Chapter 12 worked solutions – Further calculus

Solutions to Exercise 12A

1a

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$	undefined	1.40	1.16	1.02	1	1.05	1.25	1.67	undefined

1b

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$	undefined	1.40	1.16	1.02	1	1.05	1.25	1.67	undefined

2a

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

2b

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

2c

$$\frac{d}{dx}(\sin^{-1} 2x)$$
$$= \frac{1}{\sqrt{1-(2x)^2}} \times \frac{d}{dx}(2x)$$
$$= \frac{2}{\sqrt{1-4x^2}}$$

2d

$$\frac{d}{dx}(\tan^{-1} 3x)$$
$$= \frac{1}{1+(3x)^2} \times \frac{d}{dx}(3x)$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{3}{1 + 9x^2}$$

2e

$$\frac{d}{dx}(\cos^{-1} 5x)$$

$$= \frac{-1}{\sqrt{1 - (5x)^2}} \times \frac{d}{dx}(5x)$$

$$= -\frac{5}{\sqrt{1 - 25x^2}}$$

2f

$$\frac{d}{dx}(\sin^{-1} -x)$$

$$= \frac{1}{\sqrt{1 - (-x)^2}} \times \frac{d}{dx}(-x)$$

$$= \frac{-1}{\sqrt{1 - x^2}}$$

2g

$$\frac{d}{dx}(\sin^{-1} x^2)$$

$$= \frac{1}{\sqrt{1 - (x^2)^2}} \times \frac{d}{dx}(x^2)$$

$$= \frac{2x}{\sqrt{1 - x^4}}$$

2h

$$\frac{d}{dx}(\tan^{-1} x^3)$$

$$= \frac{1}{1 + (x^3)^2} \times \frac{d}{dx}(x^3)$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{3x^2}{1+x^6}$$

2i

$$\frac{d}{dx}(\tan^{-1}(x+2))$$
$$= \frac{1}{1+(x+2)^2} \times \frac{d}{dx}(x+2)$$
$$= \frac{1}{x^2+4x+5}$$

2j

$$\frac{d}{dx}(\cos^{-1}(1-x))$$
$$= \frac{-1}{\sqrt{1-(1-x)^2}} \times \frac{d}{dx}(1-x)$$
$$= \frac{-1}{\sqrt{1-(1-2x+x^2)}} \times -1$$
$$= \frac{1}{\sqrt{2x-x^2}}$$

2k

$$\frac{d}{dx}(x\sin^{-1}x)$$
$$= \sin^{-1}x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\sin^{-1}x)$$
$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

2l

$$\frac{d}{dx}\left((1+x^2)\tan^{-1}x\right)$$
$$= \tan^{-1}x \times \frac{d}{dx}(1+x^2) + (1+x^2) \times \frac{d}{dx}(\tan^{-1}x)$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \tan^{-1} x \times 2x + (1 + x^2) \times \frac{1}{1 + x^2}$$
$$= 2x \tan^{-1} x + 1$$

2m

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{5}x\right)$$
$$= \frac{1}{\sqrt{1 - \left(\frac{1}{5}x\right)^2}} \times \frac{d}{dx}\left(\frac{1}{5}x\right)$$
$$= \frac{1}{\sqrt{1 - \frac{x^2}{25}}} \times \frac{1}{5}$$
$$= \frac{1}{\sqrt{25 - x^2}}$$

2n

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{4}x\right)$$
$$= \frac{1}{1 + \left(\frac{1}{4}x\right)^2} \times \frac{d}{dx}\left(\frac{1}{4}x\right)$$
$$= \frac{1}{1 + \frac{x^2}{16}} \times \frac{1}{4}$$
$$= \frac{4}{16 + x^2}$$

2o

$$\frac{d}{dx}\left(\cos^{-1}\sqrt{x}\right)$$
$$= \frac{-1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{-1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

Uncorrected proofs

$$= \frac{-1}{\sqrt{1-\sqrt{x}^2}} \times \frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{-1}{2\sqrt{x}\sqrt{1-x}}$$
$$= -\frac{1}{2\sqrt{x-x^2}}$$

2p

$$\frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)$$
$$= \frac{1}{1+\left(\sqrt{x}\right)^2} \times \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{1+\left(\sqrt{x}\right)^2} \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{1+\left(\sqrt{x}\right)^2} \times -\frac{1}{2}x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}(1+x)}$$

2q

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{x}\right)$$
$$= \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{1+\frac{1}{x^2}} \times -\frac{1}{x^2}$$
$$= -\frac{1}{x^2+1}$$

3 For this question you only need to provide the gradient. We also provide the equation of the tangent for your benefit.

Uncorrected proofs

3a

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

At $x = 0$, $y = 2\tan^{-1} 0 = 0$ and $\frac{dy}{dx} = \frac{2}{1+0^2} = 2$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 2(x - 0)$$

$$y = 2x$$

3b

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{1-x^2}}$$

At $x = \frac{1}{2}$, $y = \sqrt{3}\sin^{-1}\frac{1}{2} = \frac{\sqrt{3}}{6}$ and $\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{1-\frac{1}{2}^2}} = \frac{\sqrt{3}}{\sqrt{\frac{3}{4}}} = 2$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{6} = 2(x - 0)$$

$$y = 2x + \frac{\sqrt{3}}{6}$$

3c

$$\frac{dy}{dx} = \frac{2}{1+4x^2}$$

At $x = -\frac{1}{2}$, $y = \tan^{-1}\left(2\left(-\frac{1}{2}\right)\right) = -\frac{\pi}{4}$ and $\frac{dy}{dx} = \frac{2}{1+4\left(-\frac{1}{2}\right)^2} = \frac{2}{2} = 1$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - (-\frac{\pi}{4}) = 1(x - 0)$$

$$y = x - \frac{\pi}{4}$$

Uncorrected proofs

3d

$$\frac{dy}{dx} = \frac{-\frac{1}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = -\frac{1}{\sqrt{4-x^2}}$$

At $x = \sqrt{3}$, $y = \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ and $\frac{dy}{dx} = -\frac{1}{\sqrt{4-\sqrt{3}^2}} = -1$. Hence, the tangent will be given by

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{6} = -1(x - 0)$

$y = -x + \frac{\pi}{6}$

4a

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(3x)^2}} \times 2 \times 3 = -\frac{6}{\sqrt{1-9x^2}}$$

When $x = 0$

$y = 2\cos^{-1} 3(0) = 2\left(\frac{\pi}{2}\right) = \pi$,

$\frac{dy}{dx} = -\frac{6}{\sqrt{1-9(0)^2}} = -6$ and $m_{norm} = -\frac{1}{\left(\frac{dy}{dx}\right)} = \frac{1}{6}$

For the tangent:

$y - y_1 = m(x - x_1)$

$y - \pi = -6(x - 0)$

$y = -6x + \pi$

For the normal:

$y - y_1 = m(x - x_1)$

$y - \pi = \frac{1}{6}(x - 0)$

$y = \frac{1}{6}x + \pi$

Uncorrected proofs

© Cambridge University Press 2019

4b

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2} = \frac{1}{\sqrt{4-x^2}}$$

When $x = \sqrt{2}$

$y = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$,

$\frac{dy}{dx} = \frac{1}{\sqrt{4-(\sqrt{2})^2}} = \frac{1}{\sqrt{2}}$ and $m_{norm} = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\sqrt{2}$

For the tangent:

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{4} = \frac{1}{\sqrt{2}}(x - \sqrt{2})$

$y = \frac{x}{\sqrt{2}} - 1 + \frac{\pi}{4}$

For the normal:

$y - y_1 = m(x - x_1)$

$y - \frac{\pi}{4} = -\sqrt{2}(x - \sqrt{2})$

$y = -x\sqrt{2} + 2 + \frac{\pi}{4}$

5a

$$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$$
$$= \frac{d}{dx}(\sin^{-1} x) + \frac{d}{dx}(\cos^{-1} x)$$
$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$
$$= 0$$

5b As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $\sin^{-1} 0 + \cos^{-1} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$, hence the function has a constant value of $\frac{\pi}{2}$.

Uncorrected proofs

6a

$$\frac{d}{dx}(\cos^{-1}x + \cos^{-1}(-x))$$
$$= -\frac{1}{\sqrt{1-x^2}} - \frac{\frac{d}{dx}(-x)}{\sqrt{1-(-x)^2}}$$
$$= -\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$
$$= 0$$

As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $\cos^{-1}0 + \cos^{-1}0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$, hence the function has a constant value of π.

6b

$$\frac{d}{dx}\left(2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)\right)$$
$$= \frac{1}{\sqrt{1-\sqrt{x}^2}} \times 2 \times \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{\sqrt{1-(2x-1)^2}} \times (-2)$$
$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{2}{\sqrt{4x-4x^2}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{2}{2\sqrt{x-x^2}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{1}{\sqrt{x-x^2}}$$
$$= 0$$

As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $2\sin^{-1}\sqrt{0} - \sin^{-1}(2(0)-1) = 2\sin^{-1}0 - \sin^{-1}(-1) = 0 - \frac{\pi}{2} = \frac{\pi}{2}$, hence the function has a constant value of $\frac{\pi}{2}$.

7a

$$f'(x) = 1 \times \tan^{-1}x + x \times \frac{1}{1+x^2} - \frac{1}{2} \times \frac{2x}{1+x^2}$$
$$= \tan^{-1}x + \frac{x}{1+x^2} - \frac{x}{1+x^2}$$
$$= \tan^{-1}x$$

Uncorrected proofs

Hence

$$f''(x) = \frac{1}{1+x^2}$$

7b $f''(-1) = \frac{1}{2} > 0$, hence the curve is concave up

8 By the quotient rule

$$\frac{dy}{dx} = \frac{x \times \frac{d}{dx}(\sin^{-1} x) - \sin^{-1} x \times \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x}{x^2}$$

When $x = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{\frac{1}{2} \times \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} - \sin^{-1}\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2}$$

$$= \frac{\frac{1}{2} \times \frac{1}{\frac{\sqrt{3}}{4}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= \frac{\frac{1}{2} \times \frac{2}{\sqrt{3}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= \frac{\frac{1}{\sqrt{3}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= 4\left(\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{3}(2\sqrt{3} - \pi)$$

Uncorrected proofs

9a

$$\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$$
$$=x\times-\frac{1}{\sqrt{1-x^2}}+\cos^{-1}x-\frac{1}{2}\times(-2x)\times(1-x^2)^{-\frac{1}{2}}$$
$$=-\frac{x}{\sqrt{1-x^2}}+\cos^{-1}x+\frac{x}{\sqrt{1-x^2}}$$
$$=\cos^{-1}x$$

9b

$$\frac{d}{dx}(\sin^{-1}e^{3x})$$
$$=\frac{1}{\sqrt{1-(e^{3x})^2}}\times\frac{d}{dx}(e^{3x})$$
$$=\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

9c

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{4}(2x-3)\right)$$
$$=\frac{1}{\sqrt{1-\left(\frac{1}{4}(2x-3)\right)^2}}\times\frac{d}{dx}\left(\frac{1}{4}(2x-3)\right)$$
$$=\frac{1}{\sqrt{1-\frac{1}{16}(4x^2-12x+9)}}\times\frac{2}{4}$$
$$=\frac{2}{4}\times\frac{1}{\frac{1}{4}\sqrt{16-(4x^2-12x+9)}}$$
$$=\frac{2}{\sqrt{7-4x^2+12x}}$$

Uncorrected proofs

© Cambridge University Press 2019

9d

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{1-x}\right)$$
$$=\frac{1}{1+\left(\frac{1}{1-x}\right)^2}\times\frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$=\frac{(1-x)^2}{(1-x)^2+1}\times -1\times -1\times(1-x)^{-2}$$
$$=\frac{1}{(1-x)^2+1}$$
$$=\frac{1}{1-2x+x^2+1}$$
$$=\frac{1}{x^2-2x+2}$$

9e

$$\frac{d}{dx}(\sin^{-1}e^x)$$
$$=\frac{1}{\sqrt{1-(e^x)^2}}\times\frac{d}{dx}(e^x)$$
$$=\frac{e^x}{\sqrt{1-e^{2x}}}$$

9f

$$\frac{d}{dx}\left(\log_e\sqrt{\sin^{-1}x}\right)$$
$$=\frac{\frac{d}{dx}\left(\sqrt{\sin^{-1}x}\right)}{\sqrt{\sin^{-1}x}}$$
$$=\frac{\frac{1}{2}\times(\sin^{-1}x)^{-\frac{1}{2}}\times\frac{d}{dx}(\sin^{-1}x)}{\sqrt{\sin^{-1}x}}$$
$$=\frac{1}{2\sin^{-1}x}\times\frac{d}{dx}(\sin^{-1}x)$$
$$=\frac{1}{2\sqrt{1-x^2}\sin^{-1}x}$$

Uncorrected proofs

© Cambridge University Press 2019

9g

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{\log_e x}\right)$$

$$= \frac{1}{\sqrt{1-\left((\log_e x)^{\frac{1}{2}}\right)^2}} \times \frac{d}{dx}\left((\log_e x)^{\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{1-\log_e x}} \times \frac{1}{2} \times (\log_e x)^{-\frac{1}{2}} \times \frac{d}{dx}(\log_e x)$$

$$= \frac{1}{\sqrt{\log_e x\,(1-\log_e x)}} \times \frac{1}{2}\left(\frac{1}{x}\right)$$

$$= \frac{1}{2x\sqrt{\log_e x\,(1-\log_e x)}}$$

9h

$$\frac{d}{dx}\left(\sqrt{x}\sin^{-1}\sqrt{1-x}\right)$$

$$= \sin^{-1}\sqrt{1-x} \times \frac{d}{dx}\left(\sqrt{x}\right) + \sqrt{x} \times \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x}\right)$$

$$= \sin^{-1}\sqrt{1-x} \times \frac{1}{2\sqrt{x}} + \sqrt{x} \times \frac{1}{\sqrt{1-\left(\sqrt{1-x}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x}\right)$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} + \sqrt{x} \times \frac{1}{\sqrt{1-(1-x)}} \times \frac{1}{2} \times -1 \times \frac{1}{\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} + \sqrt{x} \times \frac{1}{\sqrt{x}} \times \frac{1}{2} \times -1 \times \frac{1}{\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} - \frac{1}{2\sqrt{1-x}}$$

9i

$$\frac{d}{dx}\left(\tan^{-1}\frac{x+2}{1-2x}\right)$$

$$= \frac{1}{1+\left(\frac{x+2}{1-2x}\right)^2} \times \frac{d}{dx}\left(\frac{x+2}{1-2x}\right)$$

$$= \frac{1}{1+\left(\frac{x+2}{1-2x}\right)^2} \times \frac{(1-2x)-(-2)(x+2)}{(1-2x)^2}$$

Uncorrected proofs

$$= \frac{1}{1 + \left(\frac{x+2}{1-2x}\right)^2} \times \frac{1 - 2x + 2x + 4}{(1-2x)^2}$$

$$= \frac{5}{(1-2x)^2 + (x+2)^2}$$

$$= \frac{5}{1 - 4x + 4x^2 + x^2 + 4x + 4}$$

$$= \frac{5}{5x^2 + 5}$$

$$= \frac{1}{1 + x^2}$$

10a i

$$y' = 2 \times \frac{1}{\sqrt{1-x^2}} \times \sin^{-1} x = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

$$y'' = \frac{2 \times \frac{1}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} \times 2\sin^{-1} x}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{2(1 + x\sin^{-1} x)}{(1-x^2)^{\frac{3}{2}}}$$

10a ii

$$(1-x^2)y'' - xy' - 2$$

$$= (1-x^2)\left(\frac{2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}\right) - x\left(\frac{2\sin^{-1} x}{\sqrt{1-x^2}}\right) - 2$$

$$= 2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}} - \frac{2x\sin^{-1} x}{\sqrt{1-x^2}} - 2$$

$$= 0$$

Uncorrected proofs

© Cambridge University Press 2019

10b

$$y' = \frac{1}{\sqrt{1-x^2}} e^{\sin^{-1} x}$$

$$y'' = -\frac{1}{2}(-2x)(1-x^2)^{-\frac{3}{2}} e^{\sin^{-1} x} + \frac{1}{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2}} e^{\sin^{-1} x}$$

$$= \frac{x e^{\sin^{-1} x}}{(1-x^2)^{\frac{3}{2}}} + \frac{e^{\sin^{-1} x}}{1-x^2}$$

$$(1-x^2)y'' - xy' - y$$

$$= (1-x^2)\left(\frac{x e^{\sin^{-1} x}}{(1-x^2)^{\frac{3}{2}}} + \frac{e^{\sin^{-1} x}}{1-x^2}\right) - x\left(\frac{1}{\sqrt{1-x^2}} e^{\sin^{-1} x}\right) - e^{\sin^{-1} x}$$

$$= \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} + e^{\sin^{-1} x} - \frac{x}{\sqrt{1-x^2}} e^{\sin^{-1} x} - e^{\sin^{-1} x}$$

$$= 0$$

11a The range is the same as that of $\sin^{-1} x$ which is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

11b

$$y = \sin^{-1} 2x$$

$$2x = \sin y$$

$$x = \frac{1}{2} \sin y$$

11c

$\frac{dx}{dy} = \frac{1}{2} \cos y$, as the square root function always returns positive numbers, and as the division of one number by another positive is always positive, this derivative function will always be positive.

Uncorrected proofs

11d

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{\frac{1}{2}\cos y}$$

$$= \frac{2}{\cos y}$$

$$= \frac{2}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{2}{1 - (2x)^2}$$

$$= \frac{2}{\sqrt{1 - 4x^2}}$$

Note that taking the derivative normally yields

$$\frac{dy}{dx} = \frac{2}{\sqrt{1 - (2x)^2}} = \frac{2}{\sqrt{1 - 4x^2}}$$

12a

$$y = \sin^{-1}\frac{1}{2}x$$

$$\frac{x}{2} = \sin y$$

$$x = 2\sin y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{2\cos y}$$

$$= \frac{1}{2\sqrt{1 - \left(\frac{x}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{4 - x^2}}$$

Uncorrected proofs

© Cambridge University Press 2019

12b

$$y = \cos^{-1}(x-1)$$

$$x - 1 = \cos y$$

$$x = \cos y + 1$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{-\sin y}$$

$$= \frac{1}{-\sqrt{1-\cos^2 y}}$$

$$= \frac{1}{-\sqrt{1-(x-1)^2}}$$

$$= \frac{-1}{\sqrt{1-(x^2-2x+1)}}$$

$$= \frac{-1}{\sqrt{2x-x^2}}$$

12c

$$y = \tan^{-1}\sqrt{x}$$

$$\sqrt{x} = \tan y$$

$$x = \tan^2 y$$

$$\frac{dx}{dy} = 2 \times \frac{1}{1+y^2} \times \tan y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{2 \times \frac{1}{1+y^2} \times \tan y}$$

$$= \frac{1+y^2}{2\tan y}$$

$$= \frac{1+\left(\tan^{-1}\sqrt{x}\right)^2}{2\tan\left(\tan^{-1}\sqrt{x}\right)}$$

$$= \frac{1+\left(\tan^{-1}\sqrt{x}\right)^2}{2\sqrt{x}}$$

Uncorrected proofs

© Cambridge University Press 2019

13a From the domain of the inverse cosine function we know that $-1 \leq x^2 \leq 1$. Furthermore, $x^2 \geq 0$ for all real x. Hence $0 \leq x^2 \leq 1$.

Hence the domain is $-1 \leq x \leq 1$

13b $f(-x) = \cos^{-1}(-x)^2 = \cos^{-1} x^2 = f(x)$, hence the function is even and symmetric about the line $x = 0$

13c

$$f'(x) = \frac{-1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}(x^2)$$

$$= \frac{-1}{\sqrt{1-(x^2)^2}} \times 2x$$

$$= -\frac{2x}{\sqrt{1-x^4}}$$

13d $f''(x) = 2\cos^{-1} x^2 + 4x^2 \cos^{-1} x^2$

$f'(0) = 0$

$f''(0) = -\pi < 0$

Hence at $x = 0$ the derivative is zero and the curve is concave down, so it is a maximum turning point.

13e

When $x = -1, f'(-1) = -\frac{2(-1)}{\sqrt{1-(-1)^4}} = \frac{2}{\sqrt{0}}$ which is undefined.

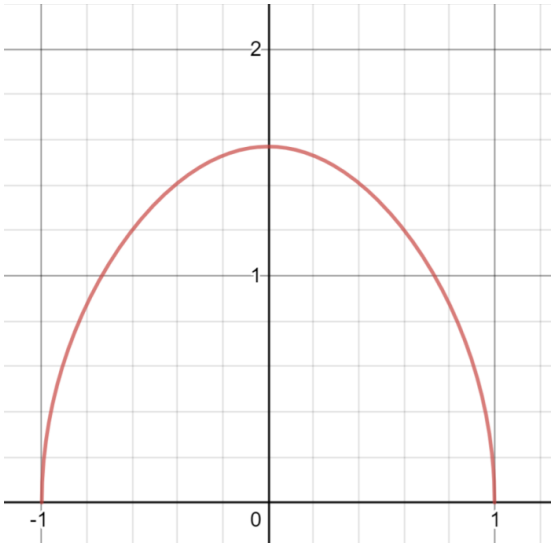
When $x = 1, f'(1) = -\frac{2(1)}{\sqrt{1-(1)^4}} = \frac{2}{\sqrt{0}}$ which is undefined.

This means that the gradient is vertical at that point.

Uncorrected proofs

© Cambridge University Press 2019

13f



14a

$$\tan \angle TEP = \frac{4}{x}$$

$$\angle TEP = \tan^{-1}\frac{4}{x}$$

$$\tan \angle BEP = \frac{3}{x}$$

$$\angle BEP = \tan^{-1}\frac{3}{x}$$

Hence

$$\theta = \angle TEP - \angle BEP = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{3}{x}$$

14b

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-\left(\frac{4}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{4}{x}\right) - \frac{1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{3}{x}\right)$$

$$= \frac{1}{\sqrt{1-\frac{16}{x^2}}} \times \left(-\frac{4}{x^2}\right) - \frac{1}{\sqrt{1-\frac{9}{x^2}}} \times \left(-\frac{3}{x^2}\right)$$

$$= \frac{-4}{x\sqrt{x^2-16}} + \frac{3}{x\sqrt{x^2-9}}$$

Uncorrected proofs

When $x = 2\sqrt{3}$

$$\frac{d\theta}{dx} = \frac{-4}{(2\sqrt{3})\sqrt{(2\sqrt{3})^2 - 16}} + \frac{3}{(2\sqrt{3})\sqrt{(2\sqrt{3})^2 - 9}}$$
$$= 0$$

Hence there is a stationary point

When $x = \sqrt{3}, \frac{d\theta}{dx} > 0$ and when $x = 3\sqrt{3}, \frac{d\theta}{dx} < 0$. Hence this is a maximum turning point.

14c When $x = 2\sqrt{3}$

$$\theta = \tan^{-1}\frac{4}{2\sqrt{3}} - \tan^{-1}\frac{3}{2\sqrt{3}}$$

$\tan\theta$

$$= \tan\left(\tan^{-1}\frac{4}{2\sqrt{3}} - \tan^{-1}\frac{3}{2\sqrt{3}}\right)$$
$$= \frac{\tan\left(\tan^{-1}\frac{4}{2\sqrt{3}}\right) - \tan\left(\tan^{-1}\frac{3}{2\sqrt{3}}\right)}{1 + \tan\left(\tan^{-1}\frac{4}{2\sqrt{3}}\right)\tan\left(\tan^{-1}\frac{3}{2\sqrt{3}}\right)}$$
$$= \frac{\frac{4}{2\sqrt{3}} - \frac{3}{2\sqrt{3}}}{1 + \frac{4}{2\sqrt{3}}\frac{3}{2\sqrt{3}}}$$
$$= \frac{\frac{1}{2\sqrt{3}}}{1 + 1}$$
$$= \frac{1}{4\sqrt{3}}$$
$$= \frac{\sqrt{3}}{12}$$

Hence

$$\theta = \tan^{-1}\frac{\sqrt{3}}{12}$$

Uncorrected proofs

15a

$\theta = \angle OPA$ (equal alternate angles on parallel lines)

$$\tan\theta = \tan\angle OPA = \frac{OA}{AP} = \frac{6}{x}$$

Hence

$$\theta = \tan^{-1}\frac{6}{x}$$

15b

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{6}{x}\right)^2} \times \frac{d}{dx}\left(\frac{6}{x}\right)$$
$$= \frac{1}{1+\frac{36}{x^2}} \times \left(-\frac{6}{x^2}\right)$$
$$= \frac{-6}{x^2+36}$$

Now $\frac{dx}{dt} = 600$

Hence

$$\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt}$$
$$= \frac{-6}{x^2+36} \times 600$$
$$= \frac{-3600}{x^2+36}$$

15c When $x = 3$,

$$\frac{d\theta}{dt} = \frac{-3600}{3^2+36} = -80$$

Hence it is travelling at 80 radians per hour which is $\frac{1}{45}$ rad/s.

16a Note that $\tan^{-1} x$ is defined for all values of x, the only value for which the function is undefined is when $x = 0$ as $\frac{1}{x}$ is undefined when $x = 0$. Hence the domain is all x such that $x \neq 0$.

Uncorrected proofs

© Cambridge University Press 2019

$$\begin{aligned} f(-x) &= \tan^{-1}(-x) + \tan^{-1}\left(\frac{1}{-x}\right) \\ &= -\tan^{-1}(x) - \tan^{-1}\left(\frac{1}{x}\right) \\ &= -\left(\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)\right) \\ &= -f(x) \end{aligned}$$

Hence the function is odd so it has rotational symmetry about the origin.

16b

$$\begin{aligned} f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{x}\right) \\ &= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times \left(-\frac{1}{x^2}\right) \\ &= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\ &= 0 \end{aligned}$$

16c

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int 0\,dx \\ &= C \end{aligned}$$

Now as this function has a point of discontinuity at $x = 0$, this constant may be different on either side of the discontinuity.

For $x > 0$, substitute in $x = 1$.

$$C = f(1) = \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

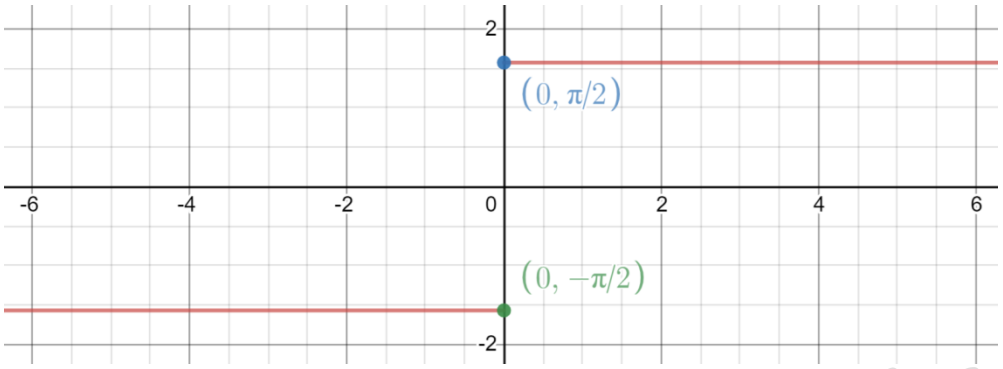
For $x < 0$, substitute in $x = -1$.

$c = f(-1) = -f(1) = -\frac{\pi}{2}$ (as the function is odd)

Hence for $x > 0$, $f(x) = \frac{\pi}{2}$ and for $x < 0$, $f(x) = -\frac{\pi}{2}$

Uncorrected proofs

© Cambridge University Press 2019



17a

$\left|\frac{1}{x}\right| \leq 1$

$|x| > 1$

Hence the domain is $x > 1$ or $x < -1$.

17b

$f'(x)$

$= \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \times -\frac{1}{x^2}$

$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$

$= \frac{1}{\sqrt{x^4}\sqrt{1-\frac{1}{x^2}}}$

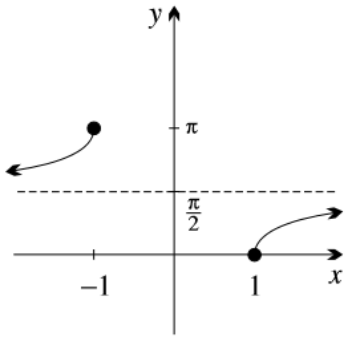
$= \frac{1}{\sqrt{x^2}\sqrt{x^2-1}}$

$= \frac{1}{|x|\sqrt{x^2-1}}$

Uncorrected proofs

© Cambridge University Press 2019

17c



17d When $x > 1$,

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

When $x < 1$,

$$f'(x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

17e For $x > 1$ we use the equation

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

Hence $x > 0$ and $\sqrt{x^2 - 1} > 0$ thus $f'(x) > 0$

For $x < -1$ we use the equation

$$f'(x) = -\frac{1}{x\sqrt{x^2 - 1}} = \frac{1}{-x\sqrt{x^2 - 1}}$$

Hence $-x > 0$ and $\sqrt{x^2 - 1} > 0$ thus $f'(x) > 0$

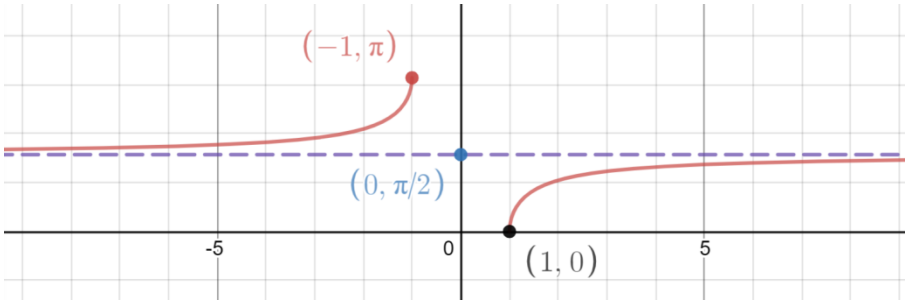
17f i $\lim\limits_{x\to\infty} \cos^{-1}\frac{1}{x} = \lim\limits_{x\to\infty} \cos^{-1} 0^{+} = \frac{\pi^{+}}{2}$

17f ii $\lim\limits_{x\to\infty} \cos^{-1}\frac{1}{x} = \lim\limits_{x\to\infty} \cos^{-1} 0^{-} = \frac{\pi^{-}}{2}$

Uncorrected proofs

© Cambridge University Press 2019

17g



18a $\frac{d}{dx}(\tan^{-1} e^{3x})$

$$= \frac{1}{1+(e^{3x})^2} \times \frac{d}{dx}(e^{3x})$$

$$= \frac{1}{1+(e^{3x})^2} \times 3e^{3x}$$

$$= \frac{3e^{3x}}{1+e^{6x}}$$

18b $\frac{d}{dx}(\sin^{-1} x^3)$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \times \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \times 3x^2$$

$$= \frac{3x^2}{\sqrt{1-x^6}}$$

18c $\frac{d}{dx}(\cos^{-1} \log_e x)$

$$= \frac{-1}{\sqrt{1-(\log_e x)^2}} \times \frac{d}{dx}(\log_e x)$$

$$= \frac{-1}{\sqrt{1-(\log_e x)^2}} \times \frac{1}{x}$$

$$= -\frac{1}{x\sqrt{1-(\log_e x)^2}}$$

Uncorrected proofs

19a The restrictions on x are

$-1 \leq x \leq 1$

and

$-1 \leq \sqrt{1-x^2} \leq 1$

But $\sqrt{1-x^2} \geq 0$, so

$0 \leq \sqrt{1-x^2} \leq 1$

$0 \leq 1-x^2 \leq 1$

$-1 \leq -x^2 \leq 0$

$0 \leq x^2 \leq 1$

Hence the domain is $-1 \leq x \leq 1$.

19b

$$g'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$
$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}} \times \left(\frac{1}{2} \times -2x \times (1-x^2)^{-\frac{1}{2}}\right)$$
$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}} \times \left(-\frac{x}{\sqrt{1-x^2}}\right)$$
$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{|x|} \times \frac{x}{\sqrt{1-x^2}}$$
$$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{|x|\sqrt{1-x^2}}$$

19c $g(x)$ is constant when $g'(x) = 0$, hence

$g'(x) = 0$

$$\frac{1}{\sqrt{1-x^2}} - \frac{x}{|x|\sqrt{1-x^2}} = 0$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{x}{|x|\sqrt{1-x^2}}$$

For $x \neq 1$ and $x \neq 1$

$|x|\sqrt{1-x^2} = x\sqrt{1-x^2}$

Uncorrected proofs

$|x| = x$

$|x| - x = 0$

This is true for $x \geq 0$. But $x \neq 0$ so the function is constant for $x > 0$.

20

$\tan^{-1}\frac{x+2}{1-2x}$ is simply $\tan^{-1} x + \tan^{-1} 2$ for $x < \frac{1}{2}$, and is $\tan^{-1} x + \tan^{-1} 2 - \pi$ for $x > \frac{1}{2}$

(this can be proven by applying tan to both equations and then using double angle formulae to show they have the same result)

21a As $\sin x$ is defined for all real x, and because $-1 \leq \sin x \leq 1$, this function will be defined for all real x.

Now since $-1 \leq \sin x \leq 1$

$\sin^{-1}(-1) \leq \sin^{-1}(\sin x) \leq \sin^{-1}(1)$

$-\frac{\pi}{2} \leq \sin^{-1}(\sin x) \leq \frac{\pi}{2}$

So the range is

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

21b

$f'(x)$

$= \frac{d}{dx}(\sin^{-1}(\sin(x)))$

$= \frac{1}{\sqrt{1-(\sin x)^2}} \times \frac{d}{dx}(\sin x)$

$= \frac{1}{\sqrt{1-\sin^2 x}} \times \cos x$

$= \frac{\cos x}{\sqrt{\cos^2 x}}$

$= \frac{\cos x}{|\cos x|}$

Uncorrected proofs

© Cambridge University Press 2019

21c When $\cos x = 0$, $f'(x) = \frac{0}{|0|}$ which is undefined. Hence $f'(x)$ is not defined when $\cos x = 0$.

21d Since $|\cos x| = \cos x$ or $|\cos x| = -\cos x$. Hence

$$f'(x) = \frac{\cos x}{\cos x} = 1$$

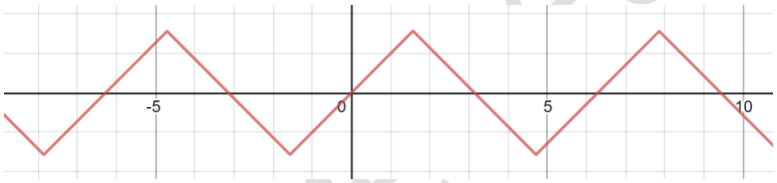
or

$$f'(x) = \frac{\cos x}{-\cos x} = -1$$

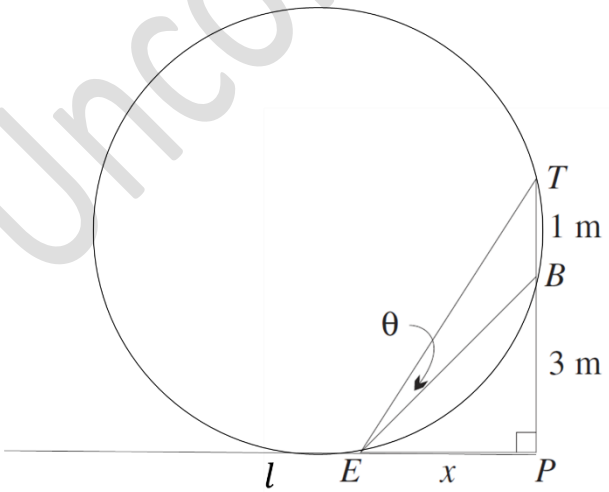
Now $f'(x) = 1$ when $\cos x > 0$ which is when $-\frac{\pi}{2} + 2\pi n < x < -\frac{\pi}{2} + 2\pi n$ for all integers n.

Now $f'(x) = -1$ when $\cos x < 0$ which is when $\frac{\pi}{2} + 2\pi n < x < \frac{3\pi}{2} + 2\pi n$ for all integers n.

21e



22a



Uncorrected proofs

Solutions to Exercise 12B

1 Answers will vary

2a

$$\int -\frac{1}{\sqrt{1-x^2}}dx = \cos^{-1}x + C$$

2b

$$\int \frac{1}{\sqrt{4-x^2}}dx$$

$$= \int \frac{1}{2\sqrt{1-\frac{x^2}{4}}}dx$$

$$= \int \frac{1}{2} \times \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

2c

$$\int \frac{1}{9+x^2}dx$$

$$= \int \frac{1}{9}\frac{1}{1+\frac{x^2}{9}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

Uncorrected proofs

© Cambridge University Press 2019

2d

$$\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx$$

$$= \int \frac{1}{\frac{2}{3}\sqrt{1-\left(\frac{3}{2}x\right)^2}}\,dx$$

$$= \int \frac{3}{2}\frac{1}{\sqrt{1-\left(\frac{3}{2}x\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{3}{2}x\right) + C$$

2e

$$\int \frac{1}{2+x^2}\,dx$$

$$= \int \frac{1}{2}\frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\,dx$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

2f

$$\int -\frac{1}{\sqrt{5-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5}}\frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}}\,dx$$

$$= \cos^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

Uncorrected proofs

© Cambridge University Press 2019

3a

$$\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= \sin^{-1}1 - \sin^{-1}0$$

$$= \frac{\pi}{2}$$

3b

$$\int_0^2 \frac{1}{4+x^2}\,dx$$

$$= \int_0^2 \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{1}{2}(\tan^{-1}1 - \tan^{-1}0)$$

$$= \frac{\pi}{8}$$

3c

$$\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$$

Uncorrected proofs

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{0}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

3d

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4} + x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + 4x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + (2x)^2}\,dx$$

$$= [\tan^{-1}(2x)]_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}}$$

$$= \tan^{-1}\left(2\sqrt{3}\right) - \tan^{-1}(1)$$

$$= \frac{\pi}{12}$$

3e

$$\int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\,dx$$

$$= \int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-1}{\frac{1}{3}\sqrt{1 - (3x)^2}}\,dx$$

$$= \int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-3}{\sqrt{1 - (3x)^2}}\,dx$$

$$= [\cos^{-1}(3x)]_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}}$$

$$= \cos^{-1}\frac{1}{2} - \cos^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6}$$

Uncorrected proofs

© Cambridge University Press 2019

3f

$$\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$$

$$= \int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\frac{3}{2}\sqrt{1-\left(\frac{2x}{3}\right)^2}}$$

$$= \int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{2}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}}$$

$$= \left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{5\pi}{12}$$

4a

$y' = (1-x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1-x^2}}$

$y = \sin^{-1} x + C$

Substituting in $(0, \pi)$

$\pi = \sin^{-1} 0 + C$

$\pi = 0 + C$

$C = \pi$

Hence

$y = \sin^{-1} x + \pi$

4b

$y' = 4(16+x^2)^{-1}$

$= \frac{1}{4(16+x^2)}$

Uncorrected proofs

$$= \frac{4}{4 \times 16} \times \frac{1}{1 + \left(\frac{x}{4}\right)^2}$$

$$= \frac{1}{4} \times \frac{1}{1 + \left(\frac{x}{4}\right)^2}$$

$$y = \tan^{-1} \frac{x}{4} + C$$

Substituting in $(-4, 0)$

$$0 = \tan^{-1} -1 + C$$

$$0 = -\frac{\pi}{4} + C$$

$$C = \frac{\pi}{4}$$

Hence

$$y = \tan^{-1} \frac{x}{4} + \frac{\pi}{4}$$

5a

$$y' = \frac{1}{\sqrt{36 - x^2}} = \frac{1}{6\sqrt{1 - \left(\frac{x}{6}\right)^2}}$$

$$y = \sin^{-1}\left(\frac{x}{6}\right) + C$$

When $x = 3, y = \frac{\pi}{6}$

$$\frac{\pi}{6} = \sin^{-1} \frac{1}{2} + C$$

$$C = 0$$

$$y = \sin^{-1}\left(\frac{x}{6}\right)$$

When $x = 3\sqrt{3}$

$$y = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Uncorrected proofs

© Cambridge University Press 2019

5b

$$y' = \frac{2}{4+x^2} = \frac{1}{2}\frac{1}{1+\left(\frac{x}{2}\right)^2}$$

$$y = \tan^{-1}\frac{x}{2} + C$$

When $x = 2, y = \frac{\pi}{3}$

$$\frac{\pi}{3} = \tan^{-1} 1 + C$$

$$\frac{\pi}{3} = \frac{\pi}{4} + C$$

$$C = \frac{\pi}{12}$$

$$y = \tan^{-1}\frac{x}{2} + \frac{\pi}{12}$$

When $x = \frac{2}{\sqrt{3}}$

$$y = \tan^{-1}\frac{1}{\sqrt{3}} + \frac{\pi}{12} = \frac{\pi}{4}$$

6a

$$\int \frac{1}{\sqrt{1-4x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

$$= \frac{1}{2}\sin^{-1} 2x + C$$

6b

$$\int \frac{1}{1+16x^2}\,dx$$

$$= \int \frac{1}{1+(4x)^2}\,dx$$

$$= \frac{1}{4}\tan^{-1} 4x + C$$

Uncorrected proofs

© Cambridge University Press 2019

6c

$$\int -\frac{1}{\sqrt{1-2x^2}}\,dx$$

$$= \int -\frac{1}{\sqrt{1-\left(\sqrt{2}x\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{2}}\cos^{-1}\sqrt{2}x + C$$

6d

$$\int \frac{1}{\sqrt{4-9x^2}}\,dx$$

$$= \int \frac{1}{2\sqrt{1-\left(\frac{3x}{2}\right)^2}}\,dx$$

$$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + C$$

6e

$$\int \frac{1}{25+9x^2}\,dx$$

$$= \int \frac{1}{25}\frac{1}{1+\left(\frac{3x}{5}\right)^2}\,dx$$

$$= \frac{1}{15}\tan^{-1}\frac{3x}{5} + C$$

6f

$$\int -\frac{1}{\sqrt{3-4x^2}}\,dx$$

$$= \int -\frac{1}{\sqrt{3}\sqrt{1-\left(\frac{2x}{\sqrt{3}}\right)^2}}\,dx$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{2}\cos^{-1}\frac{2x}{\sqrt{3}} + C$$

7a

$$\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$$

$$= \int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$$= \frac{1}{3}[\sin^{-1} 3x]_0^{\frac{1}{6}}$$

$$= \frac{\pi}{18}$$

7b

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+4x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+(2x)^2}\,dx$$

$$= [\tan^{-1}(2x)]_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}}$$

$$= \frac{\pi}{12}$$

7c

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}}\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-\left(\sqrt{3}x\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}[\sin^{-1}\sqrt{3}x]_{-\frac{1}{2}}^{\frac{1}{2}}$$

Uncorrected proofs

$$= \frac{2\pi}{9}\sqrt{3}$$

7d

$$\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9-4x^2}}\,dx$$

$$= \int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}}\,dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}}$$

$$= \frac{5\pi}{24}$$

7e

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3} \frac{1}{1+\left(\frac{2x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{6}\left[\tan^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{\pi}{12}\sqrt{3}$$

Uncorrected proofs

© Cambridge University Press 2019

7f

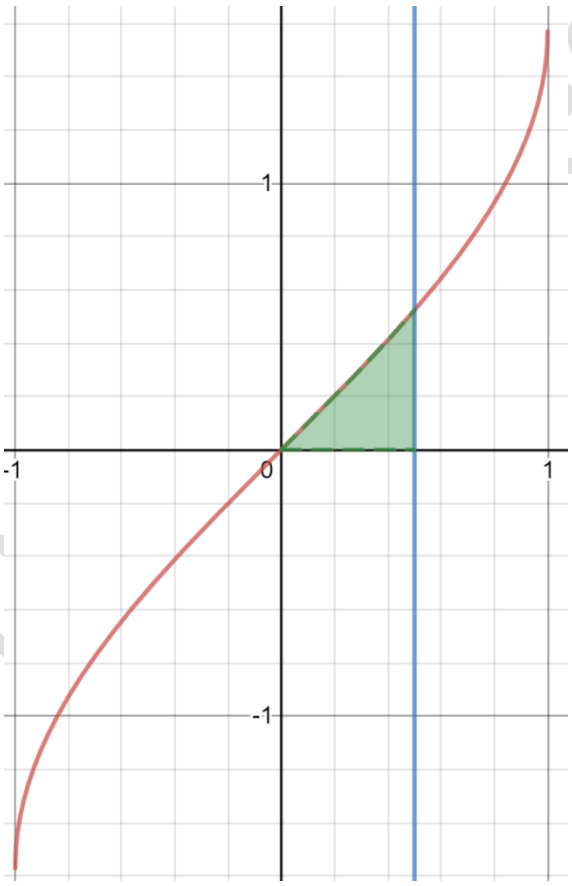
$$\int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5+2x^2}\,dx$$

$$= \int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5} \frac{1}{1+\left(\frac{\sqrt{2}x}{\sqrt{5}}\right)^2}\,dx$$

$$= \frac{\sqrt{5}}{5\sqrt{2}}\left[\tan^{-1}\frac{\sqrt{2}x}{\sqrt{5}}\right]_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}}$$

$$= \frac{\pi}{120}\sqrt{10}$$

8a



Updated proofs

8b

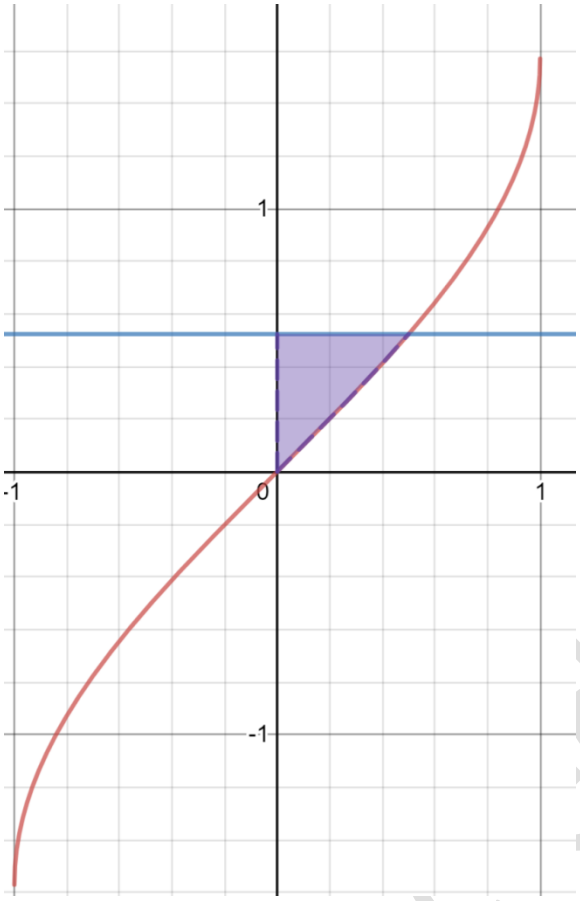
$$\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)$$
$$=\frac{d}{dx}\left(x\sin^{-1}x\right)+\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$
$$=\frac{d}{dx}(x)\sin^{-1}x+x\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right)$$
$$=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}+\frac{1}{2}\times-2x\times\frac{1}{\sqrt{1-x^2}}$$
$$=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}$$
$$=\sin^{-1}x$$

8c

$$\int_0^{\frac{1}{2}}\sin^{-1}x\,dx$$
$$=\int_0^{\frac{1}{2}}\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)dx$$
$$=\left[x\sin^{-1}x+\sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$
$$=\left(\frac{\pi}{12}+\frac{1}{2}\sqrt{3}-1\right)\text{ square units}$$

Uncorrected proofs

9a



9b

$$A = \int_0^{\frac{\pi}{6}} \sin y \, dy$$

$$= -[\cos y]_0^{\frac{\pi}{6}}$$

$$= \left(1 - \frac{1}{2}\sqrt{3}\right) \text{ square units}$$

9c Note that the area formed by combined regions of the previous two questions is that of a square with area $\frac{\pi}{6} \times \frac{1}{2} = \frac{\pi}{12}$ units^2.

Thus, the area in this question will be $\frac{\pi}{12} - (\frac{\pi}{12} + \frac{1}{2}\sqrt{3} - 1) = 1 - \frac{1}{2}\sqrt{3}$ units^2

Uncorrected proofs

© Cambridge University Press 2019

10a

$$\frac{d}{dx}(\cos^{-1}(2-x))$$
$$= \frac{-1}{\sqrt{1-(2-x)^2}} \times \frac{d}{dx}(2-x)$$
$$= \frac{-1}{\sqrt{1-(2-x)^2}} \times -1$$
$$= \frac{1}{\sqrt{1-(4-2x+x^2)}}$$
$$= \frac{1}{\sqrt{4x-x^2-3}}$$

10b

$$\int_1^2 \frac{1}{\sqrt{4x-x^2-3}}dx$$
$$= [\cos^{-1}(2-x)]_1^2$$
$$= \cos^{-1} 0 - \cos^{-1} 1$$
$$= \frac{\pi}{2}$$

11a

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{2}x^3\right)$$
$$= \frac{1}{1+\left(\frac{1}{2}x^3\right)^2} \times \frac{d}{dx}\left(\frac{1}{2}x^3\right)$$
$$= \frac{1}{1+\frac{x^6}{4}} \times \frac{3}{2}x^2$$
$$= \frac{3x^2}{2+\frac{x^6}{2}}$$
$$= \frac{6x^2}{4+x^6}$$

Uncorrected proofs

11b

$$\int \frac{x^2}{4+x^6}\,dx$$

$$= \frac{1}{6}\int \frac{6x^2}{4+x^6}\,dx$$

$$= \frac{1}{6}\int \frac{d}{dx}\left(\tan^{-1}\frac{1}{2}x^3\right)dx$$

$$= \frac{1}{6}\tan^{-1}\frac{1}{2}x^3 + C$$

12a

$$\frac{d}{dx}(x\tan^{-1}x)$$

$$= \tan^{-1}x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\tan^{-1}x)$$

$$= \tan^{-1}x + \frac{x}{1+x^2}$$

12b

$$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$$

$$\tan^{-1}x = \frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}$$

$$\int_0^1 \tan^{-1}x\,dx$$

$$= \int_0^1 \left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$$

$$= \int_0^1 \frac{d}{dx}(x\tan^{-1}x)\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= [x\tan^{-1}x]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Uncorrected proofs

© Cambridge University Press 2019

13a 0 as this is an odd function

13b 0 as this is an odd function

13c

$$\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x \, dx$$

$$= -\left[\frac{1}{\sqrt{1-x^2}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

$$= \frac{3\pi}{4}$$

13d 0 as this is an odd function

13e 0 as this is an odd function

13f

This is a semicircle of radius 6. Hence

$$\int_{-6}^{6} \sqrt{36-x^2} \, dx$$

$$= \frac{1}{2} \times (\pi \times 6^2)$$

$= 18\pi$ square units

14a i $f(0) = \frac{0}{1+0^2} - \tan^{-1} 0 = 0 - 0 = 0$

14a ii

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) - \frac{d}{dx}(\tan^{-1} x)$$

Uncorrected proofs

$$= \frac{(1+x^2) \times \frac{d}{dx}(x) - x \times \frac{d}{dx}(1+x^2)}{(1+x^2)^2} - \frac{1}{1+x^2}$$
$$= \frac{1+x^2-x(2x)}{(1+x^2)^2} - \frac{1+x^2}{(1+x^2)^2}$$
$$= \frac{-2x^2}{(1+x^2)^2}$$

14b i Firstly consider $f'(x)$

As the numerator $-2x^2 < 0$ for all $x > 0$ and the denominator $(1+x^2)^2 < 0$ for all $x \geq 0$. It follows that $f'(x) < 0$ for all $x < 0$.

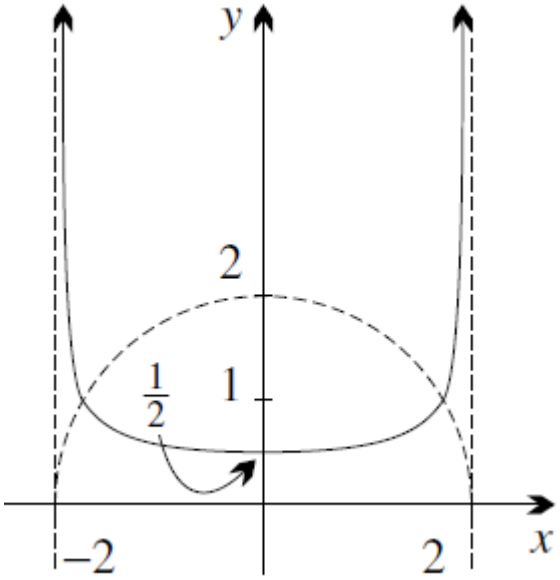
Since $f(0) = 0$, and as the function is strictly decreasing for all $x > 0$, it follows that $f(x) < 0$ for all $x > 0$.

14b ii

$$\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 \frac{-2x^2}{(1+x^2)^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 f'(x)\,dx$$
$$= -\frac{1}{2}[f(x)]_0^1$$
$$= -\frac{1}{2}[f(1) - f(0)]$$
$$= -\frac{1}{2}f(1)$$
$$= -\frac{1}{2}\left[\frac{1}{2} - \frac{\pi}{4}\right]$$
$$= \frac{1}{8}[\pi - 2]$$

Uncorrected proofs

15a,b



15c From the graph the domain is $-2 < x < 2$ (note that this is strictly less than) and the range is $x \geq \frac{1}{2}$. Its symmetry is even as $f(-x) = \frac{1}{\sqrt{4-(-x)^2}} = \frac{1}{\sqrt{4-x^2}} = f(x)$

15d

$$A = \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx$$
$$= \int_{-1}^{1} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$
$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$$
$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right)$$
$$= \frac{\pi}{3} \text{ square units}$$

Uncorrected proofs

15e

$$\begin{aligned}A &= \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx \\ &= \int_{-2}^{2} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx \\ &= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} \\ &= \sin^{-1} 1 - \sin^{-1}(-1) \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\ &= \pi \text{ square units}\end{aligned}$$

16a

$$f(-x) = \frac{4}{(-x)^2+4} = \frac{4}{x^2+4} = f(x)$$

Hence the function is even and thus has an axis of symmetry about the y-axis.

16b The function is defined for all real x so the domain is all real x.

As $x^2 + 4 \geq 4$ it follows that $0 < \frac{1}{x^2+4} \leq \frac{1}{4}$ and hence $0 < \frac{4}{x^2+4} \leq 1$ thus the range is $0 < f(x) \leq 1$.

16c

$$\begin{aligned}&f'(x) \\ &= \frac{d}{dx}(4(x^2+4)^{-1}) \\ &= -1 \times 4(x^2+4)^{-2} \times \frac{d}{dx}(x^2) \\ &= -1 \times 4(x^2+4)^{-2} \times 2x \\ &= -\frac{8x}{(x^2+4)^2}\end{aligned}$$

Hence

$f'(0) = 0$

and

$f(0) = 1$

Uncorrected proofs

So there is a turning point at $(0, 1)$.

Testing for maximum

x	-1	0	1
$f'(x)$	$\frac{8}{25}$	0	$-\frac{8}{25}$

Thus the curve is increasing before the turning point decreasing afterwards.
Thus it is a maximum

16d

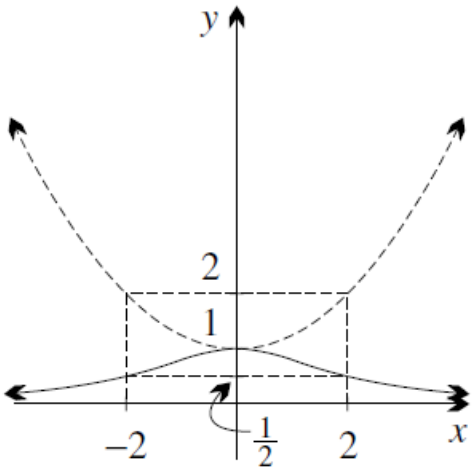
$$\lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty} \frac{4}{x^2+4}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{x^2}}{1+\frac{4}{x^2}}$$

$$= \frac{0}{1+0}$$

$$= 0$$



Uncorrected proofs

16e

$$\begin{aligned}A &= \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{4}{x^2+4}\,dx \\ &= \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \\ &= 2\left[\tan^{-1}\frac{x}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}} \\ &= 2\left(\tan^{-1}\sqrt{3}-\tan^{-1}\left(-\sqrt{3}\right)\right) \\ &= \pi \text{ square units}\end{aligned}$$

16f

$$\begin{aligned}A &= \int_{-a}^{a} \frac{4}{x^2+4}\,dx \\ &= \int_{-a}^{a} \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \\ &= 2\left[\tan^{-1}\frac{x}{2}\right]_{-a}^{a} \\ &= 2\left(\tan^{-1}\frac{a}{2}-\tan^{-1}\left(-\frac{a}{2}\right)\right) \\ &= 4\tan^{-1}\frac{a}{2} \text{ square units}\end{aligned}$$

16g

$$\begin{aligned}&\lim_{a\to\infty} 4\tan^{-1}a \\ &= 4\lim_{a\to\infty}\tan^{-1}a \\ &= 4\left(\frac{\pi}{2}\right) \\ &= 2\pi \text{ square units}\end{aligned}$$

Uncorrected proofs

© Cambridge University Press 2019

17

Since $y = \sin^{-1} x, \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

When $x = \frac{\sqrt{3}}{2}, y = \frac{\pi}{3}, \frac{dy}{dx} = 2$. Hence the tangent is

$$y - \frac{\pi}{3} = 2\left(x - \frac{\sqrt{3}}{2}\right)$$

$$y = 2x - \sqrt{3} + \frac{\pi}{3}$$

Rearranging this gives

$$x = \frac{1}{2}\left(y + \sqrt{3} - \frac{\pi}{3}\right)$$

Also note that for the tangent, when $x = 0, y = -\sqrt{3} + \frac{\pi}{3}$

Hence the area can be calculated by

$$A = \int_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} y_{tangent}\, dx - \int_0^{\frac{\pi}{3}} y_{curve}\, dx$$

$$= \int_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(y + \sqrt{3} - \frac{\pi}{3}\right) dx - \int_0^{\frac{\pi}{3}} \sin y\, dx$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + y\left(\sqrt{3} - \frac{\pi}{3}\right)\right]_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} + [\cos y]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{4} \text{ square units}$$

18a

$$I = \int_0^1 f(x)\, dx$$

$$\div \frac{b-a}{2n}(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)$$

$$= \frac{1-0}{2(4)}(f(0) + f(1) + 2(f(0.25) + f(0.5) + f(0.75))$$

$$= \frac{1}{8}\left(1 + \frac{1}{2} + 2\left(\frac{16}{17} + 0.8 + 0.64\right)\right)$$

$$= \frac{5323}{6800} \text{ square units}$$

Uncorrected proofs

18b

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

Hence

$$\frac{\pi}{4} \div \frac{5323}{6800}$$

so

$$\pi \div \frac{5323}{1700}$$

19

$$\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx$$

$$= [\tan^{-1} x]_{-\frac{1}{4}}^{\frac{3}{5}}$$

$$= \tan^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{1}{4}\right)$$

Hence

$$\tan \int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx$$

$$= \tan\left(\tan^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{1}{4}\right)\right)$$

$$= \frac{\tan\left(\tan^{-1}\frac{3}{5}\right) + \tan\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)}{1 - \tan\left(\tan^{-1}\frac{3}{5}\right)\tan\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)}$$

$$= \frac{\frac{3}{5} - \frac{1}{4}}{1 - \left(\frac{3}{5}\right)\left(-\frac{1}{4}\right)}$$

$$= 1$$

Hence

$$\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

Uncorrected proofs

© Cambridge University Press 2019

20a Let $x = u^2, dx = 2u\,du$

$$\int \frac{1}{\sqrt{x}(1-x)}dx = \int \frac{1}{\sqrt{u^2}(1+u^2)} \times 2u\,du$$

For $u > 0$

$$\int \frac{1}{\sqrt{x}(1-x)}dx$$

$$= \int \frac{1}{u(1+u^2)} \times 2u\,du$$
$$= 2\int \frac{1}{(1+u^2)}\,du$$
$$= 2\tan^{-1} u + c$$
$$= 2\tan^{-1}\sqrt{x} + \text{c}$$

For $u < 0$

$$\int \frac{1}{\sqrt{x}(1-x)}dx$$

$$= \int \frac{1}{-u(1+u^2)} \times 2u\,du$$
$$= -2\int \frac{1}{(1+u^2)}\,du$$
$$= -2\tan^{-1} u + c$$
$$= -2\tan^{-1}\sqrt{x} + \text{c}$$

20b Let $u = e^x, du = e^x\,dx$ hence $du = u\,dx$ and so $\frac{1}{u}\,du = dx$

$$\int_0^1 \frac{1}{e^{-x}+e^x}dx$$
$$= \int_1^e \frac{1}{u^{-1}+u} \times \frac{1}{u}du$$
$$= \int_1^e \frac{1}{1+u^2}du$$
$$= [\tan^{-1} u]_1^e$$
$$= \tan^{-1} e - \frac{\pi}{4}$$

Uncorrected proofs

© Cambridge University Press 2019

21a

$$\begin{aligned}
&\frac{d}{dx}\left(\tan^{-1}\left(\frac{3}{2}\tan x\right)\right)\\
&=\frac{1}{1+\left(\frac{3}{2}\tan x\right)^2}\times\frac{d}{dx}\left(\frac{3}{2}\tan x\right)\\
&=\frac{1}{1+\frac{9}{4}\tan^2 x}\times\frac{3}{2}\sec^2 x\\
&=\frac{1}{1+\frac{9}{4}\frac{\sin^2 x}{\cos^2 x}}\times\frac{3}{2}\frac{1}{\cos^2 x}\\
&=\frac{3}{2}\times\frac{1}{\cos^2 x+\frac{9}{4}\sin^2 x}\\
&=\frac{6}{4\cos^2 x+9\sin^2 x}\\
&=\frac{6}{4\sin^2 x+5(\sin^2 x+\cos^2 x)}\\
&=\frac{6}{4\sin^2 x+5}
\end{aligned}$$

21b

$$\begin{aligned}
A&=\int_0^7\frac{6}{4\sin^2 x+5}dx\\
&=\int_0^7\frac{d}{dx}\left(\tan^{-1}\left(\frac{3}{2}\tan x\right)\right)dx\\
&=\left[\tan^{-1}\left(\frac{3}{2}\tan x\right)\right]_0^7\\
&=\tan^{-1}\left(\frac{3}{2}\tan 7\right)-\tan^{-1}\left(\frac{3}{2}\tan 0\right)\\
&\doteqdot 0.153 \text{ square unit}
\end{aligned}$$

21c The integrand is well-defined in the interval $[0, 7]$, and lies between 14 and 19, so the area lies between 74 and 79, which is much larger than the answer of 0.153 that was calculated in part (b). The primitive, however, is undefined at two values within the interval, at $x=\frac{\pi}{2}$ and at $x=\frac{3\pi}{2}$, which renders the argument completely invalid.

Uncorrected proofs

© Cambridge University Press 2019

22a This is a GP with $a = 1$ and $r = -t^2$. Hence $T_k = (-t)^{k-1}$, solving for $T_k = t^{4n}$

$t^{4n} = (-t^2)^{k-1}$

$t^{4n} = (-1)^{k-1}(t^2)^{k-1}$

As $4n$ is even, $k - 1$ must also be even and hence $(-1)^{k-1} = 1$. Thus

$t^{4n} = (t^2)^{k-1}$

$4n = 2k - 2$

$k = 2n + 1$

Hence there are $2n + 1$ terms so the sum must be

$$S_{2n+1} = \frac{a(r^n - 1)}{r - 1} = \frac{(-t^2)^{2n+1} - 1}{-t^2 - 1}$$

Since $2n + 1$ is odd

$$S_{2n+1} = \frac{-t^{4n+2} - 1}{-t^2 - 1} = \frac{t^{4n+2} + 1}{t^2 + 1}$$

Hence it follows that for positive values of t

$$\frac{1}{t^2 + 1} < \frac{t^{4n+2} + 1}{t^2 + 1}$$

And thus

$$\frac{1}{t^2 + 1} < 1 - t^2 + t^4 - \cdots + t^{4n}$$

22b As we are adding one extra term to the previous sum, this will be S_{2n+2}

Hence there are $2n + 2$ terms so the sum must be

$$S_{2n+2} = \frac{a(r^n - 1)}{r - 1} = \frac{(-t^2)^{2n+2} - 1}{-t^2 - 1}$$

Since $2n + 2$ is even

$$S_{2n+2} = \frac{t^{4n+4} - 1}{-t^2 - 1} = \frac{-t^{4n+4} + 1}{t^2 + 1}$$

Hence it follows that for positive values of t

From part **a**

Uncorrected proofs

$$\frac{1}{t^2+1} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n}$$

$$\frac{1}{t^2+1} - t^{4n+2} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n} - t^{4n+2}$$

$$\frac{1}{t^2+1} - t^{4n+2} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1} + t^{4n+2}$$

$$\frac{1+t^{4n+4}}{t^2+1} < \frac{1}{t^2+1} + t^{4n+2}$$

$$1 - t^2 + t^4 - t^6 + \cdots + t^{4n} < \frac{1}{t^2+1} + t^{4n+2}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{1+t^{4n+2}+t^{4n+4}}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + \frac{t^{4n+2}+t^{4n+4}}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + \frac{t^{4n+2}(1+t^2)}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + t^{4n+2}$$

Uncorrected proofs

© Cambridge University Press 2019

22c The inequality

$$\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \cdots t^{4n} < \frac{1}{1+t^2} + t^{4n+2}$$

Is true for all $0 < t < x$, hence

$$\int_0^x \frac{1}{1+t^2}\,dt < \int_0^x (1 - t^2 + t^4 - t^6 + \cdots t^{4n})\,dt < \int_0^x (\frac{1}{1+t^2} + t^{4n+2})\,dt$$

$$[\tan^{-1} x]_0^x < \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots + \frac{t^{4n+1}}{4n+1}\right]_0^x < \left[\tan^{-1} x + \frac{t^{4n+3}}{4n+3}\right]_0^x$$

Thus

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}$$

22d

For $0 \le x \le 1$ we know that $0 \le x^{4n+3} \le 1$ and as $n \to \infty, \frac{1}{4n+3} \to 0$, hence it follows that as $n \to \infty, \frac{x^{4n+3}}{4n+3} \to 0$.

Thus as $n \to \infty$

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots < \tan^{-1} x + 0$$

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots < \tan^{-1} x$$

And so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

22e As $\tan x$ is odd, $\tan^{-1} -x = -\tan^{-1} x$

and hence for $-1 \le x < 0$, (where $0 < -x \le 1$)

$\tan^{-1} x$

$= \tan^{-1}(-(-x))$
$= -\tan^{-1}(-x)$
$= -\left((-1) - \frac{(-1)^3}{3} + \frac{(-1)^5}{5} - \frac{(-1)^7}{7} + \cdots\right)$

Uncorrected proofs

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

22f Substituting $x = 1$ into the above equation

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

22g

$$\frac{\pi}{4} = \frac{3-1}{3 \times 5} + \frac{7-5}{5 \times 7} + \frac{11-9}{9 \times 11} \ldots$$

$$\frac{\pi}{4} = \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \frac{2}{9 \times 11} \ldots$$

$$\frac{\pi}{8} = \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{9 \times 11} \ldots$$

Using 10 terms gives

$\pi \div 3.092$, error $\div 0.050$

23a All rectangles have the same width of 1 unit and their height is given by $\tan^{-1} n$ for the nth rectangle . This means that

$$S_n = \tan^{-1} 1 + \tan^{-1} 2 + \cdots + \tan^{-1} n$$

23b

$$\frac{d}{dx}(x \tan x) = \tan x + x \times \frac{1}{1 + x^2}$$

Hence

$$\int \frac{d}{dx}(x \tan x)\, dx = \int \tan x + x \times \frac{1}{1 + x^2}\, dx$$

$$x \tan x = \int \tan x\, dx + \int \frac{x}{1 + x^2}\, dx$$

$$x \tan x = \int \tan x\, dx + \frac{1}{2} \ln(1 + x^2) + C$$

$$\int \tan x\, dx = x \tan x - \frac{1}{2} \ln(1 + x^2) + C$$

Uncorrected proofs

© Cambridge University Press 2019

23c Firstly note that since

$$\int \tan x \, dx = x \tan x - \frac{1}{2}\ln(1 + x^2) + C$$

Substituting $x = u - 1$ into the equation gives

$$\int \tan(u - 1) \, du$$

$$= (u - 1)\tan(u - 1) - \frac{1}{2}\ln(1 + (u - 1)^2) + C$$
$$= (u - 1)\tan^{-1}(u - 1) - \frac{1}{2}\ln(1 + u^2 - 2u + 1) + C$$
$$= (u - 1)\tan^{-1}(u - 1) - \frac{1}{2}\ln(u^2 - 2u + 2) + C$$

The area in the boxes is overestimated by the area under $\tan^{-1} x$ but underestimated by the area under $\tan^{-1}(x - 1)$, hence

$$\int_1^{n+1} \tan(x - 1) \, dx < S_n < \int_0^{n+1} \tan(x) \, dx$$

$$\int_1^{n+1} \tan(x - 1) \, dx < S_n < \int_0^{n+1} \tan(x) \, dx$$

$$\left[(x - 1)\tan^{-1}(x - 1) - \frac{1}{2}\ln(x^2 - 2x + 2)\right]_1^{n+1} < S_n$$
$$< \left[(x)\tan^{-1}(x) - \frac{1}{2}\ln(1 + x^2)\right]_1^{n+1}$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) - (0)\tan^{-1}(0) - \frac{1}{2}\ln(1) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln(1 + (n + 1)^2) - (1)\tan^{-1}(1) - \frac{1}{2}\ln(2)$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln(1 + (n + 1)^2) - \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln\left(\frac{n^2 + 2n + 2}{2}\right) - \frac{\pi}{4}$$

$$n\tan^{-1} n - \frac{1}{2}\ln(n^2 + 1) < S_n < (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln\left(\frac{n^2}{2} + n + 1\right) - \frac{\pi}{4}$$

Uncorrected proofs

© Cambridge University Press 2019

23d Substituting $n = 1000$ into the above equation gives

$$1000\tan^{-1}1000 - \frac{1}{2}\ln(1000^2 + 1) < S_{1000}$$
$$< (1000 + 1)\tan^{-1}(1000 + 1) - \frac{1}{2}\ln\left(\frac{1000^2}{2} + 1000 + 1\right) - \frac{\pi}{4}$$

$$1562.89 < S_{100} < 1564.02$$

$$1563 < S_{1000} < 1565$$

Hence

$$1563 < \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 + \cdots + \tan^{-1}1000 < 1565$$

Uncorrected proofs

© Cambridge University Press 2019

Solutions to Exercise 12C

1a

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$2\sin^2 x = 1 - \cos 2x$$
$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

1b

$$\cos 4x = \cos 2(2x)$$
$$\cos 4x = \cos^2 2x - \sin^2 2x$$
$$\cos 4x = \cos^2 2x - (1 - \cos^2 2x)$$
$$\cos 4x = 2\cos^2 2x - 1$$
$$2\cos^2 2x = 1 + \cos 4x$$
$$\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$$

1c

$$\frac{1}{2}\sin 6x = \frac{1}{2}\sin 2(3x)$$
$$= \frac{1}{2}(2\sin 3x \cos 3x)$$
$$= \sin 3x \cos 3x$$

1d

$$\cos x = \cos 2\left(\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$
$$\cos x = 1 - \sin^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$
$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

Uncorrected proofs

© Cambridge University Press 2019

$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x$$

2a $\cos^2 15°$

$$= \frac{1}{2} + \frac{1}{2}\cos 30°$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}(2 + \sqrt{3})$$

2b

$$\sin^2 \frac{5\pi}{12}$$

$$= \frac{1}{2} - \frac{1}{2}\cos \frac{5\pi}{6}$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}(2 + \sqrt{3})$$

2c $\sin 105° \cos 105°$

$$= \frac{1}{2}\sin 210°$$

$$= \frac{1}{2} \times \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

2d

$$\sin^2 \frac{7\pi}{8}$$

$$= \frac{1}{2} - \frac{1}{2}\cos \frac{7\pi}{4}$$

Uncorrected proofs

$$= \frac{1}{2} - \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{2 - \sqrt{2}}{4}$$

3 $\sin^2 \theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$ (see 1a)

3a

$$\int \sin^2 x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

3b

$$\int \sin^2 2x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 4x}{8} + C$$

3c

$$\int \sin^2 \frac{1}{4}x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos \frac{1}{2}x\right) dx$$

$$= \frac{x}{2} - \frac{\sin \frac{1}{2}x}{2\left(\frac{1}{2}\right)} + C$$

$$= \frac{x}{2} - \sin \frac{1}{2}x + C$$

Uncorrected proofs

© Cambridge University Press 2019

3d

$$\int \sin^2 3x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 6x}{12} + C$$

4 $\cos^2 \theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ (see 1b)

4a

$$\int \cos^2 x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

4b

$$\int \cos^2 6x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 12x\right) dx = \frac{x}{2} + \frac{\sin 12x}{24} + C$$

4c

$$\int \cos^2 \frac{1}{2}x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos x\right) dx = \frac{x}{2} + \frac{\sin x}{2} + C$$

4d

$$\int \cos^2 10x \, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 20x\right) dx = \frac{x}{2} + \frac{\sin 20x}{40} + C$$

5a

$$\int_0^{\pi} \sin^2 x \, dx$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

Uncorrected proofs

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

5b

$$\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}(\pi + 2)$$

5c

$$\int_0^{\frac{\pi}{6}} \sin^2 \frac{1}{2} x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos x\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin x}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{12}(\pi - 3)$$

5d

$$\int_0^{\frac{\pi}{16}} \cos^2 2x \, dx$$

$$= \int_0^{\frac{\pi}{16}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

Uncorrected proofs

$$= \left[\frac{x}{2} + \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}(\pi + 2\sqrt{2})$$

5e

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\left(x + \frac{\pi}{12}\right) dx$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} + \frac{1}{2}\cos 2\left(x + \frac{\pi}{12}\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2\left(x + \frac{\pi}{12}\right)}{4}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{24}(4\pi + 9)$$

5f

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\left(x - \frac{\pi}{6}\right) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\left(x - \frac{\pi}{6}\right)\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2\left(x - \frac{\pi}{6}\right)}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{24}(2\pi - 3\sqrt{3})$$

Uncorrected proofs

© Cambridge University Press 2019

6a

$$\begin{aligned}
&\int \sin 3x \cos 2x \, dx \\
&= \int \left(\frac{1}{2}\sin(2x+3x) + \frac{1}{2}\sin(3x-2x)\right) dx \\
&= \int \left(\frac{1}{2}\sin 5x + \frac{1}{2}\sin x\right) dx \\
&= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C
\end{aligned}$$

6b

$$\begin{aligned}
&\int \cos 3x \sin x \, dx \\
&= \int \left(\frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x\right) dx \\
&= \frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C
\end{aligned}$$

6c

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} 2\cos 2x \cos x \, dx \\
&= \int_0^{\frac{\pi}{4}} 2 \times \frac{1}{2}(\cos x + \cos 3x) \, dx \\
&= \int_0^{\frac{\pi}{4}} (\cos x + \cos 3x) \, dx \\
&= \left[\sin x + \frac{1}{3}\sin x\right]_0^{\frac{\pi}{4}} \\
&= \frac{2\sqrt{2}}{3}
\end{aligned}$$

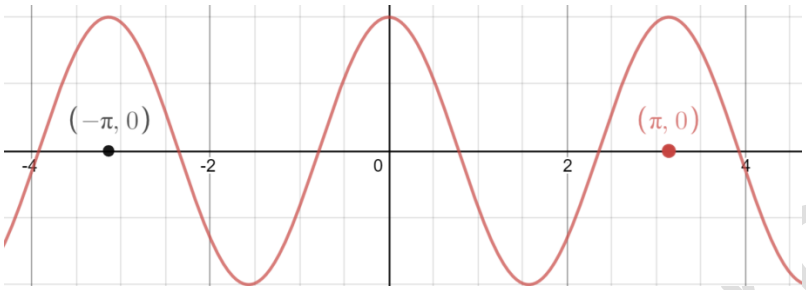
6d

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \sin 5x \sin 2x \, dx \\
&= \int_0^{\frac{\pi}{3}} \frac{1}{2}(\sin 3x - \sin 7x) \, dx
\end{aligned}$$

Uncorrected proofs

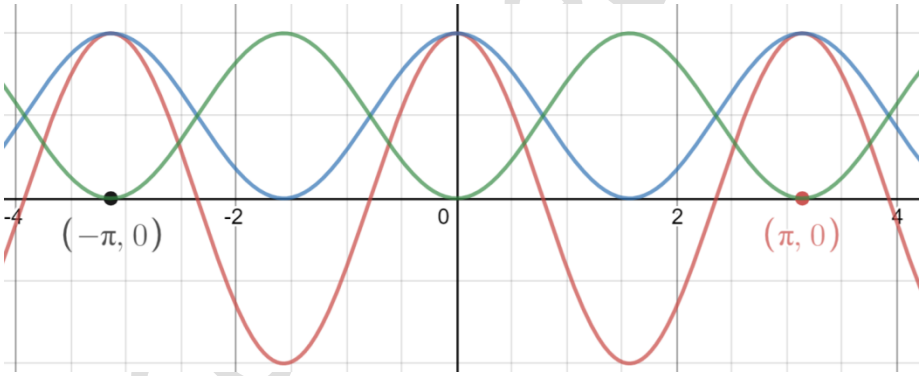
$$= \frac{1}{2}\left[-\frac{1}{3}\cos 3x + \frac{1}{7}\cos 7x\right]_0^{\frac{\pi}{3}}$$
$$= \frac{\sqrt{3}}{28}$$

7a



7b

$y = \frac{1}{2}(1 + \cos 2x)$ is shown in blue and $y = \frac{1}{2}(1 - \cos 2x)$ is shown in green.



7c

By observation firstly note that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. By observation of the graph, we see that these two graphs always add to 1 and hence $\sin^2 x + \cos^2 x = 1$.

Uncorrected proofs

8a Let $u = \sin x, du = \cos x\, dx$

$$\int \sin^3 x \cos x\, dx$$

$$= \int u^3\, du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

8b Let $u = \sin x, du = \cos x\, dx$

$$\int \sin^6 x \cos x\, dx$$

$$= \int u^6\, du$$

$$= \frac{u^7}{7} + C$$

$$= \frac{\sin^7 x}{7} + C$$

8c Let $u = \cos x, du = -\sin x\, dx$

$$\int \cos^5 x \sin x\, dx$$

$$= -\int u^5\, du$$

$$= -\frac{u^6}{6} + C$$

$$= -\frac{\cos^6 x}{6} + C$$

8d Let $u = \cos x, du = -\sin x$

$$\int \cos^8 x \sin x\, dx$$

$$= \int -u^8\, du$$

Uncorrected proofs

$$= -\frac{u^9}{9} + C$$

$$= -\frac{\cos^9 x}{9} + C$$

8e Let $u = e^x, du = e^x dx$

$$\int e^x \sin e^x \, dx$$

$$= \int \sin u \, du$$

$$= -\cos u + C$$

$$= -\cos e^x + C$$

8f Let $u = e^x, du = e^x dx$

$$\int e^x \cos 5e^x \, dx$$

$$= \int \cos 5u \, du$$

$$= \frac{1}{5}\sin 5u + C$$

$$= \frac{1}{5}\sin 5e^x + C$$

8g Let $u = \cos x, du = -\sin x$

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

$$= \int -\frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + c$$

Uncorrected proofs

8h Let $u = \sin 7x, du = 7\cos x$

$$\int \cot 7x \, dx$$

$$= \int \frac{\cos 7x}{\sin 7x} dx$$

$$= \int \frac{1}{7u} du$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln|\sin 7x| + C$$

9a

$$y = \cos x \sin x = \frac{1}{2}\sin 2x$$

Hence the range is

$$-\frac{1}{2} \leq y \leq \frac{1}{2}$$

9b i

$$\int \cos x \ \sin x \, dx$$

$$= \int \frac{1}{2}\sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + C$$

9b ii

Method 1:

Let $u = \sin x, du = \cos x \, dx$

$$\int \sin x \cos x \, dx$$

$$= \int u \, du$$

Uncorrected proofs

$= \frac{u^2}{2} + C$

$= \frac{\sin^2 x}{2} + C$

Method 2:

Let $u = \cos x, du = -\sin x\, dx$

$\int \sin x \cos x\, dx$

$= \int -u\, du$

$= -\frac{u^2}{2} + C$

$= -\frac{\cos^2 x}{2} + C$

9c Note that $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$. Thus

$-\frac{1}{4}\cos 2x + c$

$= -\frac{1}{4}(2\cos^2 x - 1)$

$= -\frac{\cos^2 x}{2} + c_2$

$= \frac{\sin^2 x}{2} + c_3$

Note at c, c_1 and c_2 are not the same constant.

10a $\sin^4 x = (\sin^2 x)^2$

$= \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2$

$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$

$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \cos 4x\right)$

Uncorrected proofs

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

10b $\cos^4 x = (\cos^2 x)^2$

$$= \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \cos 4x\right)$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

10c i

$$\int_0^{\pi} \sin^4 x \, dx$$

$$= \int_0^{\pi} \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{2}\sin 4x\right]_0^{\pi}$$

$$= \frac{3\pi}{8}$$

10c ii

$$\int_0^{\frac{\pi}{4}} \cos^4 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \left[\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{2}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}(3\pi + 8)$$

Uncorrected proofs

10d

$$\int_0^{\frac{\pi}{3}} \sin^3 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, (\sin^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, (1 - \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} (\sin x - \sin x \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \, dx - \int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, dx$$

Now for $\int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, dx$, let $u = \cos x$, $du = -\sin x$

$$\int_0^{\frac{\pi}{3}} \sin x \cos^2 x \, dx = \int_1^{\frac{1}{2}} u^2 \, du = \left[\frac{u^3}{3}\right]_1^{\frac{1}{2}} = \frac{1}{24}$$

Hence

$$\int_0^{\frac{\pi}{3}} \sin^3 x \, dx = [-\cos x]_0^{\frac{\pi}{3}} - \left[\frac{u^3}{3}\right]_1^{\frac{1}{2}} = \frac{5}{24}$$

11a

$$\int \tan^2 2x \, dx$$

$$= \int (\sec^2 2x - 1) \, dx$$

$$= \frac{1}{2}\tan 2x - x + C$$

11b

$$\int \cot^2 \frac{1}{2}x \, dx = \int (\operatorname{cosec}^2(\frac{1}{2}x) - 1) \, dx$$

Uncorrected proofs

© Cambridge University Press 2019

$$= -2\cot\frac{1}{2}x - x + C$$

11c

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3\tan^2 3x\, dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3(\sec^2 3x + 1)\, dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} (3\sec^2 3x + 3)\, dx$$

$$= [\tan 3x + 3x]_{\frac{\pi}{12}}^{\frac{\pi}{9}}$$

$$= \sqrt{3} - 1 - \frac{\pi}{12}$$

11d

$$\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x\, dx$$

$$= \int_{\frac{\pi}{24}}^{\frac{\pi}{8}} (\operatorname{cosec}^2 4x + 1)\, dx$$

$$= \left[-\frac{1}{4}\cot\frac{1}{2}x + x\right]_{\frac{\pi}{24}}^{\frac{\pi}{8}}$$

$$= \frac{1}{4}\sqrt{3} - \frac{\pi}{12}$$

12a Let $u = \tan x$, $du = \sec^2 x\, dx$

$$\int \tan x \sec^2 x\, dx$$

$$= \int u\, du$$

$$= \frac{u^2}{2} + C$$

Uncorrected proofs

$$= \frac{\tan^2 x}{2} + C$$

12b

$$\int \frac{\sin^2 x}{1 + \cos x} dx$$

$$= \int \frac{1 - \cos^2 x}{1 + \cos x} dx$$

$$= \int \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x} dx$$

$$= \int (1 - \cos x) dx$$

$$= x - \sin x + C$$

12c

$$\int \frac{1 + \cos^3 x}{\cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \cos x\right) dx$$

$$= \int (\sec^2 x + \cos x) dx$$

$$= \tan x + \sin x + C$$

13a

$$F(x) = \int_0^x \sin^2 t \, dt$$

$$= \int_0^x \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^x$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - (0 - 0)$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x$$

13b

$$F'(x) = \frac{1}{2} - \frac{1}{4} \times 2\cos 2x$$
$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$= \sin^2 x$$

13b i $F(x)$ is stationary when $F'(x) = 0$ this is when $\sin^2 x = 0$, $\sin x = 0$ and hence when $x = 0, \pi, 2\pi$.

13b ii $F(x)$ is increasing when $F'(x) > 0$ this is when $\sin^2 x > 0$. This is all x such that $\sin x \neq 0$ and hence when $x \neq 0, \pi, 2\pi$.

13b iii The curve is decreasing when $F'(x) < 0$. However, $\sin^2 x \geq 0$ for all x. Hence there are no values of x for which the curve is decreasing.

13c

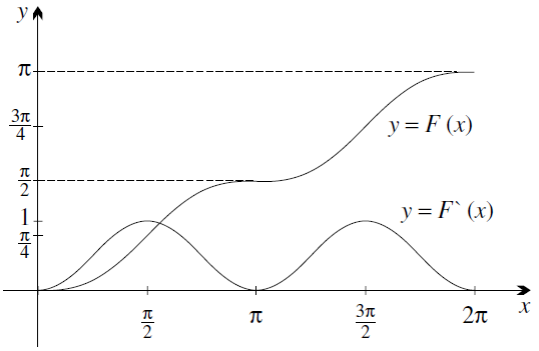
Since $-1 \leq \sin 2x \leq 1$, $-\frac{1}{4} \leq \frac{1}{4}\sin 2x \leq \frac{1}{4}$ and thus $\frac{1}{2}x - \frac{1}{4} \leq \frac{1}{2}x + \frac{1}{4}\sin 2x \leq \frac{1}{2}x + \frac{1}{4}$, thus $\frac{1}{2}x - \frac{1}{4} \leq F(x) \leq \frac{1}{2}x + \frac{1}{4}$. Thus $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.

13d We know that the stationary points are $x = 0, \pi, 2\pi$. As the curve is always increasing, these will all be points of inflection (if there is a maximum or minimum the slope would have to be decreasing on one side).

Uncorrected proofs

© Cambridge University Press 2019

13e



13f i

$$\int_0^k \sin^2 x\, dx = \frac{3\pi}{2}$$

$$\int_0^k \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{3\pi}{2}$$

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^k = \frac{3\pi}{2}$$

$$\frac{k}{2} - \frac{\sin 2k}{4} = \frac{3\pi}{2}$$

$$\frac{k}{2} = \frac{3\pi}{2}$$

$$k = 3\pi$$

13f ii

$$\int_0^k \sin^2 x\, dx = \frac{n\pi}{2}$$

$$\int_0^k \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{n\pi}{2}$$

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^k = \frac{n\pi}{2}$$

$$\frac{k}{2} - \frac{\sin 2k}{4} = \frac{n\pi}{2}$$

$$\frac{k}{2} = \frac{n\pi}{2}$$

Uncorrected proofs

© Cambridge University Press 2019

$k = n\pi$

14

$$\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\, dt\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{R}\left(\frac{R}{2} - \frac{\sin 2R}{4}\right)\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{2} - \frac{1}{4}\left(\frac{\sin 2R}{R}\right)\right)$$

Now as $-1 \leq \sin 2R \leq 0$, then as R gets large $\frac{\sin 2R}{R} \to 0$, hence

$$\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\, dt\right) = \frac{1}{2} - \frac{1}{4}(0) = \frac{1}{2}$$

Uncorrected proofs

© Cambridge University Press 2019

Solutions to Exercise 12D

1a

$$\begin{aligned} du &= \frac{d}{dx}(1+x^2) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x^2) \\ &= 0 + 2x \\ &= 2x \end{aligned}$$

1b

$$\begin{aligned} &\int 2x(1+x^2)^3\,dx \\ &= \int (1+x^2)^3\,(2x\,dx) \\ &= \int u^3\,du \qquad \text{where } u = 1+x^2 \\ &= \frac{u^4}{4} + c \\ &= \frac{(1+x^2)^4}{4} + C \end{aligned}$$

1c

$$\frac{(1+x^2)^4}{4} + C$$

1d

$$\begin{aligned} &\frac{d}{dx}\left(\frac{(1+x^2)^4}{4}\right) \\ &= \frac{4(2x)(1+x^2)^3}{4} \\ &= 2x(1+x^2)^3 \end{aligned}$$

Uncorrected proofs

2a Let $u = 2x + 3$, $du = 2\,dx$

$$\int 2(2x+3)^3\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(2x+3)^4}{4} + C$$

2b Let $u = 1 + x^3$, $du = 3x^2\,dx$

$$\int 3x^2(1+x^3)^4\,dx$$

$$= \int u^4\,du$$

$$= \frac{u^5}{5} + C$$

$$= \frac{(1+x^3)^5}{5} + C$$

2c Let $u = 1 + x^2$, $du = 2x\,dx$

$$\int \frac{2x}{(1+x^2)^2}\,dx$$

$$= \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{1+x^2} + C$$

Uncorrected proofs

© Cambridge University Press 2019

2d Let $u = 3x - 5$, $du = 3\,dx$

$$\int \frac{3}{\sqrt{3x-5}}\,dx$$

$$= \int \frac{1}{\sqrt{u}}\,du$$

$$= \int u^{-\frac{1}{2}}\,du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{3x-5} + C$$

2e Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin^3 x \cos x\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

2f Let $u = 1 + x^4$, $du = 4x^3$

$$\int \frac{4x^3}{1+x^4}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln(1+x^4) + C \qquad \text{(as } 1 + x^4 \text{ is positive)}$$

3a Let $u = 1 - x^2$, $du = -2x\,dx$ hence $x\,dx = -\frac{1}{2}du$

Uncorrected proofs

© Cambridge University Press 2019

3b

$$\int \frac{x}{\sqrt{1-x^2}}dx$$
$$= \int \frac{1}{\sqrt{u}} \times -\frac{1}{2}du$$
$$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$$

3c

$$\int \frac{x}{\sqrt{1-x^2}}dx$$
$$= \int \frac{1}{\sqrt{u}} \times -\frac{1}{2}du$$
$$= -\frac{1}{2}\int u^{-\frac{1}{2}}du$$
$$= -u^{\frac{1}{2}} + C$$
$$= -\sqrt{1-x^2} + C$$

4a Let $u = x^4 + 1, du = 4x^3\,dx$

$$\int x^3(x^4+1)^5\,dx$$
$$= \int \frac{1}{4}u^5\,du$$
$$= \frac{u^6}{24} + C$$
$$= \frac{(x^4+1)^6}{24} + C$$

Uncorrected proofs

© Cambridge University Press 2019

4b Let $u = x^3 - 1, du = 3x^2\ dx$

$$\int x^2\sqrt{x^3 - 1}\ dx$$

$$= \int \frac{1}{3}\sqrt{u}\ du$$

$$= \frac{1}{3} \times \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C$$

4c Let $u = x^3, du = 3x^2\ dx$

$$\int x^2 e^{x^3}\ dx$$

$$= \int \frac{1}{3}e^u\ du$$

$$= \frac{e^u}{3} + C$$

$$= \frac{e^{x^3}}{3} + C$$

4d Let $u = 1 + \sqrt{x}$, hence $du = \frac{1}{2}x^{-\frac{1}{2}}dx$, hence $2\ du = \frac{1}{\sqrt{x}}\ dx$

$$\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^3}dx$$

$$= \int \frac{2}{u^3}\ du$$

$$= -\frac{1}{u^2} + C$$

$$= -\frac{1}{(1 + \sqrt{x})^2} + C$$

Uncorrected proofs

© Cambridge University Press 2019

4e Let $u = \tan 2x$, $du = 2\sec^2 2x\, dx$

$$\int \tan^2 2x \sec^2 2x\, dx$$

$$= \int \frac{u^2}{2}\, du$$

$$= \frac{u^3}{6} + C$$

$$= \frac{\tan^3 2x}{6} + C$$

4f Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\, dx$

$$\int \frac{e^{\frac{1}{x}}}{x^2}\, dx$$

$$= \int -e^u\, du$$

$$= -e^u + C$$

$$= -e^{\frac{1}{x}} + C$$

5a Let $u = 2 + x^3$, $du = 3x^2\, dx$

When $x = 0, u = 2$

When $x = 1, u = 3$

$$\int_0^1 x^2(2 + x^3)^3\, dx$$

$$= \int_2^3 \frac{1}{3}u^3\, du$$

$$= \left[\frac{u^9}{9}\right]_2^3$$

$$= \frac{1}{9}(3^9 - 2^9)$$

Uncorrected proofs

© Cambridge University Press 2019

$= \frac{65}{12}$

5b When $u = 1 + x^4, du = 4x^3$

When $x = 0, u = 1$

When $x = 1, u = 2$

$\int_0^1 \frac{2x^3}{\sqrt{1+x^4}} dx$

$= \frac{1}{2}\int_0^1 \frac{4x^3}{\sqrt{1+x^4}}$

$= \frac{1}{2}\int_1^2 \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^2$

$= \left[\sqrt{u}\right]_1^2$

$= \sqrt{2} - 1$

5c Let $u = \cos x\,, du = -\sin x$

When $x = 0, u = 1$

When $x = \frac{\pi}{2}, u = 0$

$\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$

$= \int_1^0 -u^2\, du$

$= \int_0^1 u^2\, du$

$= \left[\frac{u^3}{3}\right]_0^1$

$= \frac{1}{3}$

Uncorrected proofs

5d Let $u = 1 - x^2, du = 2x\,dx$

When $x = \frac{1}{2}\sqrt{3}, u = \frac{1}{4}$

When $x = 1, u = 0$

$$\int_{\frac{1}{2}\sqrt{3}}^{1} x\sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{1}{4}} \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{3}\left[u^{\frac{3}{2}}\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}}$$

$$= \frac{1}{24}$$

5e Let $u = \ln x, du = \frac{1}{x}dx$

When $x = 1, u = \ln 1 = 0$

When $x = e^2, u = \ln e^2 = 2\ln e = 2$

$$\int_1^{e^2} \frac{\ln x}{x}\,dx$$

$$= \int_0^2 u\,du$$

$$= \left[\frac{u^2}{2}\right]_0^2$$

$$= 2$$

Uncorrected proofs

5f Let $u = \sqrt{x}, du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\ dx$

$$\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}}dx$$

$$= \int_0^2 \frac{e^u}{2}du$$

$$= \frac{1}{2}[e^u]_0^2$$

$$= \frac{1}{2}(e^2 - 1)$$

5g Let $u = \sin 2x, du = 2\cos 2x\ \ dx$

$$\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x\ dx$$

$$= \int_0^1 \frac{u^4}{2}du$$

$$= \frac{1}{2}\left[\frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{10}$$

5h Let $u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}}\ dx$

$$\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\frac{\pi}{2}} u^3\ du$$

$$= \left[\frac{u^4}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\left(\frac{\pi}{2}\right)^4}{4}$$

$$= \frac{\pi^4}{64}$$

Uncorrected proofs

© Cambridge University Press 2019

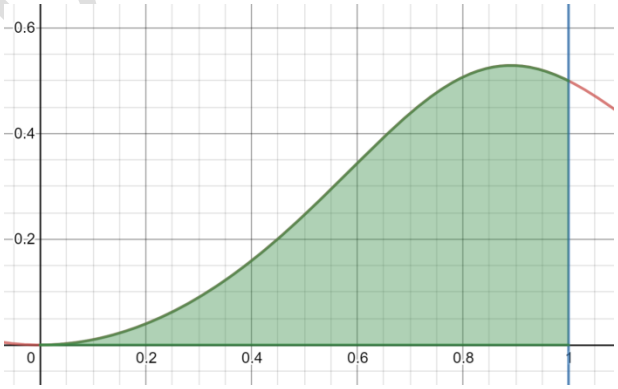
5i Let $u = x^2 + 2x, du = 2x + 2\ dx$

$$\int_0^2 \frac{x+1}{\sqrt[3]{x^2+2x}}\ dx$$
$$= \frac{1}{2}\int_0^2 \frac{2x+2}{\sqrt[3]{x^2+2x}}\ dx$$
$$= \frac{1}{2}\int_0^8 \frac{1}{\sqrt[3]{u}}\ du$$
$$= \frac{1}{2}\int_0^8 u^{-\frac{1}{3}}\ du$$
$$= \frac{1}{2}\left[\frac{3}{2}u^{\frac{2}{3}}\right]_0^8$$
$$= \frac{3}{4}\left(8^{\frac{2}{3}}\right)$$
$$= 3$$

5j Let $u = \tan x, du = \sec^2 x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\ dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u} du$$
$$= [\ln|u|]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \frac{1}{2}\ln 3$$

6 Note that the area to be found is shown in green in the diagram below



Uncorrected proofs

Let $u = x^3$, $du = 3x^2\ dx$. The area bounded will be given by

$$A = \int_0^1 \frac{x^2}{1+x^6}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^6}\,dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1+u^2}\,du$$

$$= \frac{1}{3}[\tan^{-1} u]_0^1$$

$$= \frac{\pi}{12} \text{ square units}$$

7 For all following subsections, let $u = \sin x$, $du = \cos x\ dx$

7a

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1+u}\,du$$

$$= [\ln|1+u|]_0^{\frac{1}{2}}$$

$$= \ln\frac{3}{2}$$

7b

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\,dx$$

$$= \int_0^1 \frac{1}{1+u^2}\,du$$

$$= [\tan|u|]_0^1$$

$$= \frac{\pi}{4}$$

Uncorrected proofs

7c

$$\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \, (1 - \sin^2 x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \; dx - \int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} - \int_0^1 u^2 \, du$$

$$= 1 - \left[\frac{u^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

7d

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x} dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x \, (1 - \sin^2 x)}{\sin^4 x} dx$$

$$= \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^4 x} - \frac{\cos x}{\sin^2 x} \right) dx$$

$$= \int_{\frac{1}{2}}^1 \left(\frac{1}{u^4} - \frac{1}{u^2} \right) du$$

$$= \int_{\frac{1}{2}}^1 (u^{-4} - u^{-2}) \, du$$

Uncorrected proofs

$$= \left[\frac{u^{-3}}{3} - \frac{u^{-1}}{1}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{4}{3}$$

8a Let $u = e^{2x}$, $du = 2e^{2x}\,dx$

$$\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$$

$$= \int \frac{1}{2\sqrt{1+u}}\,du$$

$$= \int \frac{1}{2}(1+u)^{-\frac{1}{2}}\,du$$

$$= (1+u)^{\frac{1}{2}} + C$$

$$= \sqrt{1+u} + C$$

$$= \sqrt{1+e^{2x}} + C$$

8b Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\int \frac{1}{x \ln x}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

8c Let $u = \ln\cos x$, $du = -\frac{\sin x}{\cos x}dx = -\tan x\,dx$

$$\int \frac{\tan x}{\ln\cos x}\,dx$$

$$= \int -\frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\ln\cos x| + C$$

Uncorrected proofs

8d Let $u = \tan x, du = \sec^2 x \ dx$

$$\int \tan^3 x \sec^4 x \, dx$$
$$= \int u^3 \sec^2 x \, du$$
$$= \int u^3(1 + \tan^2 x) \, du$$
$$= \int u^3(1 + u^2) \, du$$
$$= \int (u^3 + u^5) \, du$$
$$= \frac{u^4}{4} + \frac{u^6}{6} + C$$
$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

9a

$$\frac{dy}{dx} = \frac{e^{2x}}{1 + e^{4x}}$$
$$y = \int \frac{e^{2x}}{1 + e^{4x}} \, dx$$

Let $u = e^{2x}, du = 2e^{2x} \, dx$

$$y = \int \frac{e^{2x}}{1 + e^{4x}} \, dx$$
$$= \frac{1}{2} \int \frac{1}{1 + u^2} \, du$$
$$= \frac{1}{2} \tan^{-1} u + C$$
$$= \frac{1}{2} \tan^{-1} e^{2x} + C$$

When $x = 0, y = \frac{\pi}{8}$

$$\frac{\pi}{8} = \frac{1}{2} \tan^{-1} e^0 + C$$
$$\frac{\pi}{8} = \frac{1}{2} \tan^{-1} 1 + C$$
$$\frac{\pi}{8} = \frac{\pi}{8} + C$$

Uncorrected proofs

© Cambridge University Press 2019

Hence $C = 0$ so the equation of the line is

$$y = \frac{1}{2}\tan^{-1} e^{2x}$$

9b

$$y'' = \frac{x}{(4-x^2)^{\frac{3}{2}}}$$

Let $u = 4 - x^2$, $du = -2x\,dx$ hence $-\frac{1}{2}du = dx$

$$y' = \int \frac{x}{(4-x^2)^{\frac{3}{2}}}dx$$

$$= \int \frac{1}{u^{\frac{3}{2}}}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int u^{-\frac{3}{2}}$$

$$= u^{-\frac{1}{2}} + C$$

$$= (4-x^2)^{-\frac{1}{2}} + C$$

When $x = 0, y' = 1$

$$1 = 4^{-\frac{1}{2}} + C$$

$$1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

So

$$y' = (4-x^2)^{-\frac{1}{2}} + \frac{1}{2} = \frac{1}{\sqrt{4-x^2}} + \frac{1}{2}$$

$$y = \int \left(\frac{1}{\sqrt{4-x^2}} + \frac{1}{2}\right)dx$$

$$= \int \left(\frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}} + \frac{1}{2}\right)dx$$

Uncorrected proofs

$= \sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2} + D$

When $x = 0, y = \frac{1}{2}$

$\frac{1}{2} = 0 + 0 + D$

$D = \frac{1}{2}$

$y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2} + \frac{1}{2}$

10a

$\frac{d}{dx}(\sec x)$

$= \frac{d}{dx}(\cos^{-1} x)$

$= -1 \times -\sin x \times (\cos x)^{-2}$

$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$

$= \sec x \tan x$

10b i Let $u = \sec x, du = \sec x \tan x \, dx$

$\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x \, dx$

$= \int_1^2 2^u \, du$

$= \left[\frac{2^u}{\ln 2}\right]_1^2$

$= \frac{4}{\ln 2} - \frac{2}{\ln 2}$

$= \frac{2}{\ln 2}$

Uncorrected proofs

© Cambridge University Press 2019

10b ii

$$\int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$$

$$= \int_1^{\sqrt{2}} u^4 \, du$$

$$= \frac{1}{5}[u^5]_1^{\sqrt{2}}$$

$$= \frac{1}{5}(4\sqrt{2} - 1)$$

11a Let $u = \sin^2 x$, $du = 2\cos x \sin x \, dx = \sin 2x \, dx$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin^2 x} \, dx$$

$$= \int_0^1 \frac{1}{1 + u^2} \, du$$

$$= [\tan^{-1} u]_0^1$$

$$= \frac{\pi}{4}$$

11b Let $u = x \ln x$, $du = 1 \times \ln x + x \times \frac{1}{x} \, dx = \ln x + 1 \, dx$

$$\int_1^e \frac{\ln x + 1}{(x \ln x + 1)^2} \, dx$$

$$= \int_0^e \frac{1}{(u + 1)^2} \, du$$

$$= \left[-\frac{1}{u + 1}\right]_0^e$$

$$= -\frac{1}{e + 1} - (-1)$$

$$= 1 - \frac{1}{e + 1}$$

$$= \frac{e + 1}{e + 1} - \frac{1}{e + 1}$$

$$= \frac{e}{e + 1}$$

Uncorrected proofs

© Cambridge University Press 2019

12 Let $u = \sqrt{x-1}$, $du = \frac{1}{2\sqrt{x-1}}\,dx$

If $u = \sqrt{x-1}$, then $u^2 = x - 1$ and $x = 1 + u^2$

$$\int \frac{1}{2x\sqrt{x-1}}\,dx$$

$$= \int \frac{1}{x}\,du$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \tan^{-1} u + C$$

$$= \tan^{-1}\sqrt{x-1} + C$$

13a Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$ and $u^2 = x$

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= \int \frac{2}{\sqrt{(1-x)}}\frac{dx}{2\sqrt{x}}$$

$$= \int \frac{2}{\sqrt{1-u^2}}\,du$$

$$= 2\sin^{-1} u + C_1$$

$$= 2\sin^{-1}\sqrt{x} + C_1$$

13b Let $u = x - \frac{1}{2}$, $du = dx$, $x = \frac{1}{2} + u$

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}+u\right)\left(\frac{1}{2}-u\right)}}\,du$$

$$= \int \frac{1}{\sqrt{\frac{1}{4}-u^2}}\,du$$

$$= \int \frac{2}{\sqrt{1-(2u)^2}}\,du$$

$$= \sin^{-1}(2u) + C_2$$

$$= \sin^{-1}(2x-1) + C_2$$

Uncorrected proofs

© Cambridge University Press 2019

13c Since

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= 2\sin^{-1}\sqrt{x} + C_1$$

$$= \sin^{-1}(2x-1) + C_2$$

It follows that

$$\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} + C_3$$

Substituting $x = 0$

$$\sin^{-1}(-1) = 2\sin^{-1}\sqrt{0} + C_3$$

$$C_3 = -\frac{\pi}{2}$$

Hence we can conclude that

$$\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$$

14 Let $u = x - \frac{1}{x}$, $du = 1 + \frac{1}{x^2}\,dx$,

Hence $x^2\,du = x^2 + 1\,dx$

When $x = 1, u = 0$

When $x = \frac{1}{2}(\sqrt{6}+\sqrt{2})$,

$$u = \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{1}{\frac{1}{2}(\sqrt{6}+\sqrt{2})}$$

$$= \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{\sqrt{6}-\sqrt{2}}{\frac{1}{2}(6-2)}$$

$$= \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{1}{2}(\sqrt{6}-\sqrt{2})$$

$$= \sqrt{2}$$

$$u^2 = x^2 - 2 + \frac{1}{x^2}$$

$$u^2 = \frac{x^4+1}{x^2} - 2$$

$$u^2 + 2 = \frac{x^4+1}{x^2}$$

Uncorrected proofs

© Cambridge University Press 2019

$$\frac{1}{u^2+2} = \frac{x^2}{1+x^4}$$

$$\int_1^{\frac{1}{2}(\sqrt{6}+\sqrt{2})} \frac{1+x^2}{1+x^4}\,dx$$

$$= \int_0^{\sqrt{2}} \frac{x^2}{1+x^4}\,du$$

$$= \int_0^{\sqrt{2}} \frac{1}{u^2+2}\,du$$

$$= \frac{1}{2}\int_0^{\sqrt{2}} \frac{1}{1+\left(\frac{u}{\sqrt{2}}\right)^2}\,du$$

$$= \frac{\sqrt{2}}{2}\left[\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}(\tan^{-1}1 - 0)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{4\sqrt{2}}$$

Uncorrected proofs

© Cambridge University Press 2019

Solutions to Exercise 12E

1a $dx = \frac{d}{du}(u) + \frac{d}{du}(1)du = 1du = du$

1b $I = \int x(x-1)^5\,dx$

$= \int (u+1)(u)^5\,du$

1c $I = \int (u+1)(u)^5\,du$

$= \int (u^6 + u^5)\,du$

$= \frac{u^7}{7} + \frac{u^6}{6} + C$

$= \frac{(x-1)^7}{7} + \frac{(x-1)^6}{6} + C$

1d

$\frac{d}{dx}\left(\frac{(x-1)^7}{7} + \frac{(x-1)^6}{6}\right)$

$= (x-1)^6 + (x-1)^5$

$= (x-1+1)(x-1)^5$

$= x(x-1)^5 + C$

2a

$\int \frac{x}{\sqrt{x-1}}\,dx$

$= \int \frac{u+1}{\sqrt{u}}\,du$

$= \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$

$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2\sqrt{x-1} + C$

Uncorrected proofs

2b

$$\int \frac{x}{(x-1)^2}\,dx$$
$$= \int \frac{u+1}{u^2}\,du$$
$$= \int \left(\frac{1}{u}+\frac{1}{u^2}\right)du$$
$$= \ln|u| - \frac{1}{u} + C$$
$$= \ln|x-1| - \frac{1}{x-1} + C$$

3a $dx = 2u\,du$

3b

$$J = \int x\sqrt{x+1}\,dx$$
$$= \int (u^2-1)\sqrt{u^2-1+1}\,2u\,du$$
$$= 2\int (u^2-1)\sqrt{u^2}\,u\,du$$
$$= 2\int (u^2-1)u^2\,du$$
$$= 2\int (u^4-u^2)\,du$$

3c

$$J = 2\int (u^4-u^2)\,du$$
$$= 2\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+C$$
$$= 2\left(\frac{\left(\sqrt{x+1}\right)^5}{5}-\frac{\left(\sqrt{x+1}\right)^3}{3}\right)+C$$

Uncorrected proofs

© Cambridge University Press 2019

3d

$$\frac{d}{dx}\left(2\left(\frac{\left(\sqrt{x+1}\right)^5}{5}-\frac{\left(\sqrt{x+1}\right)^3}{3}\right)\right)$$
$$=2\left(\frac{\frac{5}{2}\left(\sqrt{x+1}\right)^3}{5}-\frac{\frac{3}{2}\left(\sqrt{x+1}\right)}{3}\right)$$
$$=\left(\sqrt{x+1}\right)^3-\left(\sqrt{x+1}\right)$$
$$=(x+1)\sqrt{x+1}-\sqrt{x+1}$$
$$=(x+1-1)\sqrt{x+1}$$
$$=x\sqrt{x+1}$$

4a

$$\int x^2\sqrt{x+1}\,dx$$
$$=\int (u^2-1)^2\sqrt{u^2}\,2u\,du$$
$$=2\int (u^2-1)^2 u\,u\,du$$
$$=2\int (u^2-1)^2u^2\,du$$
$$=2\int (u^4-2u^2+1)u^2\,du$$
$$=2\int (u^6-2u^4+u^2)\,du$$
$$=2(\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3})+C$$
$$=\frac{2(x+1)^{\frac{7}{2}}}{7}-\frac{4(x+1)^{\frac{5}{2}}}{5}+\frac{2(x+1)^{\frac{3}{2}}}{3}+C$$

4b

$$\int \frac{2x+3}{\sqrt{x+1}}dx$$
$$=\int (\frac{2x+2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+1}})\,dx$$
$$=\int (\frac{2(x+1)}{\sqrt{x+1}}+\frac{1}{\sqrt{x+1}})\,dx$$

Uncorrected proofs

$$= \int \left(\frac{2(u^2)}{\sqrt{u^2}} + \frac{1}{\sqrt{u^2}} 2u\right) du$$
$$= \int (4u^2 + 2u)\, du$$
$$= \frac{4}{3}u^3 + u^2 + C$$
$$= \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + C$$

5a Let $x = u - 2, dx = du$

$$\int \frac{x-2}{x+2} dx$$
$$= \int \frac{u-2-2}{u-2+2} du$$
$$= \int \frac{u-4}{u} du$$
$$= \int \left(1 - \frac{4}{u}\right) du$$
$$= u - 4\ln|u| + C$$
$$= x + 2 - 4\ln|x+2| + C$$

5b Let $x = \frac{1}{2}(u+1), dx = \frac{1}{2}du, u = 2x - 1$

$$\int \frac{2x+1}{\sqrt{2x-1}} dx$$
$$= \int \frac{2\left(\frac{1}{2}(u+1)\right) + 1}{\sqrt{2\left(\frac{1}{2}(u+1)\right) - 1}} \frac{1}{2}\, du$$
$$= \int \frac{u+1+1}{\sqrt{u+1-1}} \frac{1}{2}\, du$$
$$= \frac{1}{2}\int \frac{u+2}{\sqrt{u}}\, du$$
$$= \frac{1}{2}\int \left(\sqrt{u} + \frac{2}{\sqrt{u}}\right) du$$
$$= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} + 4\sqrt{u}\right) + C$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{3}u^{\frac{3}{2}} + 2\sqrt{u} + C$$
$$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + 2\sqrt{2x-1} + C$$

5c $x = \frac{1}{4}(u^2+5), dx = \frac{1}{2}u\,du, u^2 = 4x-5, u = \sqrt{4x-5}$

$$\int 3x\sqrt{4x-5}\,dx$$

$$= 3\int \frac{1}{4}(u^2+5)\sqrt{4\left(\frac{1}{4}(u^2+5)\right)-5} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{4}(u^2+5)\sqrt{u^2+5-5} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{4}(u^2+5)\sqrt{u^2} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{8}(u^2+5)u^2\,du$$
$$= \frac{3}{8}\int (u^4+5u^2)\,du$$
$$= \frac{3}{8}\left(\frac{u^5}{5} + \frac{5u^3}{3}\right) + C$$
$$= \frac{3}{8}\left(\frac{u^5}{5} + \frac{5u^3}{3}\right) + C$$
$$= \frac{3}{8}\left(\frac{(\sqrt{4x-5})^5}{5} + \frac{5(\sqrt{4x-5})^3}{3}\right) + C$$

5d $x = (u-1)^2, dx = 2(u-1)\,du$

Hence $\sqrt{x} = u-1, u = 1+\sqrt{x}$

$$\int \frac{1}{1+\sqrt{x}}dx$$
$$= \int \frac{1}{u} \times 2(u-1)\,du$$
$$= 2\int \left(1 - \frac{1}{u}\right)du$$
$$= 2(u - \ln|u|)$$
$$= 2\left((1+\sqrt{x}) - \ln\left(1+\sqrt{x}\right)\right) + C \qquad \text{(as } \sqrt{x} \text{ positive)}$$
$$= 2(1+\sqrt{x}) - 2\ln\left(1+\sqrt{x}\right) + C$$

Unconnected proofs

© Cambridge University Press 2019

6a $x = u - 1, dx = du$

$$\int_0^1 x(x+1)^3\,dx$$

$$= \int_1^2 (u-1)u^3\,du$$

$$= \int_1^2 (u^4 - u^3)\,du$$

$$= \left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^2$$

$$= \frac{49}{20}$$

6b $x = 1 - u, dx = -du$

$$\int_0^{\frac{1}{2}} \frac{1+x}{1-x}\,dx$$

$$= \int_1^{\frac{1}{2}} \frac{2-u}{u}\,(-du)$$

$$= \int_{\frac{1}{2}}^1 \left(\frac{2}{u} - 1\right)du$$

$$= [2\ln|u| - u]_{\frac{1}{2}}^1$$

$$= 2\ln 2 - \frac{1}{2}$$

6c Let $x = \frac{1}{3}(u-1), dx = \frac{1}{3}du, u = 3x + 1$

$$\int_0^1 \frac{3x}{\sqrt{3x+1}}\,dx$$

$$= \int_1^4 \frac{u-1}{\sqrt{u}}\,du$$

$$= \int_1^4 \left(\sqrt{u} - u^{-\frac{1}{2}}\right)du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$

Uncorrected proofs

$= \frac{8}{9}$

6d Let $x = u - 2, dx = du, u = x + 2$

$$\int_0^1 \frac{2-x}{(2+x)^3}\, dx$$

$$= \int_2^3 \frac{4-u}{(u)^3}\, dx$$

$$= \int_2^3 \left(\frac{4}{u^3} - \frac{1}{u^2}\right) dx$$

$$= \left[-\frac{2}{u^2} + \frac{1}{u}\right]_2^3$$

$$= \frac{1}{9}$$

6e Let $x = 4 - u^2, dx = -2u\, du, u = \sqrt{4-x}$

$$\int_0^4 x\sqrt{4-x}\, dx$$

$$= \int_2^0 (4-u^2)u\ (-2u\, du)$$

$$= \int_0^2 (4-u^2)u\,(2u\, du)$$

$$= \int_0^2 (8u^2 - 2u^4)\, du$$

$$= \left[\frac{8u^3}{3} - \frac{2u^5}{5}\right]_0^2$$

$$= \frac{128}{15}$$

6f $x = \frac{1}{2}(u^2+1), dx = u\, du, u^2 = 2x - 1, u = \sqrt{2x-1}$

$$\int_1^5 \frac{x}{(2x-1)^{\frac{3}{2}}}\, dx$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \int_1^5 \frac{x}{(2x-1)^{\frac{3}{2}}}\,dx$$
$$= \int_1^3 \frac{\frac{1}{2}(u^2+1)}{(u^2)^{\frac{3}{2}}}\,u\,du$$
$$= \frac{1}{2}\int_1^3 \frac{(u^2+1)}{u^3}\,u\,du$$
$$= \frac{1}{2}\int_1^3 \frac{(u^2+1)}{u^2}\,du$$
$$= \frac{1}{2}\int_1^3 \left(1+\frac{1}{u^2}\right)\,du$$
$$= \frac{1}{2}\left[u-\frac{1}{u}\right]_1^3$$
$$= \frac{4}{3}$$

6g $x = (u-3)^2, dx = 2(u-3)\,du, u = \sqrt{x}+3$

$$\int_0^4 \frac{1}{3+\sqrt{x}}\,dx$$
$$= \int_3^5 \frac{1}{u}\,2(u-3)\,du$$
$$= \int_3^5 2-\frac{6}{u}\,du$$
$$= [2u-6\ln|u|]_3^5$$
$$= 4-6\ln\frac{5}{3}$$

6h $x = u^3-1, dx = 3u^2\,du, u = \sqrt[3]{x+1}$

$$\int_0^7 \frac{x^2}{\sqrt[3]{x+1}}\,dx$$
$$= \int_1^2 \frac{(u^3-1)^2}{u}\,3u^2\,du$$
$$= \int_1^2 3u(u^3-1)^2\,du$$

Uncorrected proofs

$$= \int_1^2 3u(u^6 - 2u^3 + 1)du$$
$$= \int_1^2 (3u^7 - 6u^4 + 3u)\, du$$
$$= \left[\frac{3u^8}{8} - \frac{6u^5}{5} + \frac{3u^2}{2}\right]_1^2$$
$$= \frac{2517}{40}$$

7a Let $x = u - 2, dx = du$

$$I = \int \frac{1}{\sqrt{5 - 4x - x^2}} dx$$
$$= \int \frac{1}{\sqrt{5 - 4(u-2) - (u-2)^2}} du$$
$$= \int \frac{1}{\sqrt{5 - 4u + 8 - (u^2 - 4u + 4)}}$$
$$= \int \frac{1}{\sqrt{9 - u^2}}$$
$$= \int \frac{1}{3\sqrt{1 - \left(\frac{u}{3}\right)^2}} du$$
$$= \sin^{-1}\left(\frac{u}{3}\right) + C$$
$$= \sin^{-1}(\frac{x+2}{3}) + C$$

7b i Let $x = u - 1, dx = du$

$$\int \frac{1}{x^2 + 2x + 4} dx$$
$$= \int \frac{1}{(u-1)^2 + 2(u-1) + 4} du$$
$$= \int \frac{1}{u^2 - 2u + 1 + 2u - 2 + 4} du$$
$$= \int \frac{1}{u^2 + 3} du$$
$$= \frac{1}{3}\int \frac{1}{1 + \left(\frac{u}{\sqrt{3}}\right)^2} du$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + C$$

7b ii Let $x = u - 1, dx = du$

$$\int \frac{1}{\sqrt{4 - 2x - x^2}}dx$$
$$= \int \frac{1}{\sqrt{4 - 2(u-1) - (u-1)^2}}du$$
$$= \int \frac{1}{\sqrt{4 - 2u + 2 - (u^2 - 2u + 1)}}du$$
$$= \int \frac{1}{\sqrt{5 - u^2}}du$$
$$= \int \frac{1}{\sqrt{5}\sqrt{1 - \left(\frac{u}{\sqrt{5}}\right)^2}}du$$
$$= \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$
$$= \sin^{-1}\frac{x+1}{\sqrt{5}} + C$$

7b iii Let $x = u + 1, dx = du$

$$\int_1^2 \frac{1}{\sqrt{3 + 2x - x^2}}dx$$
$$= \int_0^1 \frac{1}{\sqrt{3 + 2(u+1) - (u+1)^2}}du$$
$$= \int_0^1 \frac{1}{\sqrt{3 + 2u + 2 - (u^2 + 2u + 1)}}du$$
$$= \int_0^1 \frac{1}{\sqrt{4 - u^2}}du$$
$$= \left[\sin^{-1}\frac{u}{2}\right]_0^1$$
$$= \frac{\pi}{6}$$

Uncorrected proofs

7b iv Let $x = u + 3, dx = du$

$$\int_3^7 \frac{1}{x^2 - 6x + 25}\,dx$$
$$= \int_0^4 \frac{1}{(u+3)^2 - 6(u+3) + 25}\,du$$
$$= \int_0^4 \frac{1}{(u+3)^2 - 6(u+3) + 25}\,du$$
$$= \int_0^4 \frac{1}{u^2 + 6u + 9 - 6u - 18 + 25}\,du$$
$$= \int_0^4 \frac{1}{u^2 + 16}\,du$$
$$= \frac{1}{4}\left[\tan^{-1}\frac{u}{4}\right]_0^4$$
$$= \frac{\pi}{16}$$

8a Let $x = 2\sin\theta, dx = 2\cos\theta\,d\theta$

Hence

$$J = \int \frac{1}{\sqrt{4 - x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{4 - (2\sin\theta)^2}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{1}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{1}{2\sqrt{1 - \sin^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{1}{2\sqrt{\cos^2\theta}} \times 2\cos\theta\,d\theta$$
$$= \int \frac{1}{2\cos\theta} \times 2\cos\theta\,d\theta$$
$$= \int 1\,d\theta$$
$$= \theta + C$$

Since

$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$

Hence

Uncorrected proofs

$\theta = \sin^{-1}\frac{x}{2}$

Thus

$J = \sin^{-1}\frac{x}{2} + C$

8b i Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{x}{3}$

$$\int \frac{1}{9+x^2}dx$$
$$= \int \frac{1}{9+(3\tan\theta)^2}3\sec^2\theta\, d\theta$$
$$= \int \frac{1}{9+9\tan^2\theta}3\sec^2\theta\, d\theta$$
$$= \frac{1}{3}\int \frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$
$$= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta$$
$$= \frac{1}{3}\int 1\, d\theta$$
$$= \frac{1}{3}\theta + C$$
$$= \tfrac{1}{3}\tan^{-1}\tfrac{x}{3} + C$$

8b ii Let $x = \sqrt{3}\cos\theta$, $dx = -\sqrt{3}\sin\theta\ d\theta$, $\theta = \cos^{-1}\frac{x}{\sqrt{3}}$

$$\int -\frac{1}{\sqrt{3-x^2}}dx$$
$$= \int -\frac{1}{\sqrt{3-\left(\sqrt{3}\cos\theta\right)^2}} \times -\sqrt{3}\sin\theta\, d\theta$$
$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sqrt{1-\cos^2\theta}}d\theta$$
$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sqrt{\sin^2\theta}}d\theta$$
$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sin\theta}d\theta$$
$$= \int 1\, d\theta$$

Uncorrected proofs

$= \theta + C$

$= \cos^{-1}\dfrac{x}{\sqrt{3}} + C$

8b iii Let $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta \ d\theta$ and $\theta = \sin^{-1} 2x$

$$\int \frac{1}{\sqrt{1-4x^2}}dx$$

$$= \int \frac{1}{\sqrt{1-4\left(\frac{1}{2}\sin\theta\right)^2}}\frac{1}{2}\cos\theta \, d\theta$$

$$= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$

$$= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}d\theta$$

$$= \frac{1}{2}\int \frac{\cos\theta}{\cos\theta}d\theta$$

$$= \frac{1}{2}\int 1 \, d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{2}\sin^{-1} 2x + C$$

8b iv Let $x = \frac{1}{4}\tan\theta$, $dx = \frac{1}{4}\sec^2\theta \, d\theta$ and $\theta = \tan^{-1} 4x$

$$\int \frac{1}{1+16x^2}dx$$

$$= \int \frac{1}{1+16\left(\frac{1}{4}\tan\theta\right)^2}\frac{1}{4}\sec^2\theta \, d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta$$

$$= \frac{1}{4}\int 1 \, d\theta$$

$$= \frac{1}{4}\theta + C$$

$$= \frac{1}{4}\tan^{-1} 4x + C$$

Uncorrected proofs

8bv Let $x = 6\sin\theta$, $dx = 6\cos\theta\, d\theta$ and $\theta = \sin^{-1}\frac{x}{6}$.

When $x = 0, \theta = 0$

When $x = 3, \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$$\int_0^3 \frac{1}{\sqrt{36 - x^2}}\, dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{36 - (6\sin\theta)^2}} 6\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{6\sqrt{1 - \sin^2\theta}} 6\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{6\cos\theta}{6\sqrt{\cos^2\theta}}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{6\cos\theta}{6\cos\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} 1\, d\theta$$

$$= [\theta]_0^{\frac{\pi}{6}}$$

$$= \frac{\pi}{6}$$

8b vi

Let $x = \frac{2}{3}\tan\theta$, $dx = \frac{2}{3}\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{3}{2}x$

When $x = 0, \theta = 0$

When $x = \frac{2}{3}, \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$\int_0^{\frac{2}{3}} \frac{1}{4 + 9x^2}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4 + 9\left(\frac{2}{3}\tan\theta\right)^2} \frac{2}{3}\sec^2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{4 + 4\tan^2\theta} \frac{2}{3}\sec^2\theta\, d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

Uncorrected proofs

$$= \frac{1}{6}\int_0^{\frac{\pi}{4}} 1\, d\theta$$
$$= \frac{1}{6}[\theta]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{24}$$

9a Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$ and $\theta = \sin^{-1} x$

$$I = \int \frac{1}{(1-x^2)^{\frac{3}{2}}}\, dx$$
$$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}} \cos\theta\, d\theta$$
$$= \int \frac{1}{(\cos^2\theta)^{\frac{3}{2}}} \cos\theta\, d\theta$$
$$= \int \frac{1}{\cos^3\theta} \cos\theta\, d\theta$$
$$= \int \frac{1}{\cos^2\theta}\, d\theta$$
$$= \int \sec^2\theta\, d\theta$$
$$= \tan\theta + C$$

Now since $\sin\theta = x = \frac{x}{1}$, it follows, from the Pythagorean theorem that

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

Hence

$$I = \frac{x}{\sqrt{1-x^2}} + C$$

9b i Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\, dx$$
$$= \int \frac{1}{(4+4\tan^2\theta)^{\frac{3}{2}}} 2\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(4\sec^2\theta)^{\frac{3}{2}}} 2\sec^2\theta\, d\theta$$

Uncorrected proofs

$$= \int \frac{1}{8\sec^3\theta} 2\sec^2\theta \, d\theta$$
$$= \frac{1}{4}\int \frac{1}{\sec\theta} d\theta$$
$$= \frac{1}{4}\int \cos\theta \, d\theta$$
$$= \frac{1}{4}\sin\theta + C$$
$$= \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + C$$

9b ii Let $x = \sin\theta, dx = \cos\theta \, d\theta$

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta) \, d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

9b iii Let $x = 2\sin\theta, dx = 2\cos\theta \, d\theta$

$$\int_0^2 \sqrt{4-x^2} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2} \, 2\cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\sqrt{1-\sin^2\theta} \, 2\cos\theta \, d\theta$$

Uncorrected proofs

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$
$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \pi$$

9b iv Let $x = 5\cos\theta$, $dx = -5\sin\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{25-x^2}}\, dx$$
$$= \int \frac{1}{(5\cos\theta)^2\sqrt{25-(5\cos\theta)^2}} \times -5\sin\theta\, d\theta$$
$$= \int \frac{1}{25\cos^2\theta\, 5\sqrt{1-\cos^2\theta}} \times -5\sin\theta\, d\theta$$
$$= \int \frac{1}{25\cos^2\theta\, 5\sin\theta} \times -5\sin\theta\, d\theta$$
$$= -\frac{1}{25}\int \frac{1}{\cos^2\theta}\, d\theta$$
$$= -\frac{1}{25}\int \sec^2\theta\, d\theta$$
$$= -\frac{1}{25}\tan\theta + C$$
$$= -\frac{1}{25} \times \frac{\sqrt{25-x^2}}{x} + C$$
$$= -\frac{\sqrt{25-x^2}}{25x} + C$$

9b v Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{9+x^2}}\, dx$$
$$= \int \frac{1}{(3\tan\theta)^2\sqrt{9+(3\tan\theta)^2}} 3\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(3\tan\theta)^2 3\sqrt{1+\tan^2\theta}} 3\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(3\tan\theta)^2 3\sec\theta} 3\sec^2\theta\, d\theta$$

Uncorrected proofs

$$= \int \frac{\sec\theta}{(3\tan\theta)^2}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos\theta}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta}\,d\theta$$
$$= \frac{1}{9}\int \cot\theta \operatorname{cosec}\theta\,d\theta$$
$$= -\frac{1}{9}\operatorname{cosec}\theta + C$$
$$= -\frac{1}{9} \times \frac{\sqrt{9+x^2}}{x} + C$$
$$= -\frac{\sqrt{9+x^2}}{9x} + C$$

9b vi Let $x = 2\sec\theta, dx = 2\sec\theta\tan\theta\,d\theta$

$$\int_2^4 \frac{1}{x^2\sqrt{x^2-4}}\,dx$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2\sqrt{(2\sec\theta)^2-4}}\,2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\sqrt{\sec^2\theta-1}}\,2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\sqrt{\tan^2\theta}}\,2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\tan\theta}\,2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{4\sec\theta}\,d\theta$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$$
$$= \frac{1}{4}[\sin\theta]_0^{\frac{\pi}{3}}$$
$$= \frac{\sqrt{3}}{8}$$

10 Let $x = 3\sec\theta, dx = 3\sec\theta\tan\theta\,d\theta$ and $\theta = \sec^{-1}\frac{x}{3}$

$f(x)$

Uncorrected proofs

$$= \int f'(x)\,dx$$
$$= \int \frac{\sqrt{x^2-9}}{x}\,dx$$
$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta$$
$$= \int \frac{3\tan\theta}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta$$
$$= 3\int (\sec^2\theta - 1)\,d\theta$$
$$= 3(\tan\theta - \theta) + C$$
$$= \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$$

Now note that

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$$

$$\therefore\ \theta = \tan^{-1}\frac{\sqrt{x^2-9}}{3}$$

$$f(3) = 0$$

$$\sqrt{3^2-9} - 3\sec^{-1}\frac{3}{3} + C = 0$$
$$C = 3$$

Hence,

$$f(x) = 3[\tan\theta - \theta] + C$$
$$= \sqrt{x^2-9} - 3\tan^{-1}\frac{\sqrt{x^2-9}}{3}$$

11 Let $x = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta\,d\theta$

$$A = \int_0^1 \frac{x^3}{\sqrt{3-x^2}}\,dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{3\sqrt{3}\sin^3\theta}{\sqrt{3-3\sin^2\theta}}\sqrt{3}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{3\sqrt{3}\sin^3\theta}{\sqrt{3}\cos\theta}\sqrt{3}\cos\theta\,d\theta$$

Uncorrected proofs

$$= \int_0^{\frac{\pi}{6}} 3\sqrt{3}\sin^3\theta\, d\theta$$

Let $u = \cos\theta, du = -\sin\theta\, d\theta$

$$A = \int_1^{\frac{\sqrt{3}}{2}} -3\sqrt{3}\sin^2\theta\, du$$
$$= 3\sqrt{3}\int_{\frac{\sqrt{3}}{2}}^1 1-\cos^2\theta\, du$$
$$= 3\sqrt{3}\int_{\frac{\sqrt{3}}{2}}^1 1-u^2\, du$$
$$= 3\sqrt{3}\left[u-\frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^1$$
$$= \frac{1}{3}\left(6\sqrt{3}-7\sqrt{2}\right) \text{ square units}$$

12a Let $x = r\sin\theta, dx = r\cos\theta\, d\theta$

$$A = 2\int_{-r}^{r}\sqrt{r^2-x^2}\, dx$$
$$= 2\int_{-r}^{r}\sqrt{r^2-x^2}\, dx$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{r^2-(r\sin\theta)^2}\, r\cos\theta\, d\theta$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\sqrt{1-\sin^2\theta}\, r\cos\theta\, d\theta$$
$$= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2\theta\, d\theta$$
$$= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\cos 2\theta)\, d\theta$$
$$= r^2\left[\theta-\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \pi r^2$$

12b i Let P be the x-position of chord AB

Uncorrected proofs

$$\sin\frac{\alpha}{2} = \frac{P}{OA} = \frac{P}{r}$$

Hence

$$P = r\sin\frac{\alpha}{2}$$

The area is twice the area from P to r under the semicircle

$$A = 2\int_P^r \sqrt{r^2 - x^2}\,dx = 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\,dx$$

12b ii Let $x = r\cos\theta$, $dx = -r\sin\theta\,d\theta$

$$\begin{aligned}
A &= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\,dx \\
&= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\,dx \\
&= 2\int_{r\sin\frac{\alpha}{2}}^{\frac{\pi}{2}} -\sqrt{r^2 - (r\cos\theta)^2}\,r\sin\theta\,d\theta \\
&= 2\int_{\frac{1}{2}\alpha}^0 -r\sqrt{1 - \cos^2\theta}\,r\sin\theta\,d\theta \\
&= 2\int_{\frac{1}{2}\alpha}^0 -r\sin\theta\,r\sin\theta\,d\theta \\
&= -2r^2\int_{\frac{1}{2}\alpha}^0 \sin^2\theta\,d\theta
\end{aligned}$$

12b iii

$$\begin{aligned}
A &= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\,dx \\
&= -2r^2\int_{\frac{1}{2}\alpha}^0 \sin^2\theta\,d\theta \\
&= 2r^2\int_0^{\frac{1}{2}\alpha} \sin^2\theta\,d\theta \\
&= 2r^2\int_0^{\frac{1}{2}\alpha} (1 - \cos 2\theta)\,d\theta
\end{aligned}$$

Uncorrected proofs

$$= 2r^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{1}{2}\alpha}$$
$$= \frac{1}{2}r^2(\alpha - \sin\alpha)$$

12c For an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$
$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

The area will be given by

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\, dx$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b\sqrt{1 - \frac{a^2\sin^2\theta}{a^2}}\, a\cos\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\cos\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$$
$$= ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \pi ab$$

The unit circle has area π, dilating it a units horizontally causes it to become $a\pi$ and then dilating it a further b units vertically causes the area to become πab.

Uncorrected proofs

13a Let $x = -u, dx = -du$

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx$$

$$= \int_{2}^{-2} \frac{(-u)^2}{e^{-u} + 1} (-du) \qquad \text{(using when } x = -2, u = 2 \text{ and when } x = 2, u = -2)$$

$$= \int_{-2}^{2} \frac{u^2}{e^{-u} + 1} du$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{1 + e^u} du$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{e^u + 1} du$$

Now let $x = u$

$$\int_{-2}^{2} \frac{u^2 e^u}{e^u + 1} du = \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx$$

Hence

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx = \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx$$

13b

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2}{e^x + 1} dx + \int_{-2}^{2} \frac{x^2}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2}{e^x + 1} dx + \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2 + x^2 e^x}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2(1 + e^x)}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} x^2 dx\right)$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2}$$

Uncorrected proofs

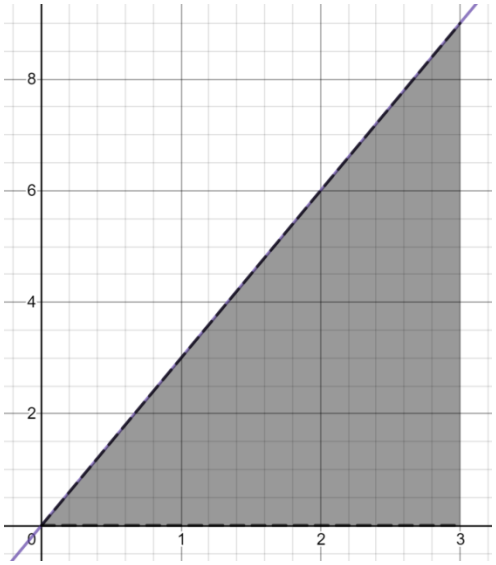
$$= \frac{1}{2}\left(\frac{8}{3} - \left(-\frac{8}{3}\right)\right)$$

$$= \frac{8}{3}$$

Uncorrected proofs

Solutions to Exercise 12F

1a



1b The radius will be $r = 3(3) = 9$ units and the height will be 3 units.

Hence

$$V = \frac{1}{3}\pi r^2$$

$$= \frac{1}{3}\pi(9)^2(3)$$

$$= 81\pi \text{ cubic units}$$

1c

$$\pi \int_0^3 9x^2\, dx$$

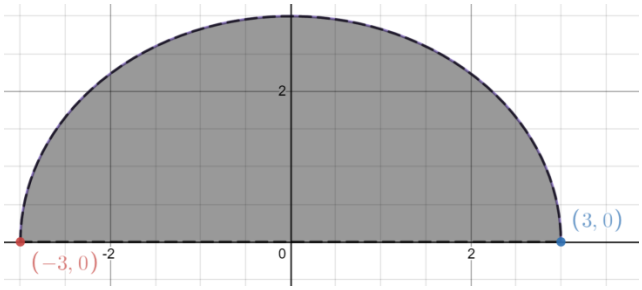
$$= 3\pi[x^3]_0^3$$

$$= 81\pi \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

2a



2b This is a sphere of radius 3, hence the volume is

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(3)^3$$
$$= 36\pi \text{ cubic units}$$

2c

$$\pi \int_{-3}^{3} (9 - x^2)\, dx$$
$$= \pi \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$
$$= 36\pi \text{ cubic units}$$

3a This is a cylinder of radius 2, height 4. Hence

$$V = \pi r^2 h$$
$$= \pi(2)^2 \times 4$$
$$= 16\pi \text{ cubic units}$$

3b This is a cone of radius 3 and height 3. Hence

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(3)^2 \times 3$$

Uncorrected proofs

$= 9\pi$ cubic units

3c

$$V = \pi \int_0^2 y^2 \, dx$$

$$= \pi \int_0^2 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{32\pi}{5} \text{ cubic units}$$

3d

$$V = \pi \int_2^4 y^2 \, dx$$

$$= \pi \int_2^4 x \, dx$$

$$= \frac{\pi}{2}[x^2]_2^4$$

$$= 6\pi \text{ cubic units}$$

3e

$$V = \pi \int_0^2 y^2 \, dx$$

$$= \int_0^2 (4 - x^2) \, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{16\pi}{3} \text{ cubic units}$$

Uncorrected proofs

3f

$$V = \pi \int_{-1}^{0} y^2 \, dx$$
$$= \pi \int_{-1}^{0} x^6 \, dx$$
$$= \pi \left[\frac{x^7}{7}\right]_{-1}^{0}$$
$$= \frac{\pi}{7} \text{ cubic units}$$

3g

$$V = \pi \int_{-5}^{-2} y^2 \, dx$$
$$= \pi \int_{-5}^{-2} (x+2)^2 \, dx$$
$$= \pi \int_{-5}^{-2} (x^2 + 4x + 4) \, dx$$
$$= \pi \left[\frac{x^3}{3} + 2x^2 + 4x\right]_{-5}^{-2}$$
$$= 9\pi \text{ cubic units}$$

3h

$$V = \pi \int_{-3}^{3} y^2 \, dx$$
$$= \pi \int_{-3}^{3} \left(4 - \frac{4}{9}x^2\right) dx$$
$$= \pi \left[4x - \frac{4}{27}x^3\right]_{-3}^{3}$$
$$= \frac{16\pi}{3} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

4a This is a cylinder of radius 1, height 3, hence

$$V = \pi r^2 h$$
$$= \pi(1)^2(3)$$
$$= 3\pi \text{ cubic units}$$

4b This is a cone with radius 2, height 4, hence

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(4)^2(2)$$
$$= \frac{256}{3}\pi \text{ cubic units}$$

4c

$$V = \pi \int_2^5 x^2 \, dy$$
$$= \pi \int_2^5 y^4 \, dy$$
$$= \left[\frac{\pi y^5}{5}\right]_2^5$$
$$= \frac{3093\pi}{5} \text{ cubic units}$$

4d

$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 y \, dy$$
$$= \pi \left[\frac{y^2}{2}\right]_0^1$$
$$= \frac{\pi}{2} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

4e This is a sphere of radius 4, hence

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3}\pi(4)^3$$
$$= \frac{256}{3}\pi \text{ cubic units}$$

4f

$$V = \pi \int_{-3}^{0} x^2\, dy$$
$$= \pi \int_{-3}^{0} (-y^2)^2\, dy$$
$$= \pi \int_{-3}^{0} y^4\, dy$$
$$= \pi \left[\frac{y^5}{5}\right]_{-3}^{0}$$
$$= \frac{243\pi}{5} \text{ cubic units}$$

4g

$$V = \pi \int_{0}^{2} x^2\, dy$$
$$= \pi \int_{0}^{2} (2y - y^2)^2\, dy$$
$$= \pi \int_{0}^{2} (4y^2 - 4y^3 + y^4)\, dy$$
$$= \pi \left[\frac{4}{3}y^3 - y^4 + \frac{y^5}{5}\right]_{0}^{2}$$
$$= \frac{16\pi}{15} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

4h

$$\begin{aligned} V &= \int_{-1}^{1} x^2\,dy \\ &= \int_{-1}^{1} (4 - 4y^2)\,dy \\ &= \left[4y - \frac{4y^3}{3}\right]_{-1}^{1} \\ &= \frac{16\pi}{3} \text{ cubic units} \end{aligned}$$

5

$$\begin{aligned} V &= \pi \int_{0}^{1} y^2\,dx \\ &= \pi \int_{0}^{1} e^{2x}\,dx \\ &= \frac{\pi}{2}[e^{2x}]_0^1 \\ &= \frac{\pi}{2}(e^2 - 1) \text{ cubic units} \end{aligned}$$

6

$$\begin{aligned} V &= \pi \int_{2}^{4} y^2\,dx \\ &= \pi \int_{2}^{4} \frac{1}{x}\,dx \\ &= \pi[\ln x]_2^4 \\ &= \pi[\ln 4 - \ln 2] \\ &= \pi \ln 2 \text{ cubic units} \end{aligned}$$

Uncorrected proofs

© Cambridge University Press 2019

7

$$V = \pi \int_1^6 x^2 \, dy$$
$$= \pi \int_1^6 \frac{1}{y} dy$$
$$= \pi [\ln y]_1^6$$
$$= \pi \ln 6 \text{ cubic units}$$

8a $\tan^2 x = \sec^2 x - 1$

8b

$$V = \pi \int_1^{\frac{\pi}{3}} y^2 \, dx$$
$$= \pi \int_1^{\frac{\pi}{3}} \tan^2 x \, dx$$
$$= \pi \int_1^{\frac{\pi}{3}} (\sec^2 x - 1) \, dx$$
$$= \pi [\tan x - x]_1^{\frac{\pi}{3}}$$
$$= \pi \left(\sqrt{3} - \frac{\pi}{3}\right) \text{ cubic units}$$

9a $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

9b

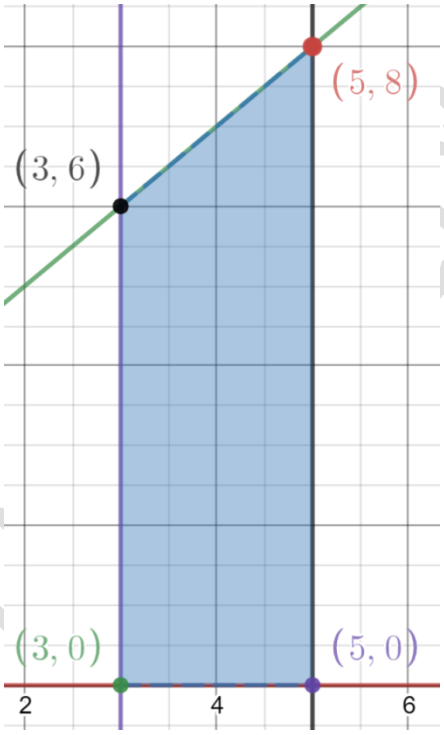
$$V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$
$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

Uncorrected proofs

$$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi^2}{4} \text{ cubic units}$$

10a

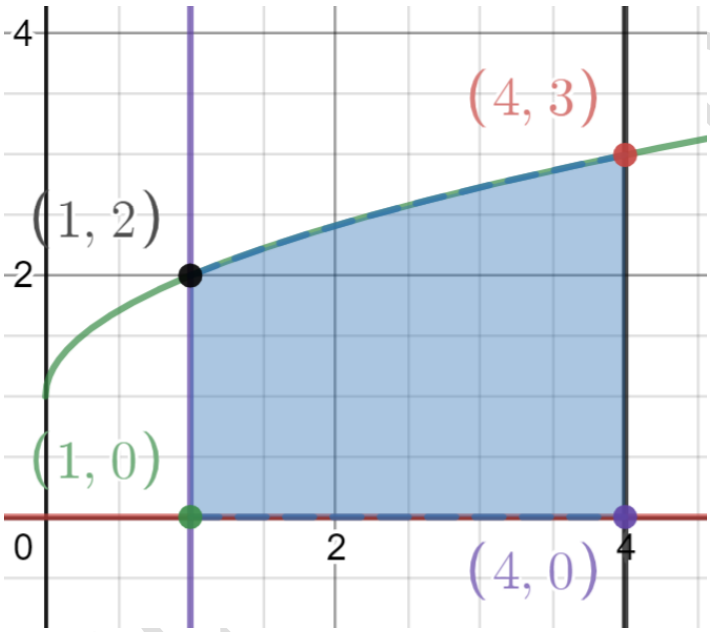
$$V = \pi \int_3^5 y^2 \, dx$$
$$= \int_3^5 (x+3)^2 \, dx$$
$$= \pi \left[\frac{(x+3)^3}{3}\right]_3^5$$
$$= \frac{296\pi}{3} \text{ cubic units}$$



Corrected proofs

10b

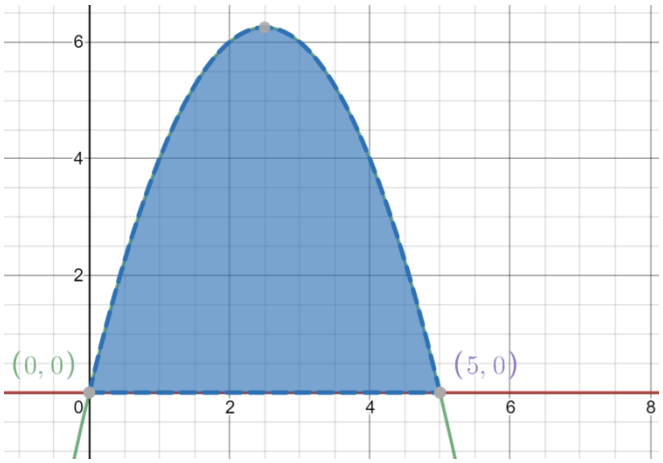
$$
\begin{aligned}
V &= \pi \int_1^4 y^2 \, dx \\
&= \pi \int_1^4 \left(1 + \sqrt{x}\right)^2 dx \\
&= \pi \int_1^4 1 + 2\sqrt{x} + x \, dx \\
&= \pi \int_1^4 1 + 2x^{\frac{1}{2}} + x \, dx \\
&= \pi \left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_1^4 \\
&= \frac{119\pi}{6} \text{ cubic units}
\end{aligned}
$$



10c

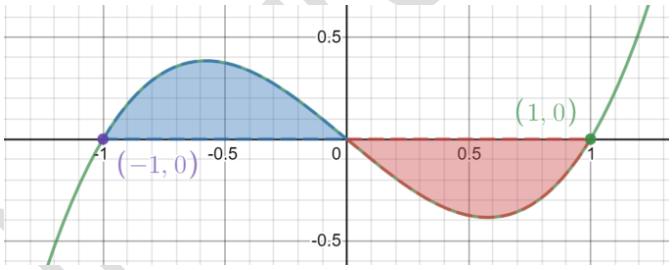
$$
\begin{aligned}
V &= \pi \int_0^5 y^2 \, dx \\
&= \pi \int_0^5 (5x - x^2)^2 \, dx \\
&= \pi \int_0^5 (25x^2 - 10x^3 + x^4) \, dx \\
&= \pi \left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5 \\
&= \frac{625\pi}{6} \text{ cubic units}
\end{aligned}
$$

Uncorrected proofs



10d

$$V = \pi \int_{-1}^{1} y^2 \, dx$$
$$= \pi \int_{-1}^{1} (1 - x^2)^2 \, dx$$
$$= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx$$
$$= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1}$$
$$= \frac{16\pi}{15} \text{ cubic units}$$

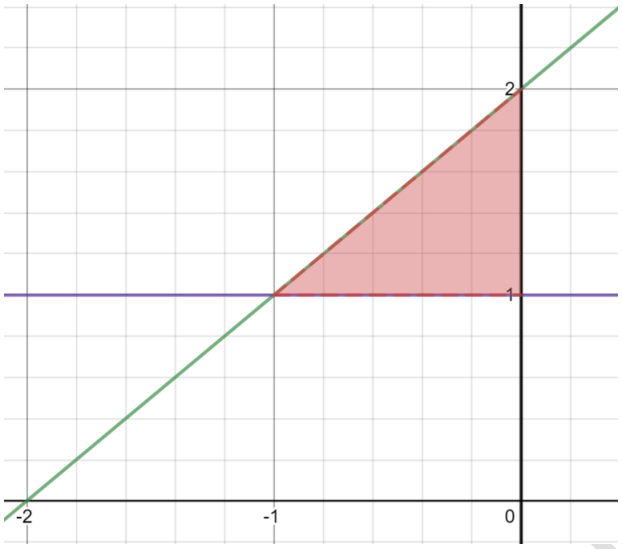


11a

$$V = \pi \int_{1}^{2} x^2 \, dy$$
$$= \pi \int_{1}^{2} (y - 2)^2 \, dy$$
$$= \pi \int_{1}^{2} (y^2 - 4y + 4) \, dy$$

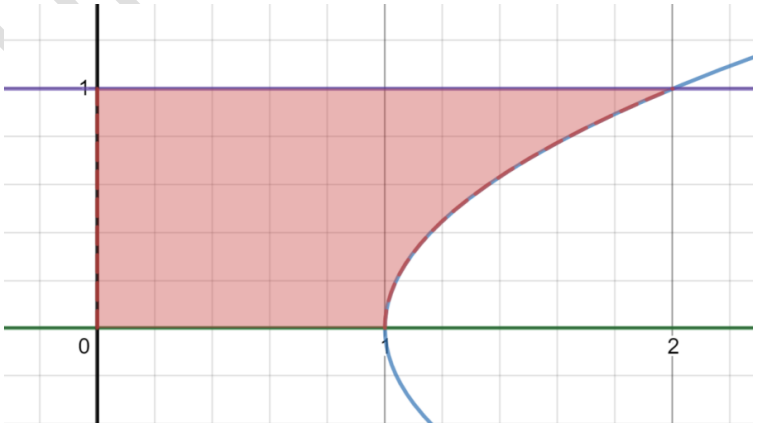
Uncorrected proofs

$$= \pi \left[\frac{y^3}{3} - 2y^2 + 4y \right]_1^2$$
$$= \frac{\pi}{3} \text{ cubic units}$$



11b

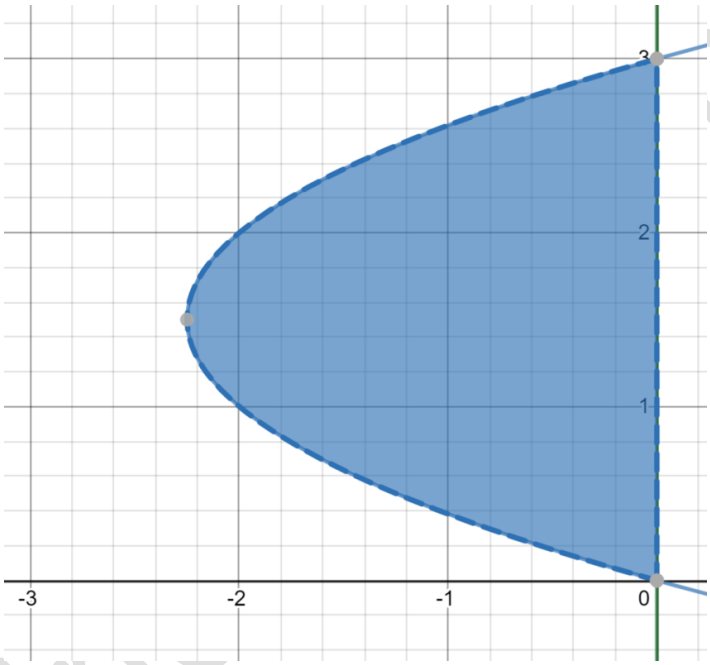
$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 (y^2 + 1)^2 \, dy$$
$$= \pi \int_0^1 (y^4 + 2y^2 + 1) \, dy$$
$$= \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y \right]_0^1$$
$$= \frac{28\pi}{15} \text{ cubic units}$$



Uncorrected proofs

11c

$$V = \pi \int_0^2 x^2 \, dy$$
$$= \pi \int_0^2 y^2 (y-3)^2 \, dy$$
$$= \pi \int_0^2 y^2 (y^2 - 6y + 9) \, dy$$
$$= \pi \int_0^2 (y^4 - 6y^3 + 9y^2) \, dy$$
$$= \pi \left[\frac{y^5}{5} - \frac{6y^4}{4} + 3y^3 \right]_0^2$$
$$= \frac{81\pi}{10} \text{ cubic units}$$

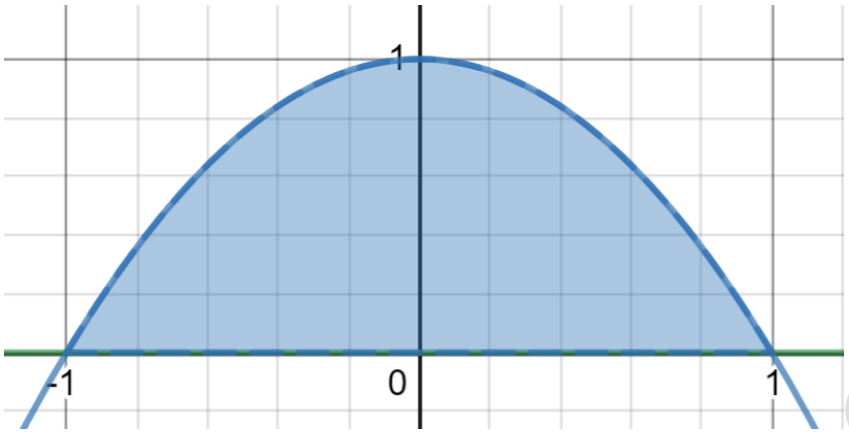


11d

$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 (1-y) \, dy$$
$$= \pi \left[y - \frac{y^2}{2} \right]_0^1$$

Unedited proofs

$= \frac{\pi}{2}$ cubic units



12

$$V = \pi \int_{\frac{1}{2}}^{3} y^2 \, dx$$
$$= \pi \int_{\frac{1}{2}}^{3} \left(1 + \frac{1}{x}\right)^2 dx$$
$$= \pi \int_{\frac{1}{2}}^{3} \left(1 + \frac{2}{x} + \frac{1}{x^2}\right) dx$$
$$= \pi \left[x + 2\ln|x| - \frac{1}{x}\right]_{\frac{1}{2}}^{3}$$
$$= \frac{\pi}{6}(25 + 12\ln 6) \text{ cubic units}$$

13

$$V = \pi \int_{0}^{\frac{1}{2}} y^2 \, dx$$
$$= \pi \int_{0}^{\frac{1}{2}} (e^x - e^{-x})^2 \, dx$$
$$= \pi \int_{0}^{\frac{1}{2}} (e^{2x} - 2 + e^{-2x}) \, dx$$
$$= \pi \left[\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2}\right]_{0}^{\frac{1}{2}}$$
$$= \frac{\pi}{2}(e - 2 - e^{-1}) \text{ cubic units}$$

Uncorrected proofs

14

$$\begin{aligned}
V &= \pi \int_{4\pi}^{6\pi} y^2\, dx \\
&= \pi \int_{4\pi}^{6\pi} \left(4 + 4\sin\frac{x}{4}\right)^2 dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(1 + 2\sin\frac{x}{4} + \sin^2\frac{x}{4}\right) dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(1 + 2\sin\frac{x}{4} + \frac{1}{2}\left(1 - \cos\frac{x}{2}\right)\right) dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(\frac{3}{2} + 2\sin\frac{x}{4} - \cos\frac{x}{2}\right) dx \\
&= 16\pi \left[x - 8\cos\frac{x}{4} - 2\sin\frac{x}{2}\right]_{4\pi}^{6\pi} \\
&= 71.62 \text{ mL}
\end{aligned}$$

15a

$$\begin{aligned}
V &= \pi \int_0^{16} y^2\, dx \\
&= \pi \int_0^{16} 4^2\, dx \\
&= \pi[4^2 x]_0^{16} \\
&= 256\pi \text{ cubic units}
\end{aligned}$$

15b

$$\begin{aligned}
V &= \pi \int_0^{16} x^2\, dy \\
&= \pi \int_0^{16} y\, dy \\
&= \pi \left[\frac{y^2}{2}\right]_0^{16} \\
&= 128\pi \text{ cubic units}
\end{aligned}$$

15c $V = 256\pi - 128\pi = 128\pi$ cubic units

Uncorrected proofs

16a i

$$V = \pi \int_0^2 y^2 \, dx$$

$$= \pi \int_0^2 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{32\pi}{5} \text{ cubic units}$$

16a ii

$$V = \pi \int_0^4 x^2 \, dy$$

$$= \pi \int_0^4 y \, dy$$

$$= \pi \left[\frac{y^2}{2}\right]$$

$$= 8\pi \text{ cubic units}$$

16b i

$$V = \pi \times 5^2 \times 1 - \pi \int_0^1 y^2 \, dx$$

$$= \pi \times 5^2 \times 1 - \pi \int_0^1 25x^2 \, dx$$

$$= \pi \times 5^2 \times 1 - \pi \left[\frac{25x^3}{3}\right]_0^1$$

$$= \frac{50\pi}{3} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

16b ii

$$V = \pi \int_0^5 x^2 \, dy$$

$$= \pi \int_0^5 \left(\frac{y}{5}\right)^2 dy$$

$$= \frac{\pi}{25} \int_0^5 y^2 \, dy$$

$$= \frac{\pi}{25} \left[\frac{y^3}{3}\right]_0^5$$

$$= \frac{5\pi}{3} \text{ cubic units}$$

16c i

$$V = \pi \int_0^4 y^2 \, dx$$

$$= \pi \int_0^4 x \, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4$$

$$= 8\pi \text{ cubic units}$$

16c ii

$$V = \pi \times 4^2 \times 2 - \pi \int_0^2 x^2 \, dy$$

$$= \pi \times 4^2 \times 2 - \pi \int_0^2 y^4 \, dy$$

$$= \pi \times 4^2 \times 2 - \pi \left[\frac{y^5}{5}\right]_0^2$$

$$= \frac{128\pi}{5} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

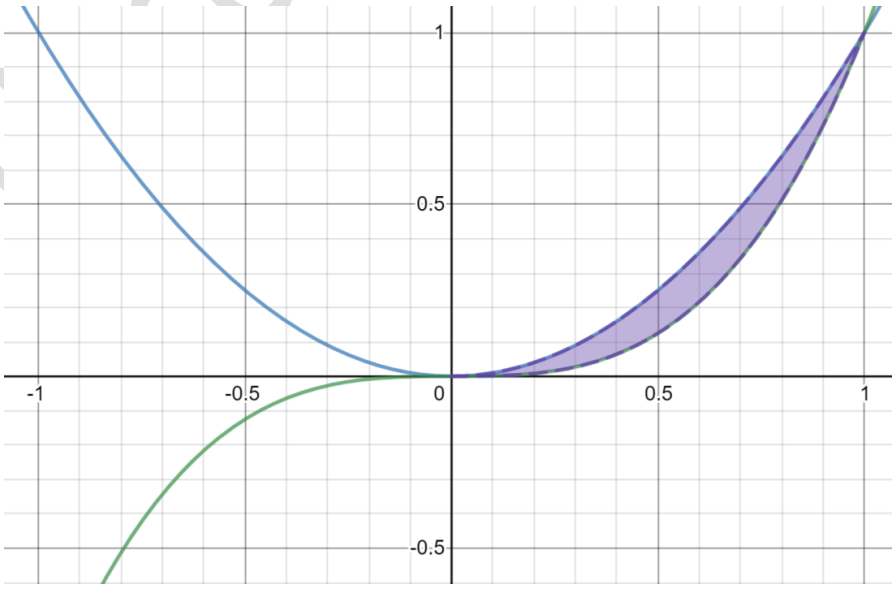
16d i

$$V = \pi \times 4^2 \times 1 - \int_0^1 y^2\,dx$$
$$= 16\pi - \pi \int_0^1 (x^2 + 3)^2\,dx$$
$$= 16\pi - \pi \int_0^1 (x^4 + 6x^2 + 9)\,dx$$
$$= 16\pi - \pi \left[\frac{x^5}{5} + 2x^3 + 9x\right]_0^1$$
$$= \frac{24\pi}{5} \text{ cubic units}$$

16d ii

$$V = \pi \int_3^4 x^2\,dy$$
$$= \pi \int_3^4 (y - 3)\,dy$$
$$= \pi \left[\frac{y^2}{2} - 3y\right]_3^4$$
$$= \frac{\pi}{2} \text{ cubic units}$$

17a



Uncorrected proofs

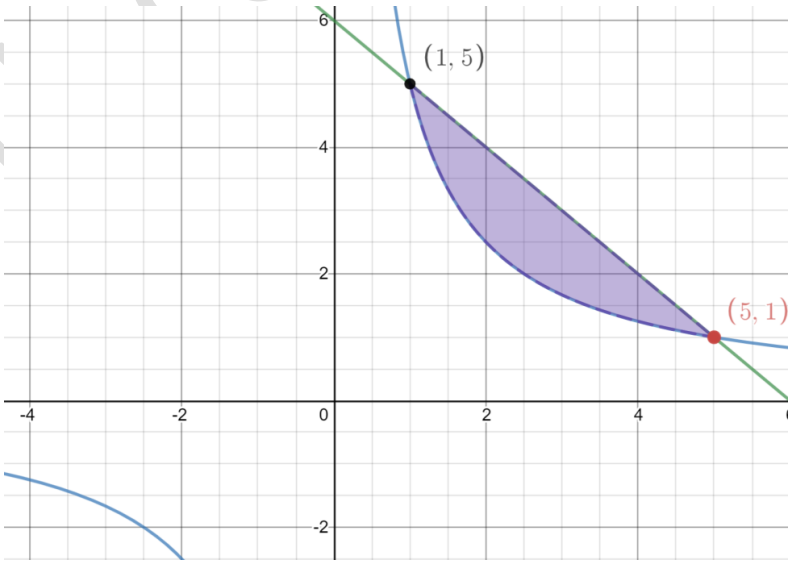
17b i

$$\begin{aligned} V &= \pi \int_0^1 y_1^2 \, dx - \pi \int_0^1 y_2^2 \, dx \\ &= \pi \int_0^1 x^4 \, dx - \pi \int_0^1 x^6 \, dx \\ &= \pi \int_0^1 (x^4 - x^6) \, dx \\ &= \pi \left[\frac{x^5}{5} - \frac{x^7}{7} \right]_0^1 \\ &= \frac{2\pi}{35} \text{ cubic units} \end{aligned}$$

17b ii

$$\begin{aligned} V &= \pi \int_0^1 x_1^2 \, dy - \pi \int_0^1 x_2^2 \, dy \\ &= \pi \int_0^1 y^1 \, dy - \pi \int_0^1 y^{\frac{3}{2}} \, dy \\ &= \pi \int_0^1 \left(y^1 - y^{\frac{3}{2}} \right) dy \\ &= \pi \left[\frac{y^2}{2} - \frac{2}{5} y^{\frac{5}{2}} \right]_0^1 \\ &= \frac{\pi}{10} \text{ cubic units} \end{aligned}$$

18a

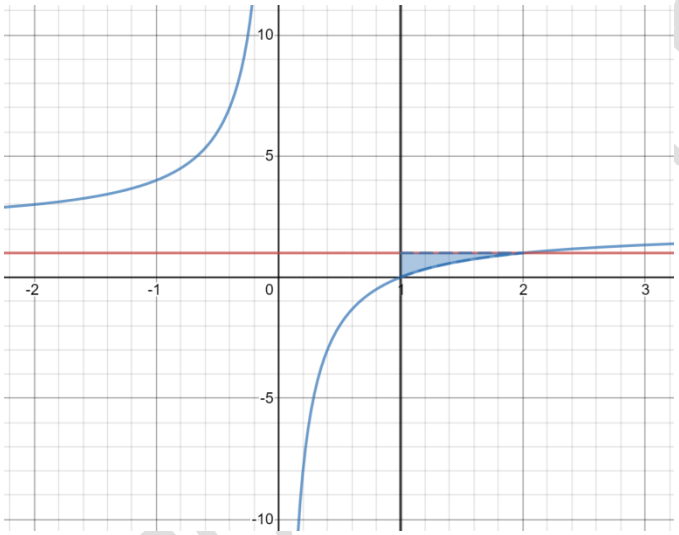


uncorrected proofs

18b

$$\begin{aligned}V &= \pi \int_1^5 (y_1^2 - y^2)\, dx \\ &= \pi \int_1^5 \left((6-x)^2 - \left(\frac{5}{x}\right)^2 \right) dx \\ &= \pi \int_1^5 \left(36 - 12x + x^2 - \frac{25}{x^2} \right) dx \\ &= \pi \left[36x - 6x^2 + \frac{x^3}{3} + \frac{25}{x} \right]_1^5 \\ &= \frac{64\pi}{3} \text{ cubic units}\end{aligned}$$

19



$$\begin{aligned}V &= \pi \int_1^2 (y_1^2 - y_2^2)\, dx \\ &= \pi \int_1^2 \left(1^2 - \left(2 - \frac{2}{x}\right)^2 \right) dx \\ &= \pi \int_1^2 (1 - \frac{4}{x^2} - 4 + \frac{8}{x})\, dx \\ &= \pi \int_1^2 (-3 - \frac{4}{x^2} + \frac{8}{x})\, dx \\ &= \pi \left[-3x + \frac{4}{x} + 8 \ln x \right]_1^2 \\ &= \pi(8 \ln 2 - 5) \text{ cubic units}\end{aligned}$$

Uncorrected proofs

20a $y = x^3 + 2$

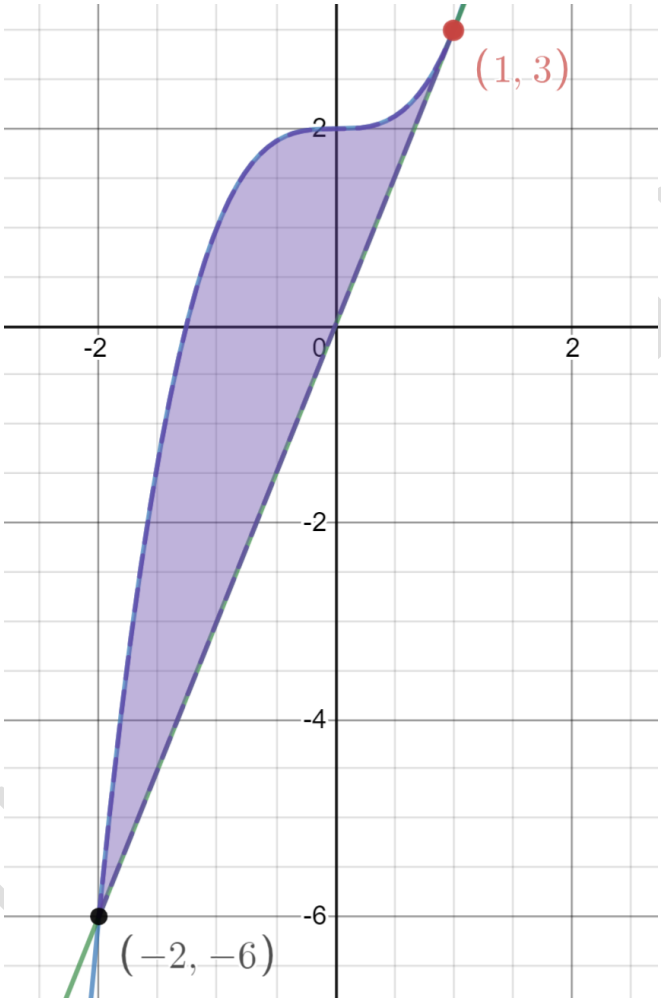
$\frac{dy}{dx} = 3x^2$

When $x = 1, y = 3$ and $\frac{dy}{dx} = 3$. Hence the equation of the tangent is

$y - 3 = 3(x - 1)$

$y = 3x$

20b



proofs

20c i

$$V = \pi \int_{-2}^{1} (y_1^2 - y_2^2)\, dx$$

$$= \pi \int_{-2}^{1} (x^6 - 9x^2)\, dx$$

$$= \pi \left[\frac{x^7}{7} - 3x^3\right]_{-2}^{1}$$

$$= \frac{15\pi}{7} \text{ cubic units}$$

20c ii

$$V = \pi \int_{-6}^{3} (x_1^2 - x_2^2)\, dy$$

$$= \pi \int_{-6}^{3} \left(y^{\frac{2}{3}} - \frac{y^2}{9}\right) dy$$

$$= \pi \left[\frac{3}{2} y^{\frac{2}{3}} - \frac{y^3}{27}\right]_{-6}^{3}$$

$$= \frac{2\pi}{5} \text{ cubic units}$$

21

$$V = \pi \times 2^2 \times \ln 2 - \pi \int_{0}^{\ln 2} x^2\, dy$$

$$= 4\pi \ln 2 - \pi \int_{0}^{\ln 2} (e^y)^2\, dy$$

$$= 4\pi \ln 2 - \pi \int_{0}^{\ln 2} e^{2y}\, dy$$

$$= 4\pi \ln 2 - \pi \left[\frac{e^{2y}}{2}\right]_{0}^{\ln 2}$$

$$= 4\pi \ln 2 - \pi \left(\frac{e^{2\ln 2}}{2} - \frac{e^0}{2}\right)$$

$$= 4\pi \ln 2 - \pi \left(\frac{e^{\ln 4}}{2} - \frac{e^0}{2}\right)$$

Uncorrected proofs

© Cambridge University Press 2019

$$= 4\pi \ln 2 - \pi\left(\frac{4}{2} - \frac{1}{2}\right)$$
$$= 4\pi \ln 2 - \pi\left(\frac{3}{2}\right)$$
$$= \frac{\pi}{2}(8 \ln 2 - 3) \text{ cubic units}$$

22a

$$V = \pi \int_0^h y^2 \, dx$$
$$= \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx$$
$$= \pi \int_0^h \frac{r^2x^2}{h^2} dx$$
$$= \pi \left[\frac{r^2x^3}{3h^2}\right]_0^h$$
$$= \pi \left[\frac{r^2h^3}{3h^2}\right]$$
$$= \frac{1}{3}\pi r^2 h$$

22b

$$V = \pi \int_0^h y^2 \, dx$$
$$= \pi \int_0^h r^2 \, dx$$
$$= \pi [r^2 x]_0^h$$
$$= \pi r^2 h$$

Uncorrected proofs

22c i

$$\begin{aligned}V &= \pi \int_{-r}^{r} y^2\,dx \\ &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\ &= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right] \\ &= \frac{4}{3}\pi r^3\end{aligned}$$

22c ii

$$\begin{aligned}V &= \pi \int_{r-h}^{r} y^2\,dx \\ &= \pi \int_{r-h}^{r} r^2 - x^2\,dx \\ &= \pi \left[r^2 x - \frac{x^3}{3}\right]_{r-h}^{r} \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right] \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^3 - r^2 h - \frac{r^3 - 3r^2 h + 3rh^2 - h^3}{3}\right)\right] \\ &= \frac{1}{3}\pi h^2 (3r - h)\end{aligned}$$

23a $\sec\theta$

$$\begin{aligned}&= \sec\theta \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \\ &= \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \\ &= \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\end{aligned}$$

$$\int \sec\theta\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\,d\theta$$

Uncorrected proofs

$= \ln(\sec\theta + \tan\theta) + C$

23b

$$y = \frac{x}{\sqrt{x^2 + 16}}$$

$$y^2 = \frac{x^2}{x^2 + 16}$$

$$x^2 + 16 = \frac{x^2}{y^2}$$

$$16 = \frac{x^2}{y^2} - x^2$$

$$16 = x^2\left(\frac{1}{y^2} - 1\right)$$

$$x^2 = \frac{16}{\left(\frac{1}{y^2} - 1\right)}$$

$$= \frac{16}{\left(\frac{1}{y^2} - \frac{y^2}{y^2}\right)}$$

$$= \frac{16}{\left(\frac{1 - y^2}{y^2}\right)}$$

$$= \frac{16y^2}{1 - y^2}$$

$$V = \pi \int_0^{\frac{1}{\sqrt{2}}} x^2\, dy$$

$$= \pi \int_0^{\frac{1}{\sqrt{2}}} \frac{16y^2}{1 - y^2}\, dy$$

Let $y = \sin\theta$, $dy = \cos\theta\, d\theta$

$$V = \pi \int_0^{\frac{1}{\sqrt{2}}} x^2\, dy$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{16\sin^2\theta}{1 - \sin^2\theta} \cos\theta\, d\theta$$

Uncorrected proofs

$$= \pi \int_0^{\frac{\pi}{4}} \frac{16 \sin^2 \theta}{\cos^2 \theta} \cos \theta \, d\theta$$
$$= \pi \int_0^{\frac{\pi}{4}} \frac{16 \sin^2 \theta}{\cos \theta} d\theta$$
$$= \pi \int_0^{\frac{\pi}{4}} \frac{16(1 - \cos^2 \theta)}{\cos \theta} d\theta$$
$$= 16\pi \int_0^{\frac{\pi}{4}} (\sec \theta - \cos \theta) \, d\theta$$
$$= 16\pi [\ln(\sec \theta + \tan \theta) + \sin \theta]_0^{\frac{\pi}{4}}$$
$$= 16\pi \left(\sqrt{2} - \ln\left(\sqrt{2} + 1\right)\right) \text{ cubic units}$$

24a

$$x^n = x^{n+1}$$
$$x^n - x^{n+1} = 0$$
$$x^n(1 - x) = 0$$

Hence $x = 0$ or 1. Substituting back into the equations gives the points of intersection as $(0, 0)$ and $(1, 1)$.

24b

$$V = \pi \int_0^1 f(x)^2 \, dx - \pi \int_0^1 f(x)^2 \, dx$$
$$= \pi \int_0^1 (x^n)^2 \, dx - \pi \int_0^1 (x^{n+1})^2 \, dx$$
$$= \pi \int_0^1 x^{2n} \, dx - \pi \int_0^1 x^{2n+2} \, dx$$
$$= \pi \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 - \pi \left[\frac{x^{2n+3}}{2n+3}\right]_0^1$$
$$= \pi \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

24c It is the cone formed by rotating the line $y = x$ from $x = 0$ to $x = 1$ about the x-axis.

Uncorrected proofs

© Cambridge University Press 2019

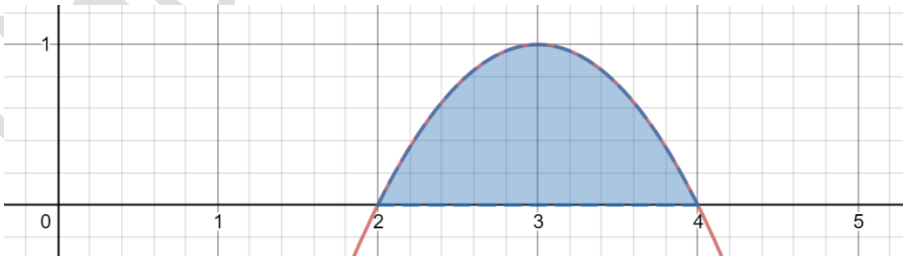
24d

$$\lim_{n\to\infty}(V_1 + V_2 + V_3 + \cdots + V_n)$$
$$= \lim_{n\to\infty}\left(\pi\left(\frac{1}{2+1} - \frac{1}{2+3}\right) + \pi\left(\frac{1}{2(2)+1} - \frac{1}{2(2)+3}\right) + \pi\left(\frac{1}{2(3)+1} - \frac{1}{2(3)+3}\right) + \cdots + \pi\left(\frac{1}{2(n)+1} - \frac{1}{2(n)+3}\right)\right)$$
$$= \pi \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots + \frac{1}{2n+1} - \frac{1}{2n+3}\right)$$
$$= \pi \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{2n+3}\right)$$
$$= \pi\left(\frac{1}{3} - 0\right)$$
$$= \frac{\pi}{3} \text{ cubic units}$$

24e

$$\frac{1}{3\times 5} + \frac{1}{5\times 7} + \frac{1}{7\times 9} + \cdots$$
$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \frac{1}{2}\left(\frac{1}{7} - \frac{1}{9}\right) + \cdots$$
$$= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots\right)$$
$$= \frac{1}{2}\left(\frac{1}{3}\right)$$
$$= \frac{1}{6}$$

25a



25b $y = -x^2 + 6x - 8$

$y = -x^2 + 6x - 9 + 1$

$y = -(x^2 - 6x + 9) + 1$

Uncorrected proofs

$y = -(x-3)^2 + 1$

$(x-3)^2 = 1 - y$

$x - 3 = \sqrt{1-y}$ for $3 \leq x \leq 4$ and $x - 3 = -\sqrt{1-y}$ for $2 \leq x \leq 3$

Hence

$x = 3 + \sqrt{1-y}$ for $3 \leq x \leq 4$ and $x = 3 - \sqrt{1-y}$ for $2 \leq x \leq 3$

25c

$$V = \pi \int_0^1 \left(3 + \sqrt{1-y}\right)^2 dy - \pi \int_0^1 \left(3 - \sqrt{1-y}\right)^2 dy$$

$$= \pi \int_0^1 \left(\left(3 + \sqrt{1-y}\right)^2 - \left(3 - \sqrt{1-y}\right)^2\right) dy$$

$$= \pi \int_0^1 \left(\left(9 + 6\sqrt{1-y} + (1-y)\right) - \left(9 - 6\sqrt{1-y} + (1-y)\right)\right) dy$$

$$= \pi \int_0^1 \left(12\sqrt{1-y}\right) dy$$

$$= \int_0^1 \left(12\pi\sqrt{1-y}\right) dy$$

$$= \int_0^1 \left(12\pi(1-y)^{\frac{1}{2}}\right) dy$$

$$= 12\pi \left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1$$

$$= 12\pi \left(-\frac{2}{3} \times 0 + \frac{2}{3} \times 1\right)$$

$= 8\pi$ cubic units

Uncorrected proofs

© Cambridge University Press 2019

Solutions to Chapter review

1a

$$\frac{d}{dx}(\sin^{-1} 3x)$$
$$= \frac{1}{\sqrt{1-(3x)^2}} \times \frac{d}{dx}(3x)$$
$$= \frac{3}{\sqrt{1-9x^2}}$$

1b

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right)\right)$$
$$= \frac{1}{1+\left(\frac{x}{3}\right)^2} \times \frac{d}{dx}\left(\frac{x}{3}\right)$$
$$= \frac{1}{1+\frac{x^2}{9}} \times \frac{1}{3}$$
$$= \frac{3}{9+x^2}$$

1c

$$\frac{d}{dx}(\cos^{-1}(1-x))$$
$$= \frac{-1}{\sqrt{1-(1-x)^2}} \times \frac{d}{dx}(1-x)$$
$$= \frac{-1}{\sqrt{1-(1-2x+x^2)}} \times -1$$
$$= \frac{1}{\sqrt{2x-x^2}}$$

Uncorrected proofs

© Cambridge University Press 2019

1d

$$\frac{d}{dx}(x^2 \tan^{-1} x)$$
$$= (\tan^{-1} x) \times \frac{d}{dx}(x^2) + x^2 \times \frac{d}{dx}(\tan^{-1} x)$$
$$= 2x \tan^{-1} x + x^2 \times \frac{1}{1+x^2}$$
$$= \frac{x^2}{1+x^2} + 2x \tan^{-1} x$$

1e

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{1}{2}x+1\right)\right)$$
$$= \frac{1}{1+\left(\frac{1}{2}x+1\right)^2} \times \frac{d}{dx}\left(\frac{1}{2}x+1\right)$$
$$= \frac{1}{1+\frac{x^2}{4}+x+1} \times \frac{1}{2}$$
$$= \frac{1}{\frac{x^2}{4}+x+2} \times \frac{1}{2}$$
$$= \frac{4}{x^2+4x+8} \times \frac{1}{2}$$
$$= \frac{2}{x^2+4x+8}$$

1f

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{\sqrt{1-\frac{1}{x^2}}} \times \left(-\frac{1}{x^2}\right)$$
$$= -\frac{1}{\sqrt{x^4}\sqrt{1-\frac{1}{x^2}}}$$
$$= -\frac{1}{\sqrt{x^4-x^2}}$$

Uncorrected proofs

2a $y = \tan^{-1} x$

$$y' = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$y'' = -1 \times (1+x^2)^{-2} \times 2x = -\frac{2x}{(1+x^2)^2}$$

Substituting for $y'$and y'':

$$(1+x^2)y'' + 2xy'$$
$$= (1+x^2) \times -\frac{2x}{(1+x^2)^2} + 2x \times \frac{1}{(1+x^2)}$$
$$= -\frac{2x}{(1+x^2)} + \frac{2x}{(1+x^2)}$$
$$= 0$$

3a

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$
$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$
$$= \frac{1}{\sqrt{1-(1-x^2)}} \times \frac{1}{2} \times (1-x^2)^{-\frac{1}{2}} \times -2x$$
$$= \frac{1}{\sqrt{x^2}} \times -\frac{x}{\sqrt{1-x^2}}$$
$$= -\frac{1}{\sqrt{x^2}} \times \frac{x}{\sqrt{1-x^2}}$$

For $0 \leq x \leq 1$, $\sqrt{x^2} = x$

Hence

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$
$$= -\frac{1}{x} \times \frac{x}{\sqrt{1-x^2}}$$
$$= -\frac{1}{\sqrt{1-x^2}}$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{d}{dx}(\cos^{-1} x)$$

3b The functions differ by at most a constant.

This constant is zero, so $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2}$ for $0 \leq x \leq 1$.

4a

$$\int \frac{1}{3+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

4b

$$\int \frac{1}{\sqrt{3-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3}\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx$$

$$= \sin^{-1}\frac{x}{\sqrt{3}} + C$$

4c

$$\int \frac{1}{9+4x^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\frac{9}{4}+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{2x}{3}\right)^2}\,dx$$

Uncorrected proofs

$$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$$

4d

$$\int -\frac{1}{\sqrt{16 - 9x^2}}\,dx$$

$$= \int -\frac{1}{4\sqrt{1 - \left(\frac{3x}{4}\right)^2}}\,dx$$

$$= \frac{1}{3}\cos^{-1}\frac{3x}{4} + C$$

5a

$$\int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 9x^2}\,dx$$

$$= \int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + (3x)^2}\,dx$$

$$= \frac{1}{3}\left[\tan^{-1} 3x\right]_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}}$$

$$= \frac{\pi}{36}$$

5b

$$\int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}}\,dx$$

$$= \int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3}\sqrt{1 - \left(\frac{2x}{\sqrt{3}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

Uncorrected proofs

$$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

$$= \frac{\pi}{3}$$

6a

$$\int \cos^2 x \, dx$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

6b

$$\int \sin^2 x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

6c

$$\int \cos^2 2x \, dx$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x + C$$

6d

$$\int \sin^2 4x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{2}x - \frac{1}{16}\sin 4x + C$$

7a

$$\int_0^{\frac{\pi}{3}} \sin^2 3x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6}$$

7b

$$\int_0^{\frac{\pi}{6}} \cos^2 \frac{1}{2}x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\sin x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{2}\cos x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{12}(\pi + 3)$$

8

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 2x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx = \left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 4x \, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 8x\right) dx = \left[\frac{1}{2}x + \frac{1}{16}\sin 8x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

Uncorrected proofs

© Cambridge University Press 2019

9a Let $u = 5x - 1$, $du = 5\,dx$

$$\int 5(5x-1)^5$$

$$= \int u^5\,du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{(5x-1)^6}{6} + C$$

9b Let $u = x^2 + 2$

$$du = 2x\,dx$$

$$\int 2x(x^2+2)^2\,dx$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{(x^2+2)^3}{3} + C$$

9c Let $u = x^4 + 1$, $du = 4x^3$

$$\int \frac{4x^3}{(x^4+1)^2}\,dx$$

$$= \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= \frac{1}{x^4+1} + C$$

Uncorrected proofs

© Cambridge University Press 2019

9d Let $u = 4x + 3$, $du = 4\,dx$

$$\int \frac{1}{\sqrt{4x+3}}\,dx$$

$$= \int \frac{1}{\sqrt{u}} \times \frac{1}{4}\,du$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4} \times 2u^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{4x+3} + C$$

9e Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin^2 x \cos x\,dx$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\sin^3 x}{3} + C$$

9f Let $u = \tan x$, $du = \sec^2 x\,dx$

$$\int \tan^3 x \sec^2 x\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\tan^4 x}{4} + C$$

Uncorrected proofs

© Cambridge University Press 2019

10a Let $u = 1 + x^3$, $du = 3x^2\ dx$ and $\frac{1}{3}du = x^2 dx$

$$\int_{-1}^{0} x^2(1 + x^3)^4\ dx$$

$$= \frac{1}{3}\int_{0}^{1} u^4\ du$$

$$= \frac{1}{3}\left[\frac{u^5}{5}\right]_{0}^{1}$$

$$= \frac{1}{15}$$

10b Let $u = \cos x$, $du = -\sin x$

$$\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin x\ dx$$

$$= \frac{1}{4}[-u^4]_{1}^{0}$$

$$= -\frac{1}{4}(0^3 - 1^2)$$

$$= \frac{1}{4}$$

10c Let $u = x^2 - 1$, $du = 2x\ dx$ and $\frac{1}{2}du = x\ dx$

$$\int_{1}^{\sqrt{2}} x\sqrt{x^2 - 1}\ dx$$

$$= \frac{1}{2}\int_{0}^{1} \sqrt{u}\ du$$

$$= \frac{1}{2}\int_{0}^{1} u^{\frac{1}{2}}\ du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{0}^{1}$$

$$= \frac{1}{2}\left(\frac{2}{3} - 0\right)$$

Uncorrected proofs

© Cambridge University Press 2019

$= \frac{1}{3}$

10d Let $u = \ln x, du = \frac{1}{x}\, dx$

$\int_1^e \frac{(\ln x)^2}{x} dx$

$= \int_0^1 u^2\, du$

$= \left[\frac{1}{3}u^3\right]_0^1$

$= \frac{1}{3}$

10e Let $u = \frac{1}{x}, du = -\frac{1}{x^2}\, dx$

$\int_{\frac{1}{2}}^1 \frac{e^{\frac{1}{x}}}{x^2} dx$

$= \int_2^1 -e^u\, du$

$= \int_1^2 e^u\, du$

$= [e^u]_1^2$

$= e^2 - e$

10f Let $u = \tan 2x, du = 2\sec^2 2x\ dx$

$\int_0^{\frac{\pi}{8}} \frac{\sec^2 2x}{1 + \tan 2x} dx$

$= \frac{1}{2}\int_0^1 \frac{1}{1+u} du$

$= \frac{1}{2}[\ln|1+u|]_0^1$

Uncorrected proofs

$= \frac{1}{2}(\ln 2 - \ln 1)$

$= \frac{1}{2}\ln 2$

11a Let $x = u + 1, du = dx$

$\int \frac{x}{x-1}dx$

$= \int \frac{u+1}{u}du$

$= \int \left(1 + \frac{1}{u}\right)du$

$= u + \ln|u| + C$

$= x - 1 + \ln|x - 1| + C$

11b Let $x = u - 2, du = dx$

$\int \frac{x-1}{\sqrt{x+2}}dx$

$= \int \frac{u-3}{\sqrt{u}}du$

$= \int (u^{\frac{1}{2}} - 3u^{-\frac{1}{2}})\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x+2)^{\frac{3}{2}} - 6\sqrt{x+2} + C$

11c Let $x = \frac{1}{2}u^2 - \frac{1}{2}, dx = u\,du$

Note that $2x + 1 = u^2$

$\int x\sqrt{2x+1}\,dx$

$= \int \left(\frac{1}{2}u^2 - \frac{1}{2}\right)\left(\sqrt{u^2}\right)u\,du$

$= \frac{1}{2}\int (u^4 - u^2)\,du$

Uncorrected proofs

© Cambridge University Press 2019

$$= \frac{1}{2}\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$
$$= \frac{1}{2}\left(\frac{(2x+1)^{\frac{5}{2}}}{5} - \frac{(2x+1)^{\frac{3}{2}}}{3}\right) + C$$
$$= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$$

11d Let $x = (u-4)^2$, $dx = 2(u-4)\,du$

$$\int \frac{1}{4+\sqrt{x}}\,dx$$
$$= \int \frac{2(u-4)}{4+u-4}\,du$$
$$= \int \left(2 - \frac{8}{u}\right)du$$
$$= 2u - 8\ln|u| + C$$
$$= 2\left(\sqrt{x}+4\right) - 8\ln\left(\sqrt{x}+4\right) + C$$

12a Let $x = u + 1$, $dx = du$

$$\int_1^2 x(x-1)^4\,dx$$
$$= \int_0^1 (u+1)u^4\,du$$
$$= \int_0^1 u^5 + u^4\,du$$
$$= \left[\frac{u^6}{6} + \frac{u^5}{5}\right]_0^1$$
$$= \frac{1}{6} + \frac{1}{5}$$
$$= \frac{11}{30}$$

Uncorrected proofs

© Cambridge University Press 2019

12b Let $x = u - 3, dx = du$

$$\int_1^5 \frac{x}{x+3}\,dx$$

$$= \int_4^8 \frac{u-3}{u}\,du$$

$$= \int_4^8 \left(1 - \frac{3}{u}\right) du$$

$$= [u - 3\ln|u|]_4^8$$

$$= (8 - 3\ln 8) - (4 - 3\ln 4)$$

$$= 4 - 3\ln 2$$

12c Let $x = u^2 - 1, dx = 2u\,du$

$$\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$$

$$= \int_1^4 \frac{u^2-1}{\sqrt{u^2}} \times 2u\,du$$

$$= \int_1^4 (2u^2 - 2)\,du$$

$$= \left[\frac{2}{3}u^3 - 2u\right]_1^4$$

$$= 36$$

12d Let $x = u^2 + 2, dx = 2u\,du$

$$\int_2^3 \frac{1}{2}x\sqrt{x-2}\,dx$$

$$= \int_0^1 \frac{1}{2}(u^2+2)\sqrt{u^2} \times 2u\,du$$

$$= \int_0^1 (u^4 + 2u^2)\,du$$

$$= \left[\frac{u^5}{5} + \frac{2}{3}u^3\right]_0^1$$

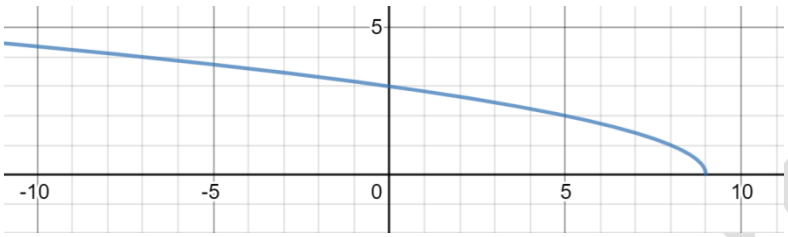
$$= \frac{1}{5} + \frac{2}{3}$$

Uncorrected proofs

$= \frac{13}{15}$

13a The domain is all x such that $9 - x \geq 0$ and hence the domain is $x \leq 9$. As the square root function always returns a positive number, the range is $y \geq 0$.

13b



13c

$$A = \int_0^9 \sqrt{9 - x}\, dx$$

$$= \int_0^9 (9 - x)^{\frac{1}{2}}\, dx$$

$$= -\frac{4}{3}\left[(9 - x)^{\frac{3}{2}}\right]_0^9$$

$= 18$ square units

13d i

$$V = \pi \int_0^9 y^2\, dx$$

$$= \pi \int_0^9 (9 - x)\, dx$$

$$= \pi \left[9x - \frac{x^2}{2}\right]_0^9$$

$= \frac{81\pi}{2}$ cubic units

Uncorrected proofs

13d ii

$$V = \pi \int_0^3 x^2 \, dy$$

$$= \pi \int_0^3 (y^2 - 9)^2 \, dy$$

$$= \pi \int_0^3 (y^4 - 18y^2 + 81) \, dy$$

$$= \pi \left[\frac{y^5}{5} - 6y^3 + 81y \right]_0^3$$

$$= \frac{648\pi}{5} \text{ cubic units}$$

14

$$V = \int_0^{3\frac{3}{4}} \pi y^2 \, dy$$

$$= \int_0^{3\frac{3}{4}} \pi \left(\frac{1}{\sqrt{4 - x}} \right)^2 dx$$

$$= \int_0^{3\frac{3}{4}} \frac{\pi}{4 - x} dx$$

$$= -\pi [\ln|4 - x|]_0^{3\frac{3}{4}}$$

$$= 4\pi \ln 2 \text{ cubic units}$$

15 $y^2 = 18(x - 6)$

$$x = \frac{y^2}{18} + 6$$

$$V = \pi \int_{-6}^6 x^2 \, dy$$

$$= \pi \int_{-6}^6 \left(\frac{y^2}{18} + 6 \right)^2 dy$$

$$= \pi \int_{-6}^6 \left(\frac{y^4}{18^2} + \frac{12}{18} y^2 + 36 \right) dy$$

Uncorrected proofs

$$= \pi\left[\frac{y^5}{5 \times 18^2} + \frac{12}{18 \times 3}y^3 + 36y\right]_{-6}^{6}$$
$$= \frac{2688\pi}{5} \text{ cubic units}$$

16 $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

16b

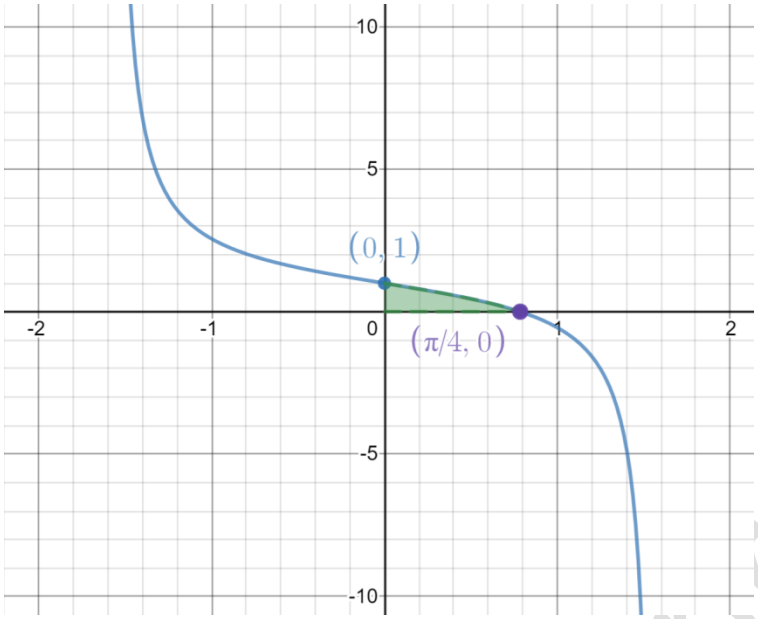
$$V = \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} y^2\,dx$$
$$= \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 2x\,dx$$
$$= \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right)dx$$
$$= \pi\left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \frac{\pi}{24}\left(4\pi + 3\sqrt{3}\right) \text{ cubic units}$$

17

$$V = \pi\int_1^3 y^2\,dx$$
$$= \pi\int_1^3 (1 + e^{-x})^2\,dx$$
$$= \pi\int_1^3 (1 + 2e^{-x} + e^{-2x})\,dx$$
$$= \pi\left[x - 2e^{-x} - \frac{1}{2}e^{-2x}\right]_1^3$$
$$= \pi\left(\left(3 - 2e^{-3} - \frac{1}{2}e^{-6}\right) - \left(1 - 2e^{-1} - \frac{1}{2}e^{-2}\right)\right)$$
$$\div 8.49 \text{ cubic units}$$

Uncorrected proofs

18a



18b

$$V = \pi \int_0^{\frac{\pi}{4}} y^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} (1 - \tan x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} (1 - 2\tan x + \tan^2 x) \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} (1 - 2\tan x + \sec^2 x - 1) \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \left(-\frac{2\sin x}{\cos x} + \sec^2 x \right) dx$$

$$= \pi [2\ln|\cos x| + \tan x]_0^{\frac{\pi}{4}}$$

$$= \pi \left(\left(2\ln\left(2^{-\frac{1}{2}}\right) + 1 \right) - (-2\ln 1 + 0) \right)$$

$$= \pi \left((-\ln 2 + 1) - (0 + 0) \right)$$

$$= \pi(1 - \ln 2) \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

19a

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

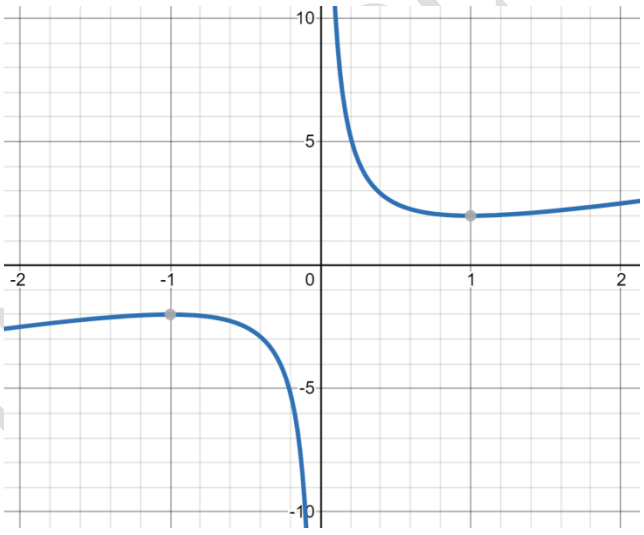
$$x^2 = 1$$

$$x = \pm 1$$

When $x = -1, y = -2, \frac{d^2y}{dx^2} = -2 < 0$, hence the curve is concave down and this is a maximum.

When $x = 1, y = 2, \frac{d^2y}{dx^2} = 2 > 0$, hence the curve is concave up and this is a minimum.

Hence there is a minimum at $(1, 2)$ and a maximum at $(-1, -2)$.



Annotated proofs

19b

$$\frac{5}{2} = x + \frac{1}{x}$$

$$5 = 2x + \frac{2}{x}$$

$$5x = 2x^2 + 2$$

$$2x^2 - 5x + 2 = 0$$

$$(2x - 1)(x - 2) = 0$$

Hence

$x = \frac{1}{2}$ or 2

19c

$$V = \pi \int_{\frac{1}{2}}^{2} \left(\frac{5}{2}\right)^2 dx - \pi \int_{\frac{1}{2}}^{2} \left(x + \frac{1}{x}\right)^2 dx$$

$$= \pi \int_{\frac{1}{2}}^{2} \left(\frac{25}{4} - x^2 - 2 - \frac{1}{x^2}\right) dx$$

$$= \pi \int_{\frac{1}{2}}^{2} \left(\frac{17}{4} - x^2 - \frac{1}{x^2}\right) dx$$

$$= \pi \left[\frac{17}{4}x - \frac{x^3}{3} + \frac{1}{x}\right]_{\frac{1}{2}}^{2}$$

$$= \pi \left(\left(\frac{17}{2} - \frac{8}{3} + \frac{1}{2}\right) - \left(\frac{17}{8} - \frac{1}{24} + 2\right)\right)$$

$$= \frac{9\pi}{4} \text{ cubic units}$$

Uncorrected proofs

© Cambridge University Press 2019

20a <To come>



21a

$$V = \pi \int_1^3 y^2 \, dx$$

$$= \pi \int_1^3 (2^{x+1})^2 \, dx$$

$$= \pi \int_1^3 2^{2x+2} \, dx$$

21b

$$V \div \pi \times \frac{3-1}{2(4)}\left(f(1) + f(3) + 2\big(f(1.5) + f(2) + f(2.5)\big)\right.$$

$$= \frac{\pi}{4}(2^2 + 2^8 + 2(2^5 + 2^6 + 2^7)$$

$$= 180\pi \text{ cubic units}$$

Uncorrected pr

© Cambridge University Press 2019

21c

$$
\begin{aligned}
V &= \pi \int_1^3 y^2 \, dx \\
&= \pi \int_1^3 (2^{x+1})^2 \, dx \\
&= \pi \int_1^3 2^{2x+2} \, dx \\
&= \pi \int_1^3 e^{\ln 2^{2x+2}} \, dx \\
&= \pi \int_1^3 e^{(2x+2)\ln 2} \, dx \\
&= \frac{\pi}{2 \ln 2} \left[e^{(2x+2)\ln 2} \right]_1^3 \\
&= \frac{120\pi}{\ln 2} \\
&\doteq 173\pi \text{ cubic units}
\end{aligned}
$$

The exact answer is smaller than the trapezoidal-rule approximation because the curve is concave up.

Uncorrected proofs