Chapter 3 worked solutions – Complex numbers II: de Moivre and Euler

Solutions to Exercise 3A Foundation questions

1a $(\cos\theta + i\sin\theta)^5$

$= (\text{cis}\,\theta)^5$

$= \text{cis}\,5\theta$

1b $(\cos\theta + i\sin\theta)^{-3}$

$= (\text{cis}\,\theta)^{-3}$

$= \text{cis}(-3\theta)$

1c $(\cos 2\theta + i\sin 2\theta)^4$

$= (\text{cis}\,2\theta)^4$

$= \text{cis}(4 \times 2\theta)$

$= \text{cis}\,8\theta$

1d $\cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

$= \text{cis}(-\theta)$

1e $(\cos\theta - i\sin\theta)^{-7}$

$= \big(\text{cis}(-\theta)\big)^{-7}$

$= \text{cis}(-7 \times -\theta)$

$= \text{cis}\,7\theta$

1f $(\cos 3\theta - i\sin 3\theta)^2$

$= \big(\text{cis}(-3\theta)\big)^2$

$= \text{cis}(2 \times -3\theta)$

$= \text{cis}(-6\theta)$

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2a

$$\frac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta - i\sin\theta)^4}$$
$$= \frac{(\text{cis}\,\theta)^6(\text{cis}\,\theta)^{-3}}{\big(\text{cis}(-\theta)\big)^4}$$
$$= \frac{\text{cis}\,6\theta \times \text{cis}(-3\theta)}{\text{cis}(-4\theta)}$$
$$= \frac{\text{cis}(6\theta - 3\theta)}{\text{cis}(-4\theta)}$$
$$= \frac{\text{cis}\,3\theta}{\text{cis}(-4\theta)}$$
$$= \text{cis}\big(3\theta - (-4\theta)\big)$$
$$= \text{cis}\,7\theta$$

2b

$$\frac{(\cos 3\theta + i\sin 3\theta)^5(\cos 2\theta - i\sin 2\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-7}}$$
$$= \frac{(\text{cis}\,3\theta)^5\big(\text{cis}(-2\theta)\big)^{-4}}{\big(\text{cis}(-4\theta)\big)^{-7}}$$
$$= \frac{\text{cis}\,15\theta \times \text{cis}\,8\theta}{\text{cis}\,28\theta}$$
$$= \frac{\text{cis}(15\theta + 8\theta)}{\text{cis}\,28\theta}$$
$$= \frac{\text{cis}\,23\theta}{\text{cis}\,28\theta}$$
$$= \text{cis}(23\theta - 28\theta)$$
$$= \text{cis}(-5\theta)$$

3a

$$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4$$
$$= \left(\text{cis}\,\frac{\pi}{4}\right)^4$$
$$= \text{cis}\left(4 \times \frac{\pi}{4}\right)$$
$$= \text{cis}\,\pi$$
$$= \cos\pi + i\sin\pi$$
$$= -1 + 0i$$
$$= -1$$

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3b

$$\begin{aligned}
&\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^3\\
&=\left(\text{cis}\,\frac{\pi}{2}\right)^3\\
&=\text{cis}\left(3\times\frac{\pi}{2}\right)\\
&=\text{cis}\,\frac{3\pi}{2}\\
&=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\\
&=0-1i\\
&=-i
\end{aligned}$$

3c

$$\begin{aligned}
&\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5\\
&=\left(\text{cis}\,\frac{\pi}{6}\right)^5\\
&=\text{cis}\left(5\times\frac{\pi}{6}\right)\\
&=\text{cis}\,\frac{5\pi}{6}\\
&=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\\
&=-\frac{\sqrt{3}}{2}+\frac{1}{2}i
\end{aligned}$$

3d

$$\begin{aligned}
&\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{-2}\\
&=\left(\text{cis}\,\frac{2\pi}{3}\right)^{-2}\\
&=\text{cis}\left(-2\times\frac{2\pi}{3}\right)\\
&=\text{cis}\left(\frac{-4\pi}{3}\right)\\
&=\cos\frac{4\pi}{3}-i\sin\frac{4\pi}{3}\\
&=-\frac{1}{2}+\frac{\sqrt{3}}{2}i
\end{aligned}$$

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3e

$$\left(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right)^{-6}$$
$$= \left(\text{cis}\left(\frac{-3\pi}{8}\right)\right)^{-6}$$
$$= \text{cis}\left(-6 \times \frac{-3\pi}{8}\right)$$
$$= \text{cis}\frac{9\pi}{4}$$
$$= \cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

3f

$$\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)^{4}$$
$$= \left(\text{cis}\left(-\frac{5\pi}{12}\right)\right)^{4}$$
$$= \text{cis}\left(-\frac{5\pi}{3}\right)$$
$$= \cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

4a $1 + i$

$$= \sqrt{1^2 + 1^2}\,\text{cis}\left(\tan^{-1}\left(\frac{1}{1}\right)\right)$$
$$= \sqrt{2}\text{cis}\frac{\pi}{4}$$

4b $(1 + i)^{17}$

$$= \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^{17}$$
$$= \left(\sqrt{2}\right)^{17}\text{cis}\left(\frac{\pi}{4} \times 17\right)$$
$$= 256\sqrt{2}\text{cis}\left(\frac{17\pi}{4}\right)$$

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$$= 256\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$$
$$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= 256\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$
$$= 256 + 256i$$

5a $z = 1 + i\sqrt{3}$

$$= \sqrt{1^2 + \left(\sqrt{3}\right)^2}\,\text{cis}\left(\tan^{-1}\frac{\sqrt{3}}{1}\right)$$
$$= 2\text{cis}\,\frac{\pi}{3}$$

5b z^{11}

$$= \left(2\text{cis}\,\frac{\pi}{3}\right)^{11}$$
$$= 2^{11}\text{cis}\left(\frac{\pi}{3} \times 11\right)$$
$$= 2048\,\text{cis}\,\frac{11\pi}{3}$$
$$= 2048\,\text{cis}\,\frac{5\pi}{3}$$
$$= 2048\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$
$$= 2048\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= 1024 - 1024\sqrt{3}i$$

6a $z = -\sqrt{3} + i$

$|z|$
$$= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2}$$
$$= \sqrt{3 + 1}$$
$$= \sqrt{4}$$
$$= 2$$

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$$\begin{aligned}&\text{Arg}(z)\\&= \pi - \tan^{-1}\frac{1}{\sqrt{3}}\\&= \pi - \frac{\pi}{6}\\&= \frac{5\pi}{6}\end{aligned}$$

6b $z^7 + 64z$

$$\begin{aligned}&= \left(2\text{cis}\frac{5\pi}{6}\right)^7 + 64\left(2\text{cis}\frac{5\pi}{6}\right)\\&= 2^7\text{cis}\left(\frac{5\pi}{6}\times 7\right) + 128\text{cis}\left(\frac{5\pi}{6}\right)\\&= 128\text{cis}\frac{35\pi}{6} + 128\text{cis}\frac{5\pi}{6}\\&= 128\left(\text{cis}\frac{35\pi}{6} + \text{cis}\frac{5\pi}{6}\right)\\&= 128\left(\cos\frac{35\pi}{6} + i\sin\frac{35\pi}{6} + \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\\&= 128\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\\&= 0\end{aligned}$$

7a $\sqrt{3} - i$

$$\begin{aligned}&= \sqrt{\left(\sqrt{3}\right)^2 + 1^2}\,\text{cis}\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)\\&= 2\text{cis}\left(-\frac{\pi}{6}\right)\end{aligned}$$

7b $\left(\sqrt{3} - i\right)^7$

$$\begin{aligned}&= \left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^7\\&= 2^7\text{cis}\left(-\frac{\pi}{6}\times 7\right)\\&= 128\text{cis}\left(-\frac{7\pi}{6}\right)\end{aligned}$$

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$= 128\text{cis}\dfrac{5\pi}{6}$

7c $\left(\sqrt{3}-i\right)^7$

$= 128\text{cis}\dfrac{5\pi}{6}$

$= 128\left(\cos\dfrac{5\pi}{6}+i\sin\dfrac{5\pi}{6}\right)$

$= 128\left(-\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)$

$= -64\sqrt{3}+64i$

8a $\left(-1-i\sqrt{3}\right)$

$= \sqrt{1^2+\left(\sqrt{3}\right)^2}\,\text{cis}\left(-\pi+\tan^{-1}\dfrac{\sqrt{3}}{1}\right)$

$= 2\text{cis}\left(-\dfrac{2\pi}{3}\right)$

8b $\left(-1-i\sqrt{3}\right)^5$

$= \left(2\text{cis}\left(-\dfrac{2\pi}{3}\right)\right)^5$

$= 2^5\,\text{cis}\left(-\dfrac{2\pi}{3}\times 5\right)$

$= 32\,\text{cis}\left(-\dfrac{10\pi}{3}\right)$

$= 32\,\text{cis}\dfrac{2\pi}{3}$

8c $\left(-1-i\sqrt{3}\right)^5$

$= 32\,\text{cis}\dfrac{2\pi}{3}$

$= 32\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$

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$$= 32\left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)$$

$$= -16 + 16i\sqrt{3}$$

9a $\sqrt{2} - i\sqrt{2}$

$$= \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2}\,\text{cis}\left(\tan^{-1}\left(-\frac{\sqrt{2}}{\sqrt{2}}\right)\right)$$

$$= \sqrt{2+2}\,\text{cis}\left(-\frac{\pi}{4}\right)$$

$$= 2\text{cis}\left(-\frac{\pi}{4}\right)$$

9b z^{22}

$$= \left(2\text{cis}\left(-\frac{\pi}{4}\right)\right)^{22}$$

$$= 2^{22}\,\text{cis}\left(-\frac{\pi}{4}\times 22\right)$$

$$= 2^{22}\,\text{cis}\left(-\frac{11\pi}{2}\right)$$

$$= 2^{22}\,\text{cis}\frac{\pi}{2}$$

$$= 2^{22}i$$

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Chapter 3 worked solutions – Complex numbers II

Solutions to Exercise 3A Development questions

10a $(1+i)^{10}$

$= \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{10}$

$= \left(\sqrt{2}\right)^{10}\text{cis}\left(\frac{\pi}{4}\times 10\right)$

$= 2^5\,\text{cis}\left(\frac{5\pi}{2}\right)$

$= 2^5\,\text{cis}\left(\frac{5\pi}{2}-2\pi\right)$

$= 2^5\,\text{cis}\left(\frac{\pi}{2}\right)$

$= 2^5 i$

which is purely imaginary

10b $\left(1-i\sqrt{3}\right)^9$

$= \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^9$

$= 2^9\,\text{cis}\left(-\frac{\pi}{3}\times 9\right)$

$= 2^9\,\text{cis}(-3\pi)$

$= 2^9\,\text{cis}(-\pi)$

$= -2^9$

which is real

10c $(-1+i)^4$

$= \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^4$

$= \left(\sqrt{2}\right)^4\text{cis}(3\pi)$

$= 2^2\,\text{cis}(\pi)$

$= 4\,\text{cis}(\pi)$

Uncorrected proofs

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$= -4$

Hence $-1 + i$ is a fourth root of -4.

10d $\left(-\sqrt{3} - i\right)^6$

$= \left(2 \text{ cis}\left(-\frac{5\pi}{6}\right)\right)^6$

$= 2^6 \text{ cis}(-5\pi)$

$= 2^6 \text{ cis}(\pi)$

$= -2^6$

$= -64$

Hence $-\sqrt{3} - i$ is a sixth root of -64.

11 If k is a multiple of 4 then $k = 4n$ where n is an integer. Thus,

$(-1 + i)^k$

$= (-1 + i)^{4n}$

$= \left(\sqrt{2} \text{ cis}\left(\frac{3\pi}{4}\right)\right)^{4n}$

$= \left(\left(\sqrt{2}\right)^4 \text{ cis}(3\pi)\right)^n$

$= (-2^2)^n$

$= (-4)^n$

which is real as required

12a i $\left(\sqrt{3} + i\right)^m$

$= \left(2 \text{ cis}\left(\frac{\pi}{6}\right)\right)^m$

$= 2^m \text{ cis}\left(\frac{m\pi}{6}\right)$

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which is real when $\frac{m\pi}{6}$ is a multiple of π. The lowest positive integer for which this is true is when $m = 6$.

12a ii $\left(\sqrt{3} + i\right)^m$

$= \left(2 \text{ cis}\left(\frac{\pi}{6}\right)\right)^m$

$= 2^m \text{ cis}\left(\frac{m\pi}{6}\right)$

which is imaginary when $\frac{m\pi}{6}$ is of the form $n\pi \pm \frac{\pi}{2}$ where n is an integer. The lowest positive integer for which this is true is when $m = 3$.

12b i $\left(\sqrt{3} + i\right)^6$

$= 2^6 \text{ cis}\left(\frac{6\pi}{6}\right)$

$= -2^6$

$= -64$

12b ii $\left(\sqrt{3} + i\right)^3$

$= 2^3 \text{ cis}\left(\frac{3\pi}{6}\right)$

$= 2^3 i$

$= 8i$

13a $(1 + i)^n + (1 - i)^n$

$= \left(\sqrt{2} \text{ cis}\left(\frac{\pi}{4}\right)\right)^n + \left(\sqrt{2} \text{ cis}\left(-\frac{\pi}{4}\right)\right)^n$

$= \left(\sqrt{2}\right)^n \text{ cis}\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^n \text{ cis}\left(-\frac{n\pi}{4}\right)$

$= \left(\sqrt{2}\right)^n \left[\text{cis}\left(\frac{n\pi}{4}\right) + \text{cis}\left(-\frac{n\pi}{4}\right)\right]$

$= \left(\sqrt{2}\right)^n \left[\cos\left(\frac{n\pi}{4}\right) + i \sin\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{4}\right) - i \sin\left(\frac{n\pi}{4}\right)\right]$

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$= (\sqrt{2})^n \left(2\cos\left(\frac{n\pi}{4}\right)\right)$

$= 2(\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right)$

which is real

13b $2(\sqrt{2})^n \cos\left(\frac{n\pi}{4}\right) = 0$

$\frac{n\pi}{4} = 2\lambda\pi \pm \frac{\pi}{2}$

$n = 8\lambda \pm 2\pi$

Since n is a positive integer, $n = 2, 6, 10, 14, 18\ldots$

14 $(-\sqrt{3} + i)^n - (-\sqrt{3} - i)^n$

$= \left(2\,\text{cis}\left(\frac{5\pi}{6}\right)\right)^n - \left(2\,\text{cis}\left(-\frac{5\pi}{6}\right)\right)^n$

$= 2^n \text{cis}\left(\frac{5n\pi}{6}\right) - 2^n \text{cis}\left(-\frac{5n\pi}{6}\right)$

$= 2^n \left[\text{cis}\left(\frac{5n\pi}{6}\right) - \text{cis}\left(-\frac{5n\pi}{6}\right)\right]$

$= 2^n \left[\text{cis}\left(\frac{5n\pi}{6}\right) - \text{cis}\left(-\frac{5n\pi}{6}\right)\right]$

$= 2^n \left[\cos\left(\frac{5n\pi}{6}\right) + i\sin\left(\frac{5n\pi}{6}\right) - \left(\cos\left(\frac{5n\pi}{6}\right) - i\sin\left(\frac{5n\pi}{6}\right)\right)\right]$

$= 2^n \left[\cos\left(\frac{5n\pi}{6}\right) + i\sin\left(\frac{5n\pi}{6}\right) - \cos\left(\frac{5n\pi}{6}\right) + i\sin\left(\frac{5n\pi}{6}\right)\right]$

$= 2^n \left[2i\sin\left(\frac{5n\pi}{6}\right)\right]$

$= 2^{n+1} i \sin\left(\frac{5n\pi}{6}\right)$

$= 2^{n+1} \sin\left(\frac{5n\pi}{6}\right) i$

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15a $\left(1+\sqrt{3}i\right)^{2n}+\left(1-\sqrt{3}i\right)^{2n}$

$=\left(2\,\text{cis}\left(\frac{\pi}{3}\right)\right)^{2n}+\left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^{2n}$

$=2^{2n}\,\text{cis}\left(\frac{2n\pi}{3}\right)+2^{2n}\,\text{cis}\left(-\frac{2n\pi}{3}\right)$

$=2^{2n}\left[\text{cis}\left(\frac{2n\pi}{3}\right)+\text{cis}\left(-\frac{2n\pi}{3}\right)\right]$

$=2^{2n}\left[\cos\left(\frac{2n\pi}{3}\right)+i\sin\left(\frac{2n\pi}{3}\right)+\cos\left(\frac{2n\pi}{3}\right)-i\sin\left(\frac{2n\pi}{3}\right)\right]$

$=2^{2n}\left[2\cos\left(\frac{2n\pi}{3}\right)\right]$

$=2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$

If n is divisible by 3 then $n=3\lambda$ where λ is an integer. Hence

$\left(1+\sqrt{3}i\right)^{2n}+\left(1-\sqrt{3}i\right)^{2n}$

$=2^{2n+1}\cos\left(\frac{2(3\lambda)\pi}{3}\right)$

$=2^{2n+1}\cos(2\pi\lambda)$

$=2^{2n+1}$

15b $2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$ from part **a**, since n is not divisible by 3,

$2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$

$=2^{2n+1}\cos\left(\pm\frac{2\pi}{3}\right),2^{2n+1}\cos\left(\pm\frac{4\pi}{3}\right),2^{2n+1}\cos\left(\pm\frac{8\pi}{3}\right)\ldots$

$=2^{2n+1}\left(-\frac{1}{2}\right)$

$=-2^{2n}$

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$$\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{1+\cos^2\theta - \sin^2\theta + i\sin 2\theta}{1+\cos^2\theta - \sin^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{\cos^2\theta + \cos^2\theta + i\sin 2\theta}{\cos^2\theta + \cos^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{2\cos^2\theta + i\sin 2\theta}{2\cos^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{2\cos^2\theta + 2i\cos\theta\sin\theta}{2\cos^2\theta - 2i\cos\theta\sin\theta}\right)^n$$

$$= \left(\frac{2\cos\theta\,(\cos\theta + i\sin\theta)}{2\cos\theta\,(\cos\theta - i\sin\theta)}\right)^n$$

$$= \left(\frac{(\cos\theta + i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)^n$$

$$= \left(\frac{\text{cis}(\theta)}{\text{cis}(-\theta)}\right)^n$$

$$= \big(\text{cis}(2\theta)\big)^n$$

$$= \text{cis}(2n\theta)$$

17 $(1+\cos\alpha + i\sin\alpha)^k + (1+\cos\alpha - i\sin\alpha)^k$

$$= \left(1+\cos^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\alpha + i\sin\alpha\right)^k + \left(1+\cos^2\frac{1}{2}\alpha - \sin^2\frac{1}{2}\alpha - i\sin\alpha\right)^k$$

$$= \left(\cos^2\frac{1}{2}\alpha + 1 - \sin^2\frac{1}{2}\alpha + i\sin\alpha\right)^k + \left(\cos^2\frac{1}{2}\alpha + 1 - \sin^2\frac{1}{2}\alpha - i\sin\alpha\right)^k$$

$$= \left(\cos^2\frac{1}{2}\alpha + \cos^2\frac{1}{2}\alpha + i\sin\alpha\right)^k + \left(\cos^2\frac{1}{2}\alpha + \cos^2\frac{1}{2}\alpha - i\sin\alpha\right)^k$$

$$= \left(2\cos^2\frac{1}{2}\alpha + i\sin\alpha\right)^k + \left(2\cos^2\frac{1}{2}\alpha - i\sin\alpha\right)^k$$

$$= \left(2\cos^2\frac{1}{2}\alpha + 2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\right)^k + \left(2\cos^2\frac{1}{2}\alpha - 2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\right)^k$$

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$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left(\cos\frac{1}{2}\alpha + i\sin\frac{1}{2}\alpha\right)^k + \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left(\cos\frac{1}{2}\alpha - i\sin\frac{1}{2}\alpha\right)^k$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left(\text{cis}\frac{1}{2}\alpha\right)^k + \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left(\text{cis}\left(-\frac{1}{2}\alpha\right)\right)^k$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left[\left(\text{cis}\frac{1}{2}\alpha\right)^k + \left(\text{cis}\left(-\frac{1}{2}\alpha\right)\right)^k\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left[\text{cis}\left(\frac{1}{2}k\alpha\right) + \text{cis}\left(-\frac{1}{2}k\alpha\right)\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left[\cos\left(\frac{1}{2}k\alpha\right) + i\sin\left(\frac{1}{2}k\alpha\right) + \cos\left(\frac{1}{2}k\alpha\right) - i\sin\left(\frac{1}{2}k\alpha\right)\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k \left[2\cos\left(\frac{1}{2}k\alpha\right)\right]$$

$$= 2^{k+1}\cos^k\left(\frac{1}{2}\alpha\right)\cos\left(\frac{1}{2}k\alpha\right)$$

$$= 2^{k+1}\cos\left(\frac{1}{2}k\alpha\right)\cos^k\left(\frac{1}{2}\alpha\right)$$

Uncorrected proofs

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Ext 2 Ch 3

Ex 3A

Q18 a) $1+\zeta+\zeta^2+\cdots+\zeta^{2n-1} = \dfrac{\zeta^{2n}-1}{\zeta-1}$ $(\zeta \neq 1$, by G.P. theory$)$

$= \dfrac{\text{cis } 2\pi - 1}{\zeta - 1}$ $\left(\text{since } (\text{cis}\tfrac{\pi}{n})^{2n} = \text{cis } 2\pi \text{ by de Moivre}\right)$

$= \dfrac{1-1}{\zeta-1}$

$= 0$

Alternatively $1+\zeta+\zeta^2+\cdots+\zeta^{2n-1}$

$= 1+\zeta+\zeta^2+\cdots+\zeta^{n-1}+\zeta^n+\zeta^{n+1}+\zeta^{n+2}+\cdots+\zeta^{2n-1}$

$= (1+\zeta+\zeta^2+\cdots+\zeta^{n-1}) + \zeta^n(1+\zeta+\zeta^2+\cdots+\zeta^{n-1})$

$= (1+\zeta+\zeta^2+\cdots+\zeta^{n-1})(1+\zeta^n)$

but $1+\zeta^n = 1+(\text{cis}\tfrac{\pi}{n})^n$

$= 1+\text{cis}\,\pi$ (by de Moivre)

$= 1-1$

$= 0$

hence $1+\zeta+\zeta^2+\cdots+\zeta^{2n-1} = (1+\zeta+\zeta^2+\cdots+\zeta^{n-1}) \times 0$

$= 0.$

b) LHS $= \dfrac{\zeta^n - 1}{\zeta - 1}$ $(\zeta \neq 1$, by G.P. theory$)$

$= \dfrac{-1-1}{\zeta-1}$ (shown in (a))

$= \dfrac{2}{1-\zeta}$

$= \dfrac{2}{1-\text{cis}\frac{\pi}{n}} \times \dfrac{\text{cis}\frac{\pi}{2n}}{\text{cis}\frac{\pi}{2n}}$ (using the half angle on the RHS)

(Alternatively, use double angle formulae)

$= \dfrac{2\,\text{cis}\frac{\pi}{2n}}{\text{cis}\frac{\pi}{2n} - \text{cis}\frac{\pi}{2n}}$

$= \dfrac{2(\cos\frac{\pi}{2n} - i\sin\frac{\pi}{2n})}{-2i\sin\frac{\pi}{2n}}$

$= -\frac{1}{i}\cot\frac{\pi}{2n} + 1 = 1 + i\cot\frac{\pi}{2n}$