sin\alpha - 20\cos^3\alpha\sin^3\alpha + 6\sin\alpha\cos^5\alpha)$

Equating real and imaginary components

$\cos 6\alpha$

$= \cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha$

$= \cos^6\alpha - 15\cos^4\alpha(1-\cos^2\alpha) + 15\cos^2\alpha(1-\cos^2\alpha)^2 - (1-\cos^2\alpha)^3$

$= \cos^6\alpha - 15\cos^4\alpha(1-\cos^2\alpha) + 15\cos^2\alpha(1-2\cos^2\alpha+\cos^4\alpha)$
$-(1-3\cos^2\alpha+3\cos^4\alpha-\cos^6\alpha)$

$= 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$

6b $32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1 = 0$

is solved when

$\cos 6\alpha = 0$

which is when

$\alpha = \frac{n\pi}{12}$ for $n = 1, 3, 4, 5, 9, 11$

Hence, letting $x = \cos\alpha$, the equation

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$32x^6 + 4x^3 - 6x^2 - 4x + 1 = 0$

has solutions when

$x = \cos\frac{n\pi}{12}$ for $n = 1, 3, 4, 5, 9, 11$

6c The product of the roots is

$$\cos\frac{\pi}{12}\cos\frac{3\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{9\pi}{12}\cos\frac{11\pi}{12} = \frac{-1}{32}$$

$$\left(\cos\frac{\pi}{12}\right)\left(\cos\frac{\pi}{4}\right)\left(\cos\frac{5\pi}{12}\right)\left(-\cos\frac{5\pi}{12}\right)\left(\cos\frac{3\pi}{4}\right)\left(-\cos\frac{\pi}{12}\right) = \frac{-1}{32}$$

$$\left(\cos\frac{\pi}{12}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\cos\frac{5\pi}{12}\right)\left(\cos\frac{5\pi}{12}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\cos\frac{\pi}{12}\right) = \frac{1}{32}$$

$$\frac{1}{2}\left(\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\right)^2 = \frac{1}{32}$$

$$\left(\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\right)^2 = \frac{1}{16}$$

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \pm\frac{1}{4}$$

Since $\cos\frac{\pi}{12} > 0$ and $\cos\frac{5\pi}{12} > 0$ it follows that

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12} = \frac{1}{4}$$

7a Let $x = \tan\theta$

$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$

$$\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$$

$$\tan^4\theta - 6\tan^2\theta + 1 = 4\tan^3\theta - 4\tan\theta$$

$$1 = \frac{4\tan^3\theta - 4\tan\theta}{\tan^4\theta - 6\tan^2\theta + 1}$$

$$1 = \tan 4\theta$$

$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$$

$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$$

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Hence the equation is solved when $x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$

7b The sum of the roots is

$$\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\frac{9\pi}{16} + \tan\frac{13\pi}{16} = -4$$

Hence

$$\left(\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\frac{9\pi}{16} + \tan\frac{13\pi}{16}\right)^2 = 16$$

$$\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16}$$

$$+2\left(\tan\frac{\pi}{16}\tan\frac{5\pi}{16} + \tan\frac{\pi}{16}\tan\frac{9\pi}{16} + \tan\frac{\pi}{16}\tan\frac{13\pi}{16} + \tan\frac{5\pi}{16}\tan\frac{9\pi}{16} + \tan\frac{9\pi}{16}\tan\frac{13\pi}{16}\right) = 16$$

However, the term in the brackets is just the sum of the products of the roots, hence

$$\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16} + 2(-6) = 16$$

$$\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{9\pi}{16} + \tan^2\frac{13\pi}{16} = 28$$

$$\tan^2\frac{\pi}{16} + \tan^2\frac{5\pi}{16} + \left(-\tan\frac{7\pi}{16}\right)^2 + \left(-\tan\frac{3\pi}{16}\right)^2 = 28$$

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

8a $\cos 5\theta + i\sin 5\theta$

$= \text{cis}(5\theta)$

$= \left(\text{cis}(\theta)\right)^5$

$= (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$

$+5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

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$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$+i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating the imaginary parts of this equation gives:

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

8b Let $x = \sin\theta$

$16x^5 - 20x^3 + 5x - 1 = 0$

$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta - 1 = 0$

$\sin 5\theta - 1 = 0$

$\sin 5\theta = 1$

$5\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

$x = \sin\frac{\pi}{10}, \sin\frac{\pi}{2}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$

$x = 1, \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$

as required

8c $(4x^2+bx+c)^2$

$= 16x^4 + 4bx^3 + 4cx^2 + 4bx^3 + b^2x^2 + bcx + 4cx^2 + bcx + c^2$

$= 16x^4 + (4b+4b)x^3 + (4c+b^2+4c)x^2 + (bc+bc)x + c^2$

$= 16x^4 + 8bx^3 + (8c+b^2)x^2 + 2bcx + c^2$

Since

$16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2+bx+c)^2$

$16x^4 + 16x^3 - 4x^2 - 4x + 1 = 16x^4 + 8bx^3 + (8c+b^2)x^2 + 2bcx + c^2$

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Equating coefficients of x gives

$16 = 8b$ (1)

$-4 = 8c + b^2$ (2)

From (1), $b = 2$.

Substituting this into (2) gives $c = -1$.

Any root of $16x^4 + 16x^3 - 4x^2 - 4x + 1$ is necessarily a root of $(4x^2 + bx + c)$. Since there are two factors of $(4x^2 + bx + c)$ in the original equation, any root will be a root of the quadratic and will be a double root of the initial equations. Since quadratics have two roots, it follows that $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ must have two double roots (or one quadruple root).

8d Dividing $16x^5 - 20x^3 + 5x - 1$ by $(x - 1)$ yields $16x^4 + 16x^3 - 4x^2 + 1$.

Hence it follows that since $x = 1, \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$ are roots of the former equation, and since $x = 1$ is a root of the divisor,

$x = \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$ must be roots of the quotient $16x^4 + 16x^3 - 4x^2 + 1$ as required.

There is no contradiction to part c as $\sin\frac{9\pi}{10} = \sin\left(\pi - \frac{\pi}{10}\right) = \sin\frac{\pi}{10}$ and

$\sin\frac{13\pi}{10} = \sin\left(3\pi - \frac{13\pi}{10}\right) = \sin\frac{17\pi}{10}$ and so indeed there are two double roots.

8e The sum of the roots is

$$\sin\frac{\pi}{10} + \sin\frac{\pi}{10} + \sin\frac{13\pi}{10} + \sin\frac{13\pi}{10} + 1 = 0$$

$$\sin\frac{\pi}{10} + \sin\frac{\pi}{10} - \sin\frac{3\pi}{10} - \sin\frac{3\pi}{10} + 1 = 0$$

$$2\sin\frac{\pi}{10} - 2\sin\frac{3\pi}{10} + 1 = 0$$

$$\sin\frac{\pi}{10} - \sin\frac{3\pi}{10} + \frac{1}{2} = 0$$

Hence

$$\sin\frac{\pi}{10} = \sin\frac{3\pi}{10} - \frac{1}{2} \quad (1)$$

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The product of the roots is

$$\sin\frac{\pi}{10}\sin\frac{\pi}{10}\sin\frac{13\pi}{10}\sin\frac{13\pi}{10}(1) = \frac{1}{16}$$

$$\sin\frac{\pi}{10}\sin\frac{\pi}{10}\left(-\sin\frac{3\pi}{10}\right)\left(-\sin\frac{3\pi}{10}\right) = \frac{1}{16}$$

$$\sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10} = \frac{1}{16} \quad (2)$$

Hence

$$\sin\frac{\pi}{10}\sin\frac{3\pi}{10} = \frac{1}{4}$$

(noting that we take the positive solution as $\sin\frac{\pi}{10} > 0$ and $\sin\frac{3\pi}{10} > 0$)

Substituting (1) in (2) gives:

$$\left(\sin\frac{3\pi}{10} - \frac{1}{2}\right)\sin\frac{3\pi}{10} = \frac{1}{4}$$

$$\sin^2\frac{3\pi}{10} - \frac{1}{2}\sin\frac{3\pi}{10} - \frac{1}{4} = 0$$

$$\sin\frac{3\pi}{10} = \frac{-\left(-\frac{1}{2}\right) \pm \sqrt{\left(-\frac{1}{2}\right)^2 - 4(1)\left(-\frac{1}{4}\right)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{4}$$

Since $\sin\frac{3\pi}{10} > 0$

$$\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4}$$

$$\sin\frac{\pi}{10} = \frac{1+\sqrt{5}}{4} - \frac{1}{2}$$

$$= \frac{-1+\sqrt{5}}{4}$$

$$= \frac{\sqrt{5}-1}{4}$$

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9a $\cos 7\theta + i \sin 7\theta$

$= \text{cis}(7\theta)$

$= \left(\text{cis}(\theta)\right)^7$

$= (\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + 7i\cos^6\theta\sin\theta + 21i^2\cos^5\theta\sin^2\theta + 35i^3\cos^4\theta\sin^3\theta$

$+35i^4\cos^3\theta\sin^4\theta + 21i^5\cos^2\theta\sin^5\theta + 7i^6\cos^2\theta\sin^6\theta + i^7\sin^7\theta$

$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$

$+21i\cos^2\theta\sin^5\theta - 7\cos^2\theta\sin^6\theta - i\sin^7\theta$

Equating the real components on both sides of the equation

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos^2\theta\sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2$

$-7\cos^2\theta(1-\cos^2\theta)^3$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)$

$-7\cos^2\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

9b Let $x = 4\cos^2\theta$

$(4\cos^2\theta)^3 - 7(\cos^2\theta)^2 + 14(\cos^2\theta) - 7 = 0$

$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta = 0$

$\cos 7\theta = 0$

$7\theta = 2n\pi \pm \frac{\pi}{2}$ where n is an integer, hence

$7\theta = \left(\frac{4n \pm 1}{2}\right)\pi$

$\theta = \left(\frac{4n \pm 1}{14}\right)\pi$

So

$x = 4\cos^2\left(\frac{4n \pm 1}{14}\right)\pi$ where n is an integer, hence:

$x = 4\cos^2\frac{\pi}{14}, 4\cos^2\frac{3\pi}{14}, 4\cos^2\frac{5\pi}{14}$

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9c i The sum of the roots in the above equation is

$$4\cos^2\frac{\pi}{14}+4\cos^2\frac{3\pi}{14}+4\cos^2\frac{5\pi}{14}=\frac{-(-7)}{1}=7$$

Hence

$$\cos^2\frac{\pi}{14}+\cos^2\frac{3\pi}{14}+\cos^2\frac{5\pi}{14}=\frac{7}{4}$$

9c ii $\left(\cos^2\frac{\pi}{14}+\cos^2\frac{3\pi}{14}+\cos^2\frac{5\pi}{14}\right)^2=\left(\frac{7}{4}\right)^2$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}$$
$$+2\left(\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}+\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}+\cos^2\frac{\pi}{14}\cos^2\frac{5\pi}{14}\right)$$
$$=\left(\frac{7}{4}\right)^2$$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}$$
$$+\frac{2}{16}\left(4\cos^2\frac{\pi}{14}4\cos^2\frac{3\pi}{14}+4\cos^2\frac{3\pi}{14}4\cos^2\frac{5\pi}{14}+4\cos^2\frac{\pi}{14}4\cos^2\frac{5\pi}{14}\right)$$
$$=\left(\frac{7}{4}\right)^2$$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}+\frac{2}{16}(14)=\left(\frac{7}{4}\right)^2$$

(from the product of pairs of roots)

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}=\left(\frac{7}{4}\right)^2-\frac{28}{16}$$
$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}=\frac{21}{16}$$

10a Let $z=\cos\theta+i\sin\theta=\text{cis}(\theta)$

$$z-z^{-1}$$
$$=\text{cis}(\theta)-\frac{1}{\text{cis}(\theta)}$$

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$$= \text{cis}(\theta) - \frac{1}{\text{cis}(\theta)}$$

$$= \text{cis}(\theta) - \text{cis}(-\theta)$$

$$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$$

$$= 2i\sin\theta$$

$$z^n - z^{-n}$$

$$= \text{cis}(n\theta) - \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) - \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) - \text{cis}(-n\theta)$$

$$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$

$$= 2i\sin n\theta$$

$$(z - z^{-1})^5$$

$$= z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$$

$$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$$

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$2^5 i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$2^5 i\sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

10b From above we know that

$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

Hence, $16\sin^5\theta = \sin 5\theta$ if and only if

$$-5\sin 3\theta + 10\sin\theta = 0$$

$$10\sin\theta = 5\sin 3\theta$$

Uncorrected proofs

$$\sin\theta = \frac{1}{2}\sin 3\theta$$

$$\sin\theta = \frac{1}{2}(\sin\theta\cos 2\theta + \sin 2\theta\cos\theta)$$

$$\sin\theta = \frac{1}{2}(\sin\theta\,(1 - 2\sin^2\theta) + (2\sin\theta\cos\theta)\cos\theta)$$

$$\sin\theta = \frac{1}{2}(\sin\theta\,(1 - 2\sin^2\theta) + 2\sin\theta\cos^2\theta)$$

$$\sin\theta = \frac{1}{2}\big(\sin\theta\,(1 - 2\sin^2\theta) + 2\sin\theta\,(1 - \sin^2\theta)\big)$$

$$\sin\theta = \frac{1}{2}\sin\theta\,\big((1 - 2\sin^2\theta) + 2(1 - \sin^2\theta)\big)$$

$$\sin\theta = \frac{1}{2}\sin\theta\,(3 - 4\sin^2\theta)$$

$$\frac{1}{2}\sin\theta\,(3 - 4\sin^2\theta) - \sin\theta = 0$$

$$\sin\theta\,(3 - 4\sin^2\theta) - 2\sin\theta = 0$$

$$\sin\theta\,(1 - 4\sin^2\theta) = 0$$

$$\sin\theta\,(1 - 2\sin\theta)(1 + 2\sin\theta) = 0$$

Hence $\sin\theta = 0, \pm\frac{1}{2}$ and so $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$

11a Let $z = \cos\theta + i\sin\theta = \text{cis}(\theta)$

So

$$\cos 5\theta + i\sin 5\theta$$

$$= \text{cis}(5\theta)$$

$$= \big(\text{cis}(\theta)\big)^5$$

$$= (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$$
$$+5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

$$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$$
$$+i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Equating real and imaginary components,

Uncorrected proofs

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$

$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$

$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \div \frac{\cos^5\theta}{\cos^5\theta}$$

$$= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

11b Let $x = \tan\theta$

$$x^4 - 10x^2 + 5 = 0$$

$$\tan^4\theta - 10\tan^2\theta + 5 = 0$$

$$\tan\theta\,(\tan^4\theta - 10\tan^2\theta + 5) = \tan\theta \times 0$$

$$\tan^5\theta - 10\tan^3\theta + 5\tan\theta = 0$$

$$\frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = 0$$

$$\tan 5\theta = 0$$

$$5\theta = \pm\pi, \pm 2\pi$$

$$\theta = \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$$

Hence the roots are

$$x = \tan\left(\pm\frac{\pi}{5}\right), \tan\left(\pm\frac{2\pi}{5}\right)$$

$$x = \pm\tan\left(\frac{\pi}{5}\right), \pm\tan\left(\frac{2\pi}{5}\right)$$

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11c The product of pairs of the roots is

$$\left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) = 5$$

$$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$$

$$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$$

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5} \quad \left(\text{taking the positive solution as } \tan\frac{\pi}{5} > 0 \text{ and } \tan\frac{2\pi}{5} > 0\right)$$

The product of the pairs of roots is

$$\left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{\pi}{5}\right)\right) + \left(\tan\left(\frac{2\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right)$$
$$+ \left(-\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) + \left(-\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right)$$
$$= 10$$

$$\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$$

12a

$$z^n + \frac{1}{z^n}$$

$$= \left(\text{cis}(\theta)\right)^n + \frac{1}{\left(\text{cis}(\theta)\right)^n}$$

$$= \text{cis}(n\theta) + \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) + \text{cis}(-n\theta)$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

Uncorrected proofs

$$z^n - \frac{1}{z^n}$$

$$= \left(\text{cis}(\theta)\right)^n - \frac{1}{\left(\text{cis}(\theta)\right)^n}$$

$$= \text{cis}(n\theta) - \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) - \text{cis}(-n\theta)$$

$$= \cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta)$$

$$= 2i \sin n\theta$$

12b $128 \cos^3 \theta \sin^4 \theta$

$$= (2 \cos \theta)^3 (2i \sin \theta)^4$$

$$= \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^4$$

$$= \left(z^3 + \frac{3}{z} - \frac{3}{z} + \frac{1}{z^3}\right)\left(z^4 - \frac{4}{z^3} + \frac{16}{z} - 16z + 4z^3 - z^4\right)$$

$$= \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

12c From part b,

$$128 \cos^3 \theta \sin^4 \theta = \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

Using the result from part a,

$$128 \cos^3 \theta \sin^4 \theta = 2 \cos 7\theta - 2 \cos 5\theta - 3 \times 2 \cos 3\theta + 3 \times 2 \cos \theta$$

Hence

$$\cos^3 \theta \sin^4 \theta = \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3 \cos 3\theta + 3 \cos \theta)$$

as required

Uncorrected proofs

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13a $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$5z^2 - 11z + 16 - \frac{11}{z} + \frac{5}{z^2} = 0$

$5\left(z^2 + \frac{1}{z^2}\right) - 11\left(z + \frac{1}{z}\right) + 16 = 0$

$5(2\cos 2\theta) - 11(2\cos\theta) + 16 = 0$

$5\cos 2\theta - 11\cos\theta + 8 = 0$

13b $5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$

$\cos\theta = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(10)(3)}}{2(10)}$

$= \frac{11 \pm \sqrt{1}}{20}$

$= \frac{1}{2}$ or $\frac{3}{5}$

When $\cos\theta = \frac{1}{2}, \sin\theta = \pm\frac{\sqrt{2^2 - 1}}{2} = \pm\frac{\sqrt{3}}{2}$

When $\cos\theta = \frac{3}{5}, \sin\theta = \pm\frac{\sqrt{5^2 - 3^2}}{5} = \pm\frac{\sqrt{16}}{5} = \pm\frac{4}{5}$

$z = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

$z = \frac{3}{5} \pm \frac{4}{5}i$

14a

$\frac{\sin 8\theta}{\sin\theta\cos\theta}$

$= \frac{2\sin 4\theta\cos 4\theta}{\sin\theta\cos\theta}$

$= \frac{2(2\sin 2\theta\cos 2\theta)(1 - 2\sin^2 2\theta)}{\sin\theta\cos\theta}$

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$$= \frac{2\big(2(2\sin\theta\cos\theta)(1-2\sin^2\theta)\big)(1-2(1-2\sin^2\theta)^2)}{\sin\theta\cos\theta}$$

$$= 8(1-2\sin^2\theta)(1-2(1-2\sin^2\theta)^2)$$

$$= 8(1-2s^2)(1-2(1-2s^2)^2)$$

$$= 8(1-2s^2)\big(1-2(1-4s^2+4s^4)\big)$$

$$= 8(1-2s^2)(-1+8s^2-8s^4)$$

$$= 8(-1+8s^2-8s^4+2s^2-16s^4+16s^4)$$

$$= 8(1-10s^2+24s^4-16s^6)$$

14b $x^6-6x^4+10x^2-4=0$

Let $x = 2\sin\theta = 2s$

$$(2s)^6-6(2s)^4+10(2s)^2-4=0$$

$$64s^6-6\times16s^4+40s^2-4=0$$

$$4(16s^6-24s^4+10s^2-1)=0$$

$$-\frac{1}{2}\big(8(1-10s^2+24s^4-16s^6)\big)=0$$

$$-\frac{1}{2}\left(\frac{\sin 8\theta}{\sin\theta\cos\theta}\right)=0$$

$$\sin 8\theta = 0$$

$8\theta = n\pi$ for $n = \pm1, \pm2, \pm3$

Thus

$$x = 2\sin\frac{n\pi}{8} \text{ for } n = \pm1, \pm2, \pm3$$

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Ext 2 Ch 3

Ex 3B

Q 15 a) $\text{cis}\,(2n+1)\theta = (\text{cis}\,\theta)^{2n+1}$ (by de Moivre)

$$= {}^{2n+1}C_0 \cos^{2n+1}\theta + i\,{}^{2n+1}C_1 \cos^{n}\theta \sin\theta$$
$$- {}^{2n+1}C_2 \cos^{2n-1}\theta \sin^2\theta - i\,{}^{2n+1}C_3 \cos^{2n-2}\theta \sin^3\theta$$
$$+ {}^{2n+1}C_4 \cos^{2n-3}\theta \sin^4\theta + i\,{}^{2n+1}C_5 \cos^{2n-4}\theta \sin^5\theta$$
$$+ \cdots + i^{2n+1} \times {}^{2n+1}C_{2n+1} \sin^{2n+1}\theta$$

Now take imaginary parts & note $i^{2n} = (-1)^n$ to get

$$\sin(2n+1)\theta = {}^{2n+1}C_1 \cos^{2n}\theta \sin\theta - {}^{2n+1}C_3 \cos^{2n-2}\theta \sin^3\theta$$
$$+ {}^{2n+1}C_5 \cos^{2n-4}\theta \sin^5\theta + \cdots + (-1)^n \sin^{2n+1}\theta$$

b) divide through by $\sin^{2n+1}\theta$ for $\sin\theta \neq 0$

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = {}^{2n+1}C_1 \cot^{2n}\theta - {}^{2n+1}C_3 \cot^{2n-2}\theta$$
$$+ {}^{2n+1}C_5 \cot^{2n-4}\theta + \cdots + (-1)^n$$

Put $x = \cot^2\theta$ so that

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = {}^{2n+1}C_1 x^n - {}^{2n+1}C_3 x^{n-1} + {}^{2n+1}C_5 x^{n-2} + \cdots + (-1)^n$$
$$= P(x)$$

Now $P(x) = 0$ when $\sin(2n+1)\theta = 0$ with $\sin\theta \neq 0$

which has solutions $(2n+1)\theta = k\pi$

and for principal values $-\pi < \frac{k\pi}{2n+1} \leq \pi$, $k \neq 0$ $(\sin\theta \neq 0)$

so $k = \pm 1, \pm 2, \cdots, \pm n$.

and for distinct solutions of x, $k = 1, 2, 3, \cdots, n$,

i.e. $\theta = \frac{k\pi}{2n+1}$, $k = 1, 2, 3, \cdots, n$.

so $P(x) = 0$ at $x = \cot^2\left(\frac{k\pi}{2n+1}\right)$; $k = 1, 2, 3, \cdots, n$.

c) Summing the roots of the polynomial

$$\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{{}^{2n+1}C_3}{{}^{2n+1}C_1} \qquad \left(\frac{\text{coeff of } x^{n-1}}{\text{coeff of } x^n}\right)$$

Ext 2

3B

Q15 (ctd) c) $\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) = \frac{(2n+1)!}{3!\,(2n-2)!} \cdot \frac{1!\,(2n)!}{(2n+1)!}$

$= \frac{2n(2n-1)(2n-2)!}{3\times 2\times (2n-2)!}$

$= \frac{n(2n-1)}{3}$

d) $\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$, so for $\theta = \frac{k\pi}{2n+1}$

$$\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) < \sum_{k=1}^{n} \left(\frac{2n+1}{k\pi}\right)^2$$

hence $\left(\frac{2n+1}{\pi}\right)^2 \sum_{k=1}^{n} \frac{1}{k^2} > \frac{2n(2n-1)}{6}$

$$\frac{(2n+1)^2}{2n(2n-1)} \sum_{k=1}^{n} \frac{1}{k^2} > \frac{\pi^2}{6}$$

i.e. $\frac{(2n+1)^2}{2n(2n-1)}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) > \frac{\pi^2}{6}$.